THE HIGHEST WEIGHT REPRESENTATIONS OF THE CONTACT LIE SUPERALGEBRA ON 1|6-DIMENSIONAL SUPERCIRCLE *)**)

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Among simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth there is only one without Cartan matrix but with an invariant nondegenerate supersymmetric bilinear form. This is $\mathfrak{k}^L(1|6)$, the Lie superalgebra of vector fields on the $(1|6)$-dimensional supercircle preserving the Pfaff equation $\alpha = 0$, where $\alpha = dt + \sum_{1\leq i\leq 3}(\xi_i d\eta_i + \eta_i d\xi_i)$ and where ξ, η are the odd variables, $t = \exp(i\varphi)$ for the angle parameter φ on the circle. For $\mathfrak{k}^L(1|6)$ we compute the Casimir element wherefrom we deduce the Shapovalov determinant and the description of the irreducible Verma modules over $\mathfrak{k}^L(1|6)$.

Introduction

The Cartan matrix of any simple $\mathbb{Z}$-graded Lie algebra of polynomial growth (SZGLAPG, for short) is symmetrizable. These Cartan matrices correspond to Dynkin diagrams and extended Dynkin diagrams. (More exactly, the algebras corresponding to extended diagrams are not simple, they are central extensions, called Kac-Moody algebras, of simple ones; in applications Kac-Moody algebras are even more interesting than simple ones.) The Cartan matrices A corresponding to the Lie algebras of class SZGLAPG are very special: normed so that the diagonal elements are equal to 2, the other elements A_{ij} are such that $A_{ij} \in -\mathbb{Z}_+$, $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$ and satisfy several other relations that guarantee simplicity. For Lie algebras $\mathfrak{g}(A)$ with such a Cartan matrix Shapovalov [1] found a powerful method (the *Shapovalov determinant*) for description of its irreducible highest weight modules (Verma modules and their quotients).

Kac and Kazhdan [2] generalized Shapovalov's result to all Lie algebras with an *arbitrary* (indecomposable, since this does not affect generality) symmetrizable Cartan matrix. They further obtained a description of irreducible modules that occur in the Jordan-Hölder series of an arbitrary Verma module. The invariant nondegenerate symmetric bilinear form and the Casimir element associated to it are crucial in the description. Moreover, thanks to the existence of the Casimir element, the Shapovalov determinant becomes equal to the product of *linear* func-

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tions. Similar results, clearly, hold for Lie superalgebras with Cartan matrix, for the conjectural formulas, see [3].

One can also define the Shapovalov determinant in the absence of the Casimir element provided there is an anti-involution, but the determinant fails to factor into the product of linear functions. For some distinguished stringy superalgebras (for their complete list see [4]) the Shapovalov determinant is calculated and turned out to be the product of quadratic polynomials, cf. [5–7] and refs. therein.

The result of this note: calculation of the Casimir element for a remarkable Lie superalgebra $\mathfrak{k}^{\mathrm{L}}(1|6)$ and the corollaries: calculation of the Shapovalov determinant and the description of the irreducible Verma modules over $\mathfrak{k}^{\mathrm{L}}(1|6)$.

1 A description of $\mathfrak{k}^{\mathrm{L}}(1|6)$

1.1 Supercircle

A *supercircle* or (for a physicist) a *superstring* of dimension $1|n$ is the real supermanifold $S^{1|n}$ associated with the rank n trivial vector bundle over the circle. Let $x = \mathrm{e}^{\mathrm{i}\varphi}$, where φ is the angle parameter on the circle, be the even indeterminate of the Fourier transforms; let $\theta = (\theta_1, \dots, \theta_n)$, be the odd coordinates on the supercircle formed by a basis of the fiber of the trivial bundle over the circle. Then (x, θ) are the coordinates on $(\mathbb{C}^*)^{1|n}$, the complexification of $S^{1|n}$.

Let the contact form be

$$\tilde{\alpha} = \mathrm{d}x - \sum_{1\leq i\leq n} \theta_i \mathrm{d}\theta_i .$$

Usually, if $\left[\frac{n}{2}\right] = k$ we rename the first $2k$ indeterminates and express $\tilde{\alpha}$ as follows for $n = 2k$ and $n = 2k + 1$, respectively:

$$\alpha = \mathrm{d}x - \sum_{1\leq i\leq k} (\xi_i \mathrm{d}\eta_i + \eta_i \mathrm{d}\xi_i) \quad \text{or} \quad \alpha = \mathrm{d}x - \sum_{1\leq i\leq k} (\xi_i \mathrm{d}\eta_i + \eta_i \mathrm{d}\xi_i) - \theta \mathrm{d}\theta.$$

For the classification of simple "stringy" Lie superalgebras of vector fields and their nontrivial central extensions see [4]. Among the "main" series are: $\mathfrak{vect}^{\mathrm{L}}(1|n) = \mathfrak{Der}\,\mathbb{C}[x^{-1}, x, \theta]$, of all vector fields and $\mathfrak{k}^{\mathrm{L}}(1|n)$ that preserves the Pfaff equation $\alpha = 0$. The superscript $^{\mathrm{L}}$ indicates that we consider vector fields with Laurent coefficients, not polynomial ones.

1.2 The modules of tensor fields

To advance further, we have to recall the definition of the modules of tensor fields over the general vectoral Lie superalgebra $\mathfrak{vect}(m|n)$ and its subalgebras, see [8]. Let $\mathfrak{g} = \mathfrak{vect}(m|n)$ (for any other $\mathbb{Z}$-graded vectoral Lie superalgebra the construction is identical) and $\mathfrak{g}_{\geq} = \underset{i\geq 0}{\oplus} \mathfrak{g}_i$. Clearly, $\mathfrak{vect}_0(m|n) \cong \mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda = \mathrm{lwt}(V)$. Make V into a $\mathfrak{g}_{\geq}$-module setting

$\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \underset{i>0}{\oplus} \mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates $x = (u, \xi)$. The superspace $T(V) = \mathrm{Hom}_{U(\mathfrak{g}_\geq)}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré-Birkhoff-Witt theorem, to $\mathbb{C}[[x]] \otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V (or λ). When $\lambda = (a, \dots, a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$. We usually consider irreducible $\mathfrak{g}_0$-modules.

Examples: $T(\vec{0})$ is the superspace of functions; $Vol(m|n) = T(1, \dots, 1; -1, \dots$ $\dots, -1)$ (the semicolon separates the first m coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $Vol(m|n)$ corresponding to the ordered set of coordinates x by $vol(x)$. The space of λ-densities is $Vol^\lambda(m|n) = T(\lambda, \dots, \lambda; -\lambda, \dots, -\lambda)$. In particular, $Vol^\lambda(m|0) = T(\vec{\lambda})$ but $Vol^\lambda(0|n) = T(\overrightarrow{-\lambda})$.

1.3 Modules of tensor fields over stringy superalgebras

Denote by $T^{\mathrm{L}}(V) = \mathbb{C}[t^{-1}, t] \otimes V$ the $\mathfrak{vect}(1|n)$-module that differs from $T(V)$ by allowing the Laurent polynomials as coefficients of its elements instead of just polynomials. Clearly, $T^{\mathrm{L}}(V)$ is a $\mathfrak{vect}^{\mathrm{L}}(1|n)$-module. Define the *twisted with weight* μ version of $T^{\mathrm{L}}(V)$ by setting:

$$T^{\mathrm{L}}_\mu(V) = \mathbb{C}[t^{-1}, t]t^\mu \otimes V.$$

• **The "simplest" modules – the analogues of the standard or identity representation of the matrix algebras**. The simplest modules over the Lie superalgebras of series $\mathfrak{vect}$ are, clearly, the modules of λ-densities. These modules are characterized by the fact that they are of rank 1 over $\mathcal{F}$, the algebra of functions. Over stringy superalgebras, we can also twist these modules and consider Vol^λ_μ. Observe that for $\mu \notin \mathbb{Z}$ this module has only one submodule, the image of the exterior differential d, see [8], whereas for $\mu \in \mathbb{Z}$ there is, additionally, the kernel of the residue:

$$\begin{aligned} &\mathrm{Res} : Vol^{\mathrm{L}} \longrightarrow \mathbb{C}, \\ &\quad f vol_{t,\xi} \;\mapsto\; \text{the coeff. of } \frac{\xi_1 \dots \xi_n}{t} \text{ in the power series expansion of } f. \end{aligned}$$

• Over contact superalgebras $\mathfrak{k}(2n+1|m)$, it is more natural to express the simplest modules not in terms of λ-densities but via powers of the form α:

$$\mathcal{F}_\lambda = \begin{cases} \mathcal{F}\alpha^\lambda & \text{for } n = m = 0\,, \\ \mathcal{F}\alpha^{\lambda/2} & \text{otherwise.} \end{cases}$$

Observe that $Vol^\lambda \cong \mathcal{F}_{\lambda(2n+2-m)}$ as $\mathfrak{k}(2n+1|m)$-modules. In particular, the Lie superalgebra of series $\mathfrak{k}$ does not distinguish between $(\partial/\partial t)$ and α^{-1}: their transformation rules are identical. Hence, $\mathfrak{k}(2n+1|m) \cong \mathcal{F}_{-1}$ if $n = m = 0$ or $\mathcal{F}_{-2}$ otherwise.

1.4 Convenient formulas

A laconic way to describe the Lie superalgebras of series $\mathfrak{k}$ is via *generating functions*. For $f \in \mathbb{C}[t, \theta]$ set :

$$K_f = \Delta(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E,$$

where $E = \sum_i \theta_i(\partial/\partial\theta_i)$, $\Delta(f) = 2f - E(f)$, and H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha$:

$$H_f = -(-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial\theta_j}\frac{\partial}{\partial\theta_j}\right), \quad f \in \mathbb{C}[\theta].$$

The choice of the form α instead of $\tilde{\alpha}$ only affects the form of H_f that we give for $m = 2k$:

$$H_f = -(-1)^{p(f)}\sum_{j\leq k}(\frac{\partial f}{\partial\xi_j}\frac{\partial}{\partial\eta_j} + \frac{\partial f}{\partial\eta_j}\frac{\partial}{\partial\xi_j}), \quad f \in \mathbb{C}[\xi, \eta].$$

Since

$$L_{K_f}(\alpha) = K_1(f)\alpha,$$

it follows that $K_f \in \mathfrak{k}(2n+1|m)$.

To the (super)commutator $[K_f, K_g]$ there corresponds the *contact bracket* of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{\text{k.b.}}}.$$

The explicit formula for the contact brackets is as follows. Let us first define the brackets on functions that do not depend on t (resp. τ). The *Poisson bracket* $\{\cdot,\cdot\}_{\text{P.b.}}$ is given by the formula

$$\{f, g\}_{\text{P.b.}} = -(-1)^{p(f)}\left[\sum_{j\leq m}(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial\eta_j} + \frac{\partial f}{\partial\eta_j}\frac{\partial g}{\partial\xi_j})\right].$$

Now, the contact bracket is

$$\{f, g\}_{\text{k.b.}} = \Delta(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\Delta(g) - \{f, g\}_{\text{P.b.}}.$$

It is not difficult to prove the following isomorphism of superspaces:

$$\mathfrak{k}(2n+1|m) \cong Span(K_f : f \in \mathbb{C}[t, p, q, \xi]).$$

2 The Shapovalov determinant for $\mathfrak{k}^{\mathrm{L}}(1|6)$

In the realization of $\mathfrak{k}^{\mathrm{L}}(1|6)$ with the help of generating functions the invariant nondegenerate bilinear form B is given by the formula

$$B(K_f, K_g) = \text{Res } fg, \text{ where } \text{Res}(f) = \text{the coefficient of } \frac{\xi_1 \dots \xi_3 \eta_1 \dots \eta_3}{t}.$$

Designate the elements of degree 0 and weight 0 with respect B and the standard grading as

$$\begin{gathered} H_1 = K_{\xi_1\eta_1}, \qquad H_2 = K_{\xi_2\eta_2}, \qquad H_3 = K_{\xi_3\eta_3}, \qquad H_4 = K_t; \\ H_5 = K_{\frac{1}{t}\xi_2\xi_3\eta_3\eta_2}, \ H_6 = K_{\frac{1}{t}\xi_1\xi_3\eta_3\eta_1}, \ H_7 = K_{\frac{1}{t}\xi_1\xi_2\eta_2\eta_1}, \ H_8 = K_{\frac{1}{t^2}\xi_1\xi_2\xi_3\eta_3\eta_2\eta_1}. \end{gathered}$$

It is easy to verify directly that

$$H_1^* = H_5, \quad H_2^* = H_6, \quad H_3^* = H_7, \quad H_4^* = H_8.$$

2.1. Lemma . *The following element corresponds to the form B and, therefore, belongs to (the completion of) the center of the enveloping algebra of $\mathfrak{k}^{\mathrm{L}}(1|6)$:*

$$\Delta = \sum_{\alpha > 0} e_\alpha^* e_\alpha + \sum_{i=1}^{4} H_i H_i^* + 4H_5 + 2H_6 - 4H_8,$$

where $B(e_\alpha, e_\alpha^*) = 1$.

2.2. Theorem . *Let $(a_1, \dots, a_8)$ be the highest weight of the Verma module M^a over $\mathfrak{k}^{\mathrm{L}}(1|6)$ and $(b_1, \dots, b_8)$ be one of the weights of M^a. The module M^a is irreducible if and only if*

$$\begin{aligned} & a_1a_5 + a_2a_6 + a_3a_7 + a_4a_8 \neq \\ & \neq a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8 + b_1a_5 + b_2a_6 + b_3a_7 + b_4a_8 \\ & \quad + b_1b_5 + b_2b_6 + b_3b_7 + b_4b_8 - 4b_5 - 2b_6 + 4b_8 \,. \end{aligned}$$

Proof of Lemma is a direct verification, Theorem is proved along the lines of [2].

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