INTEGRABLE SYSTEMS AND LIE SUPERALGEBRAS

D. A. Leites and M. A. Semenov-Tyan-Shanskii UDC 519.4

The point functor allows us to associate with each Lie superalgebra the class of ordinary Lie algebras, composed of its points over Grassman algebras. To these Lie algebras we extend the Alder–Kostant geometric scheme of constructing non-linear Lax equations.

0. A scheme of constructing integrable systems associated with Lie algebras was proposed by Kostant and Adler. Cf. [1] for a detailed account of an improved version of it and for the literature (cf. also the paper of Semenov-Tyan-Shanskii in the present issue). In this note we consider the transfer of the Adler–Kostant scheme to Lie superalgebras. The technical means for such a generalization and for the construction of examples are known from the structure theory of simple Lie superalgebras. The psychological difficulties are more serious: it is not easy to imagine what a differential equation on a ringed space with nilpotents in the structural sheaf means (and precisely such a definition is usually used in work on supermanifolds).

In the present note we adopt a completely elementary point of view: instead of super-objects (supergroup, supermanifolds) we consider the ordinary objects associated with them (Lie groups, manifolds, etc.). The "language of points" which allows us to pass to such a description is introduced in Sec. 1. (Of necessity the account here is more algebraized, since it is necessary to establish a connection with the "nonelementary" definitions of [2].) The method of constructing set-theoretic models of objects with nilpotents in the structure sheaf ("point functor") is well known in algebraic geometry. The specifics of the problem compel us to consider immediately a whole collection of set-theoretic models, and not one as is usually the case.

1. Let $\mathcal{M}=(\mathcal{M},\mathcal{O})$ be a supermanifold [2]. (In this paper we consider only algebraic supermanifolds over $\mathbb{C}$.) For any commutative superalgebra C we define the set $\mathcal{M}(C)=\mathcal{M}or(C,\mathcal{M})$. To a morphism of supermanifolds $\alpha:\mathcal{M}\longrightarrow\mathcal{N}$ corresponds a map of sets $\alpha(C):\mathcal{M}(C)\longrightarrow\mathcal{N}(C)$; here to morphism $\varphi:C\longrightarrow C'$ correspond maps $\varphi^{\mathcal{M}}:\mathcal{M}(C')\longrightarrow\mathcal{M}(C)$, $\varphi^{N}:\mathcal{N}(C')\longrightarrow\mathcal{N}(C)$ such that $\varphi^{\mathcal{M}}\alpha(C)=\alpha(C')\varphi^{\mathcal{N}}$. The supermanifold $\mathcal{G}$ is a supergroup if and only if the sets $\mathcal{G}(C)$ are groups and the maps $\varphi^{\mathcal{G}}$, $\varphi\in\mathcal{M}or\,(C,C')$ are group homomorphisms.

By an action a of a supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$ we mean a collection of actions $a(C):\mathcal{G}(C)\times\mathcal{M}(C)\longrightarrow\mathcal{M}(C)$, compatible with the substitutions $\varphi:C\longrightarrow C'$.

Examples. 1. The linear supermanifold of dimension (n,m) $\mathcal{V}^{n,m}=(V_{\bar 0},\mathcal{O}_{V_0}\otimes_{\mathbb{C}}\Lambda(V_{\bar 1}))$, where $V_{\bar 0},V_{\bar 1}$ are linear spaces of dimension n,m, respectively. Let $V=V_{\bar 0}\oplus V_{\bar 1}$ be a $\mathbb{Z}_2$-graded space. Then $\mathcal{V}(C)=(V\otimes_{\mathbb{C}}C)_{\bar 0}$.

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2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over $\mathbb{C}$. Then $\mathfrak{g}(C) = (\mathfrak{g} \otimes_{\mathbb{C}} C)_{\bar{0}}$; obviously $\mathfrak{g}(C)$ has an imposed (canonical) Lie algebra structure over $\mathbb{C}$. The co-adjoint representations of the Lie algebras $\mathfrak{g}(C)$ are compatible with the morphisms $\varphi : C \longrightarrow C'$, and define a representation of the Lie superalgebra $\mathfrak{g}$ in the sense of the definition given above.

3. The supergroup $\mathcal{GL}(p|q)$ is a functor which associates with a commutative superalgebra C the group $GL(p|q;C)$ of even invertible matrices of order (p,q) with elements from C.

Remark. For a given supermanifold $\mathcal{M}$, instead of an arbitrary commutative superalgebra C, it suffices to consider a sufficiently large Grassman algebra Λ. For each supermanifold the number of variables in Λ is its own, so it is convenient not to fix it, but to consider it "very large." We shall make more precise in what sense a supermanifold is defined by its Λ-points.

LEMMA. Let $\alpha, \beta : \mathcal{M} \longrightarrow \mathcal{N}$ be morphisms of supermanifolds, $\alpha(n), \beta(n) : \mathcal{M}(\Lambda(n)) \longrightarrow \mathcal{N}(\Lambda(n))$ be the corresponding mappings of sets. (i) If $\dim \mathcal{M} = (p,q)$, $n \geq q$, then it follows from $\alpha(n) = \beta(n)$ that $\alpha = \beta$. (ii) Let the collection of maps $\tilde{\alpha}(n)$: $\mathcal{M}(\Lambda(n)) \longrightarrow \mathcal{N}(\Lambda(n))$ be defined for $n \geq q$ and suppose to each homomorphism $\varphi : \Lambda(n) \longrightarrow \Lambda(n')$ there corresponds a map of sets $\varphi^{\mathcal{M}} : \mathcal{M}(\Lambda(n')) \longrightarrow \mathcal{M}(\Lambda(n))$, $\varphi^{\mathcal{N}} : \mathcal{N}(\Lambda(n')) \longrightarrow \mathcal{N}(\Lambda(n))$ such that $\varphi^{\mathcal{M}} \tilde{\alpha}(n) = \tilde{\alpha}(n') \varphi^{\mathcal{N}}$. Then there exists a morphism of supermanifolds $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ such that $\alpha(n) = \tilde{\alpha}(n)$.

The use of the language of Λ-points allows us to avoid the consideration of super-objects almost completely. For example, the Adler—Konstant scheme for Lie superalgebras is the ordinary scheme applicable to the special class of Lie algebras which are Λ-points of Lie superalgebras. The structure theory of Lie superalgebras is used to describe the decomposition into subalgebras, the construction of invariants of the coadjoint action, orbits, etc.

The convenience of the language of Λ-points compared with the classical language of ringed spaces is clearly evident upon considering actions of supergroups on supermanifolds. The orbits of such an action furnish examples of objects which one wants to consider sub-supermanifolds, but which are not (they are not ringed spaces).

Example. We consider $SO(n)$-orbits in the standard n-dimensional representation. We shall now assume the space of the representation is $(0,n)$-dimensional (and the action of the group is understood, for example, in the language of Λ-points, described above). Then all the orbits except the trivial one in general have no points over $\mathbb{C}$!

By a G-supermanifold we mean a subfunctor of a functor from the category CSA of commutative superalgebras to the category of sets, $Sets$, of the supermanifold represented.* Thus, an orbit which is not a supermanifold will be a G-supermanifold.

*Apparently it is precisely such objects which are meant in [3]. We have not succeeded in attaching a precise menaing to the difficult definitions of this paper (it is not excluded that the author means not G-supermanifolds, but simply any functors $CSA \longrightarrow Sets$; the meaninglessness of such a definition is already evident for ordinary manifolds).

The possibility of transferring all differential-goemetric constructions to -supermanifolds is unclear. This does not lead to difficulties in studying differential equations: in accord with our general approach we consider only ordinary differential equations on the set of Λ -points. For G -supermanifolds the set of Λ -points is a well-defined ordinary manifold. (For example, if one is concerned with an orbit of the coadjoint action, this is an ordinary orbit of the group G_Λ.) Here the following questions arise: on the "functoriality" of dynamics, i.e., on the connection of solutions of a differential equation in different Λ-hulls and on the representability of different functors in our categories. Fortunately, these questions are not important for the elementary investigation of differential equations.

It is important to note that any differential equation on $\mathfrak{g}_\Lambda^*$ can be reduced by expansions with respect to a basis in the Grassman algebra to a system of differential equations such that all the equations except the underlying ones (i.e., those connected with the even part of the superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$) are linear nonautonomous equations [4]. (Cf. the examples of Toda superlattices in [11].) Thus, passage to Lie superalgebras does not increase our possibilities too much: the new nonlinear equations are extensions of old ones by means of linear nonautonomous equations. Such a mechanism is well-known to specialists in the method of the inverse problem. The justification for the superization of the Adler—Kostant scheme is in its geometric character and the possibility of a coordinate-free treatment of rather complex systems (which in the coordinates associated with a basis in Λ are not so simple even to write out.)

2. To apply the Adler—Kostant scheme and the Lie algebra $\mathfrak{g}_\Lambda$ we need the following objects.

(1) The decomposition of $\mathfrak{g}_\Lambda$ into the linear sum of two Lie subalgebras. We shall consider the natural decompositions generated by the decomposition of the Lie superalgebra $\mathfrak{g}$ into the linear sum of two Lie subsuperalgebras, $\mathfrak{g}=\mathfrak{a}\dotplus\mathfrak{b}$. Here we obviously have $\mathfrak{g}_\Lambda=\mathfrak{a}_\Lambda+\mathfrak{b}_\Lambda$.

(2) The description of the orbits of the algebra $(\mathfrak{g}_\Lambda)_0=(\mathfrak{g}_0)_\Lambda=\mathfrak{a}_\Lambda\oplus\mathfrak{b}_\Lambda$.

(3) The description of the invariants of the coadjoint action of the algebra $\mathfrak{g}_\Lambda$.

Let $\mathfrak{g}$ be a contragradient simple Lie superalgebra. As examples of such superalgebras over $\mathbb{C}$ one has the classical finite-dimensional Lie superalgebras and the exterior automorphisms of infinite-dimensional Lie superalgebras associated with them which are the analogs of the Kac-Moody algebras, cf. [5], and also the compressions of some of them and the Lie superalgebras of stringed theories $\mathcal{W}(2)$, $\mathcal{Y}(2)$, $\mathcal{K}(n)$ for $0\leqslant n\leqslant 3$, cf. [6]. The decompositions of the Lie superalgebra $\mathfrak{g}$ into a sum of Lie subsuperalgebras are canonically connected with its $\mathbb{Z}$ -gradings. We note that in contrast with Lie algebras for simple finite-dimensional Lie superalgebras there are several inequivalent systems of simple roots and consequently essentially different principal $\mathbb{Z}$ -gradings. All $\mathbb{Z}$ -gradings are listed in [7].

In describing the invariants of finite-dimensional simple Lie superalgebras it is useful to identify the superalgebra with its dual space (when such an identification is possible).

Cf. [8] for a list of invariant polynomials on finite-dimensional simple Lie superalgebras. In large part they have the form $\mathrm{str}\,\rho(X)^n$ or $\mathrm{otr}\,\rho(X)^n$ for some representation ρ, where str is the supertrace [2], and $\mathrm{otr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}\,B$ is the odd or strange trace. On the Lie superalgebras of the series $W, S, S\Pi$ there are no invariant polynomials except constants.

One should dwell especially on the superalgebras $\mathfrak{g}(n)$. For them the invariant bilinear form on $\mathfrak{g}(n)$ is odd; hence the canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by this form is the operator Π of change of parity. Here the Kirillov bracket on $\mathfrak{g}^*$ goes into the so-called Byutan bracket or odd bracket

$$\{f, g\}(X) = (X(\Pi([\Pi df_X, \Pi dg_X])),$$
$$f, g \in E(\mathfrak{g}) \stackrel{\mathrm{def}}{=} S(\Pi(\mathfrak{g})),\ X \in \mathfrak{g}.$$

The passage from invariants of a Lie superalgebra to invariants of the coadjoint representation of the Lie algebra $\mathfrak{g}_\Lambda$ is obvious due to the following simple assertion.

Lemma. Let $\varphi \in I(\mathfrak{g}^*)$ be an invariant polynomial on the dual space of the Lie superalgebra $\mathfrak{g}$. For $X \in \mathfrak{g}_\Lambda$, $\varepsilon \in \Lambda$ let $\varphi_\varepsilon(X) = \int_\Lambda \varepsilon\varphi(X)$. Then φ_ε is an invariant polynomial on $\mathfrak{g}^*_\Lambda$; and the map $\Lambda \times I(\mathfrak{g}^*) \longrightarrow I(\mathfrak{g}^*_\Lambda)$: $(\varepsilon, \varphi) \longmapsto \varphi_\varepsilon$ is surjective.

Examples of the simplest integrable systems associated with finite-dimensional simple superalgebras are given in the paper of R. Yu. Kirillova published in this collection. More interesting systems are connected with infinite-dimensional simple Lie superalgebras and two-dimensionalized systems of the type considered in [9]. The consideration of these systems in the spirit of the Adler–Kostant method uses central extension of the current superalgebra (for ordinary current algebras the two-dimensionalization is described in [10]). The most important problem here is the choice of interesting equations.

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