Invitation

to Partial Differential Equations

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ABSTRACT

This book contains a course on partial differential equations which was first taught twice to students of experimental groups in the Department of Mechanics and Mathematics of the Moscow State University. All problems presented here were used in recitations accompanying the lectures. Later this course was also taught in Freie University (Berlin) and, in a shortened version, in Northeastern University (Boston).

There are about 100 problems in the book. They are not just exercises. Some of them add essential information though require no new ideas to solve them. Hints and, sometimes, solutions are provided at the end of the book.
I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I –
I took the one less travelled by,
And that has made all the difference.

From *The Road not Taken*
by Robert Frost

Preface

It will not be an overstatement to say that everything we see around us (as well as hear, touch, and so on) can be described by partial differential equations or, in a slightly more restricted way, by equations of mathematical physics. Among the processes and phenomena whose adequate models are based on such equations, we encounter, for example, string vibrations, waves on the surface of water, sound, electromagnetic waves (in particular, light), propagation of heat and diffusion, gravitational attraction of planets and galaxies, behavior of electrons in atoms and molecules, and also the Big Bang that led to the creation of our Universe.

It is no wonder, therefore, that the theory of partial differential equations (PDE) forms today a vast branch of mathematics and mathematical physics that uses methods of all the remaining parts of mathematics (from which the PDE theory is inseparable). In turn, PDE influence numerous parts of mathematics, physics, chemistry, biology and other sciences.

A disappointing conclusion is that no book, to say nothing of a textbook, can be sufficiently complete in describing this area. The only possibility remains: to show the reader pictures at an exhibition — several typical methods and problems — chosen by the author’s taste. In doing so, the author has to severely restrict the choice of material.

This book contains, practically intact, a transcript of the PDE course which I taught twice to students of experimental groups at the Department of Mechanics and Mathematics of the Moscow State University, and later, in
a shortened version, in Northeastern University (Boston). Professor Louhiwaara also taught this course in Freie Universität (Berlin).

In Moscow, all problems, presented in this book, were used in recitations accompanying the lectures. There was one lecture per week during the whole academic year (two semesters), recitations were held once a week during the first semester and twice a week during the second one.

My intention was to create a course both concise and modern. These two requirements proved to be contradictory. To make the course concise, I was, first, forced to explain some ideas on simplest examples, and, second, to omit many topics which should be included in a modern course but would have expanded the course beyond a reasonable length. With a particular regret I omit the Schrödinger equation and related material; I hoped the gap would be filled by a parallel course of quantum mechanics.

In any modern course of equations of mathematical physics, something should, perhaps, have been said about nonlinear equations but I found it extremely difficult to select this something.

The bibliography at the end of the book contains a number of textbooks where the reader can find information complementing this course. Naturally, the material of these books overlap, but it seemed difficult to do a more precise selection, so I let this selection to be done according to the reader’s taste.

The problems in the book are not just exercises. Some of them add essential information but require no new ideas to solve them.

Acknowledgments

I am very grateful to Prof. S. P. Novikov who organized the experimental groups and invited me to teach PDE to them. I am also greatly indebted to professors of the Chair of Differential Equations of the Moscow State University for many valuable discussions.

I owe a lot to Prof. Louhiwaara who provided unexpectedly generous help in editing of an early version of this text, and encouraged me to publish this book.

I thank Ognen Milatovic for help in verifying problems.

I am also grateful to Prof. D. A. Leites who translated the book into English.
Selected notational conventions

Cross-references inside the book are natural: Theorem 6.2 stands for Theorem 2 in Sect. 6, Problem 6.2 is Problem 2 from Sect. 6, etc.

:= means “denotes by definition”.
≡ means “identically equal to”

By \( \mathbb{Z} \), \( \mathbb{Z}_+ \), \( \mathbb{N} \), \( \mathbb{R} \), and \( \mathbb{C} \) we will denote the sets of all integers, non-negative integers, positive integers, real, and complex numbers respectively.

\( I \) denotes the identity operator.

The Heaviside function is \( \theta(z) = \begin{cases} 1 & \text{for } z \geq 0 \\ 0 & \text{for } z < 0. \end{cases} \)
Chapter 1

Linear differential operators

1.1. Definition and examples

We will start by introducing convenient notations for functions of several variables and differential operators. By $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ we will denote the sets of integer, real and complex numbers respectively. A *multi-index* $\alpha$ is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_j \in \mathbb{Z}_+$ (i.e. $\alpha_j$ are non-negative integers). If $\alpha$ is a multi-index, then we set $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = \alpha_1! \ldots \alpha_n!$; for a vector $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ we define $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. If $\Omega$ is an open subset of $\mathbb{R}^n$, then $C^\infty(\Omega)$ denotes the space of complex-valued infinitely differentiable functions on $\Omega$, and $C^\infty_0(\Omega)$ denotes the subspace of infinitely differentiable functions with compact support, i.e. $\varphi \in C^\infty_0(\Omega)$ if $\varphi \in C^\infty(\Omega)$ and there exists a compact $K \subset \Omega$ such that $\varphi$ vanishes on $\Omega \setminus K$. (Here $K$ depends on $\varphi$.)

Denote $\partial_j = \frac{\partial}{\partial x_j} : C^\infty(\Omega) \rightarrow C^\infty(\Omega); D_j = -i\partial_j$, where $i = \sqrt{-1}$.

$$\partial = (\partial_1, \ldots, \partial_n); D = (D_1, \ldots, D_n);$$

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}; D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}.$$  

Therefore, $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ is a mixed derivative operator $\partial^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega), D^\alpha = i^{-|\alpha|}\partial^\alpha$.

If $f \in C^\infty(\Omega)$, then instead of $\partial^\alpha f$ we will sometimes write $f^{(\alpha)}$. 

1
Exercise 1.1. Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \), i.e. \( f \) is a polynomial of \( n \) variables \( x_1, \ldots, x_n \). Prove the Taylor formula

\[
f(x) = \sum_{\alpha} \frac{f^{(\alpha)}(x_0)}{\alpha!} (x - x_0)^\alpha,
\]

where the sum runs over all multi-indices \( \alpha \). (Actually, the sum is finite, i.e. only a finite number of terms may be non-vanishing.)

A **linear differential operator** is an operator \( A : C^\infty(\Omega) \rightarrow C^\infty(\Omega) \) of the form

\[
A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,
\]

where \( a_\alpha(x) \in C^\infty(\Omega) \). Clearly, instead of \( D^\alpha \) we may write \( \partial^\alpha \), but the expression in terms of \( D^\alpha \) is more convenient as we will see in the future. Here \( m \in \mathbb{Z}_+ \), and we will say that \( A \) is an operator of order \( \leq m \). We say that \( A \) is an operator of order \( m \) if it can be expressed in the form (1.2) and there exists a multi-index \( \alpha \) such that \( |\alpha| = m \) and \( a_\alpha(x) \not\equiv 0 \).

**Examples.**

1) The **Laplace operator** or Laplacian \( \Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} = -(D_1^2 + \ldots + D_n^2) \).

2) The **heat operator** \( \frac{\partial}{\partial t} - \Delta \) (here the number of independent variables is equal to \( n + 1 \) and they are denoted by \( t, x_1, \ldots, x_n \)).

3) The **wave operator** or d’Alembertian \( \Box = \frac{\partial^2}{\partial t^2} - \Delta \).

4) The **Sturm-Liouville operator** \( L \) defined for \( n = 1 \) and \( \Omega = (a, b) \) by the formula

\[
Lu = \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u,
\]

where \( p, q \in C^\infty((a, b)) \). Operators in Examples 1)-3) are operators with constant coefficients. The Sturm-Liouville operator has variable coefficients.

1.2. The total and the principal symbols

The **symbol** or the **total symbol** of the operator (1.2) is

\[
a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha, \quad x \in \Omega, \xi \in \mathbb{R}^n,
\]
1.2. The total and the principal symbols

and its principal symbol is the function

\begin{equation}
    a_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \quad x \in \Omega, \xi \in \mathbb{R}^n.
\end{equation}

The symbol belongs to \( C^\infty(\Omega)[\xi] \), i.e., it is a polynomial in \( \xi_1, \ldots, \xi_n \) with coefficients in \( C^\infty(\Omega) \) and the principal symbol is a homogeneous polynomial of degree \( m \) in \( \xi_1, \ldots, \xi_n \) with coefficients in \( C^\infty(\Omega) \).

**Examples.**

1) The total and the principal symbols of the Laplace operator coincide and are equal to
\(-\xi^2 = -(\xi_1^2 + \ldots + \xi_n^2)\).

2) The total symbol of the heat operator \( \frac{\partial}{\partial \tau} - \Delta \) is
\( a(t, x, \tau, \xi) = i\tau + \xi^2 \) and its principal symbol is \( a_2(t, x, \tau, \xi) = \xi^2 \).

3) The total and the principal symbols of the wave operator coincide and are equal to
\( a(t, x, \tau, \xi) = a_2(t, x, \tau, \xi) = -\tau^2 + \xi^2 \).

4) The total symbol of the Sturm-Liouville operator is
\( a(x, \xi) = -p(x)\xi^2 + ip'(x)\xi + q(x) \);
its principal symbol is \( a_2(x, \xi) = -p(x)\xi^2 \).

The symbol of an operator \( A \) is recovered from the operator by the formula
\begin{equation}
    a(x, \xi) = e^{-ix \cdot \xi} A(e^{ix \cdot \xi}),
\end{equation}
where \( A \) is applied with respect to \( x \). Here we used the notation \( x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n \). Formula (1.5) is obtained from the relation
\( D^\alpha e^{ix \cdot \xi} = \xi^\alpha e^{ix \cdot \xi} \),
which is easily verified by induction with respect to \( |\alpha| \). Formula (1.5) also implies that the coefficients \( a_\alpha(x) \) of \( A \) are uniquely defined by this operator, because the coefficients of the polynomial \( a(x, \xi) \) for each fixed \( x \) are uniquely defined by this polynomial viewed as a function of \( \xi \).

It is convenient to apply \( A \) to a more general exponent than \( e^{ix \cdot \xi} \), namely to \( e^{i\lambda \varphi(x)} \), where \( \varphi \in C^\infty(\Omega) \) and \( \lambda \) is a parameter. Then we get

**Lemma 1.1.** If \( f \in C^\infty(\Omega) \), then \( e^{-i\lambda \varphi(x)} A(f(x)e^{i\lambda \varphi(x)}) \) is a polynomial of \( \lambda \) of degree \( \leq m \) with coefficients in \( C^\infty(\Omega) \) and
\begin{equation}
    e^{-i\lambda \varphi(x)} A(f(x)e^{i\lambda \varphi(x)}) = \lambda^m f(x)a_m(x, \varphi_x) + \lambda^{m-1}(... + \ldots,
\end{equation}
i.e., the highest coefficient of this polynomial (that of $\lambda^m$) is equal to $f(x) a_m(x, \varphi_x)$, where $\varphi_x = (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n}) = \text{grad} \varphi$ is the gradient of $\varphi$.

**Proof.** We have
\[ D_j [f(x) e^{i\lambda \varphi(x)}] = \lambda (\partial_j \varphi)(x) e^{i\lambda \varphi(x)} + (D_j f) e^{i\lambda \varphi(x)}, \]
so that the Lemma holds for the operators of order $\leq 1$. In what follows, to find the derivatives $D^\alpha e^{i\lambda \varphi(x)}$ we will have to differentiate products of the form $f(x) e^{i\lambda \varphi(x)}$, where $f \in C^\infty(\Omega)$.

A new factor $\lambda$ will only appear when we differentiate the exponent. Clearly,
\[ D^\alpha e^{i\lambda \varphi(x)} = \lambda |\alpha| \varphi_\alpha^\alpha e^{i\lambda \varphi(x)} + \lambda |\alpha|-1(\ldots) + \ldots, \]
implying the statement of the Lemma. □

**Corollary 1.2.** Let $A$, $B$ be two linear differential operators in $\Omega$, let $k, l$ be their orders, $a_k, b_l$ their principal symbols, $C = A \circ B$, their composition, $c_{k+l}$ the principal symbol of $C$. Then
\[ c_{k+l}(x, \xi) = a_k(x, \xi) b_l(x, \xi). \]

**Remark.** Clearly, $C$ is a differential operator of order $\leq k + l$.

**Proof of Corollary 1.2.** We have
\[ C(e^{i\lambda \varphi(x)}) = A(\lambda^l b_l(x, \varphi_x) e^{i\lambda \varphi(x)} + \lambda^{l-1}(\ldots) + \ldots) = \]
\[ = \lambda^{k+l} a_k(x, \varphi_x) b_l(x, \varphi_x) e^{i\lambda \varphi(x)} + \lambda^{k+l-1}(\ldots) + \ldots. \]

By Lemma 1.1, we obtain
\[ c_{k+l}(x, \varphi_x) = a_k(x, \varphi_x) b_l(x, \varphi_x), \quad \text{for any } \varphi \in C^\infty(\Omega). \]

But then, by choosing $\varphi(x) = x \cdot \xi$, we get $\varphi_x = \xi$ and (1.8) becomes $c_{k+l}(x, \xi) = a_k(x, \xi) b_l(x, \xi)$, as required. □

**1.3. Change of variables**

Let $\kappa : \Omega \longrightarrow \Omega_1$ be a diffeomorphism between two open subsets $\Omega, \Omega_1$ in $\mathbb{R}^n$, i.e. $\kappa$ is a $C^\infty$ map having an inverse $C^\infty$ map $\kappa^{-1} : \Omega_1 \longrightarrow \Omega$. We will denote the coordinates in $\Omega$ by $x$ and coordinates in $\Omega_1$ by $y$. The map $\kappa$ can be defined by a set of functions $y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)$, $y_j \in C^\infty(\Omega)$, whose values at $x$ are equal to the respective coordinates of the point $\kappa(x)$.
If $f \in C^\infty(\Omega_1)$, set $\kappa^* f = f \circ \kappa$, i.e., $(\kappa^* f)(x) = f(\kappa(x))$ or $(\kappa^* f)(x) = f(y_1(x), \ldots, y_n(x))$. Function $\kappa^* f$ is obtained from $f$ by a change of variables or by passage to coordinates $x_1, \ldots, x_n$ from coordinates $y_1, \ldots, y_n$. Recall that the diffeomorphism $\kappa$ has the inverse $\kappa^{-1}$ which is also a diffeomorphism. Clearly, $\kappa^*$ is a linear isomorphism $\kappa^*: C^\infty(\Omega_1) \rightarrow C^\infty(\Omega)$ and $(\kappa^*)^{-1} = (\kappa^{-1})^*$.

Given an operator $A : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$, define the operator $A_1 : C^\infty(\Omega_1) \rightarrow C^\infty(\Omega_1)$ with the help of the commutative diagram

$$
\begin{array}{ccc}
C^\infty(\Omega) & \xrightarrow{A} & C^\infty(\Omega) \\
\uparrow \kappa^* & & \uparrow \kappa^* \\
C^\infty(\Omega_1) & \xrightarrow{A_1} & C^\infty(\Omega_1)
\end{array}
$$

i.e., by setting $A_1 = (\kappa^*)^{-1} A \kappa^* = (\kappa^{-1})^* A \kappa^*$. In other words,

$$A_1 v(y) = \left\{ A[v(y(x))] \right\}_{x=x(y)},$$

where $y(x) = \kappa(x), x(y) = \kappa^{-1}(y)$. Therefore, the operator $A_1$ is, actually, just the expression of $A$ in coordinates $y$.

If $A$ is a linear differential operator, then so is $A_1$, as easily follows from the chain rule and (1.9). Let us investigate a relation between the principal symbols of $A$ and $A_1$. To this end we will use the notions of tangent and cotangent vectors in $\Omega$, as well as tangent and cotangent bundles over $\Omega$ (see Appendix to this Chapter).

**Theorem 1.3.** The value of the principal symbol $a_m$ of an operator $A$ at the cotangent vector $(x, \xi)$ is the same as that of the principal symbol $a'_m$ of $A_1$ at the corresponding vector $(y, \eta)$, i.e.,

$$a_m(x, \xi) = a'_m(\kappa(x), (\kappa^*)^{-1}\eta).$$

In other words, the principal symbol is a well-defined function on $T^*\Omega$ (it does not depend on the choice of $C^\infty$-coordinates in $\Omega$).

**Proof.** A cotangent vector at $x$ can be expressed as the gradient $\varphi_x(x)$ of a function $\varphi \in C^\infty(\Omega)$ at $x$. It is clear from Lemma 1.1 that $a_m(x, \varphi_x(x))$ does not depend on the choice of coordinates. But on the other hand, it is clear that this value does not depend on the choice of a function $\varphi$ with a prescribed differential at $x$. Therefore, the principal symbol is well-defined on $T^*\Omega$. □
1. The canonical form of second order operators with constant coefficients

By a change of variables we can try to transform an operator to a simpler one. Consider a second order operator with constant coefficients of the principal part:

\[ A = - \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{l=1}^{n} b_l \frac{\partial}{\partial x_l} + c, \]

where \(a_{ik}\) are real constants such that \(a_{ik} = a_{ki}\) (the latter can always be assumed since \(\frac{\partial^2}{\partial x_i \partial x_k} = \frac{\partial^2}{\partial x_k \partial x_i}\)). Ignoring the terms of lower order, we will transform the highest order part of \(A\)

\[ A_0 = - \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} \]

to a simpler form with the help of a linear change of variables:

\[ y_k = \sum_{l=1}^{n} c_{kl} x_l, \]

where \(c_{kl}\) are real constants. Consider the quadratic form

\[ Q(\xi) = \sum_{i,k=1}^{n} a_{ik} \xi_i \xi_k, \]

which only differs from the principal symbol of \(A\) by a sign.

With the help of a linear change of variables \(\eta = F\xi\), where \(F\) is a nonsingular constant matrix, we can transform \(Q(\xi)\) to the following form

\[ Q(\xi) = (\pm \eta_1^2 \pm \eta_2^2 \pm \ldots \pm \eta_r^2)|_{\eta=F\xi}. \]

Denote by \(C\) the matrix \((c_{kl})_{k,l=1}^{n}\) in (1.11). By Theorem 1.3, the operator \(A\) in coordinates \(y\) has the form of a second order operator \(A_1\) whose principal symbol is a quadratic form \(Q_1(\eta)\) such that

\[ Q(\xi) = Q_1(\eta)|_{\eta=(^tC)^{-1}\xi}, \]

where \(^tC\) is the transpose of \(C\). From this and (1.12) it follows that we should choose \(C\) so that \((^tC)^{-1} = F\) or

\[ C = (^tF)^{-1}. \]
Then the principal part of \( A \) under the change of variables \( y = Cx \) of the form (1.11) will be reduced to the form

\[
\pm \frac{\partial^2}{\partial y_1^2} \pm \frac{\partial^2}{\partial y_2^2} \pm \ldots \pm \frac{\partial^2}{\partial y_r^2},
\]

called the canonical form.

**Remark.** An operator with variable coefficients may be reduced to the form (1.13) at any fixed point by a linear change of variables.

### 1.5. Characteristics. Ellipticity and hyperbolicity

Let \( A \) be an \( m \)-th order differential operator, \( a_m(x, \xi) \) its principal symbol. A nonzero cotangent vector \((x, \xi)\) is called a characteristic vector if \( a_m(x, \xi) = 0 \). A surface (of codimension 1) in \( \Omega \) is called characteristic at \( x_0 \) if its normal vector at this point is a characteristic vector. The surface is called a characteristic if it is characteristic at every point.

If a surface \( S \) is defined by the equation \( \varphi(x) = 0 \), where \( \varphi \in C^1(\Omega) \), \( \varphi_x|_S \neq 0 \), then \( S \) is characteristic at \( x_0 \) if \( a_m(x_0, \varphi_x(x_0)) = 0 \). The surface \( S \) is a characteristic if \( a_m(x, \varphi_x(x)) \) identically vanishes on \( S \).

All level surfaces \( \varphi = \text{const} \) are characteristics if and only if \( a_m(x, \varphi_x(x)) \equiv 0 \).

Theorem 1.3 implies that the notion of a characteristic does not depend on the choice of coordinates in \( \Omega \).

**Examples.**

1) The Laplace operator \( \Delta \) has no real characteristic vectors.

2) For the heat operator \( \frac{\partial}{\partial t} - \Delta \) the vector \((\tau, \xi) = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \) is a characteristic one. The surfaces \( t = \text{const} \) are characteristics. The surface \( t = |x|^2 \) (paraboloid) is characteristic at a single point only (the origin).

3) Consider the wave operator \( \frac{\partial^2}{\partial t^2} - \Delta \). Its characteristic vectors at each point \((t, x)\) constitute a cone \( \tau^2 = |\xi|^2 \). Any cone \((t - t_0)^2 = |x - x_0|^2 \) is a characteristic. In particular, for \( n = 1 \) (i.e. for \( x \in \mathbb{R}^1 \)) the lines \( x + t = \text{const} \) and \( x - t = \text{const} \) are characteristics.

**Definition.**

a) An operator \( A \) is called elliptic if \( a_m(x, \xi) \neq 0 \) for all \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \setminus \{0\} \), i.e., if \( A \) has no non-zero characteristic vectors.
b) An operator $A$ in the space of functions of $t, x$, where $t \in \mathbb{R}^1, x \in \mathbb{R}^n$, is hyperbolic with respect to $t$ if the equation $a_m(t, x, \tau, \xi) = 0$ considered as an equation in $\tau$ for any fixed $t, x, \xi$ has exactly $m$ distinct real roots if $\xi \neq 0$. In this case one might say that the characteristics of $A$ are real and distinct.

**Examples.**

a) The Laplace operator $\Delta$ is elliptic.

b) The heat operator is neither elliptic nor hyperbolic with respect to $t$.

c) The wave operator is hyperbolic with respect to $t$ since the equation $\tau^2 = |\xi|^2$ has two different real roots $\tau = \pm|\xi|$ if $\xi \neq 0$.

d) The Sturm-Liouville operator $Lu \equiv \frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u$ is elliptic on $(a, b)$ if $p(x) \neq 0$ for $x \in (a, b)$.

By Theorem 1.3, the ellipticity of an operator does not depend on the choice of coordinates. The hyperbolicity with respect to $t$ does not depend on the choice of coordinates in $\mathbb{R}^n$.

Let us see what it means for the surface $x_1 = \text{const}$ to be characteristic. The coordinates of the normal vector are $(1, 0, \ldots, 0)$. Substituting them into the principal symbol, we get

$$a_m(x; 1, 0, \ldots, 0) = \sum_{|\alpha|=m} a_\alpha(x)(1, 0, \ldots, 0)^\alpha = a_{(m,0,\ldots,0)}(x),$$

i.e., the statement “the surface $x_1 = \text{const}$ is characteristic at $x$” means that $a_{(m,0,\ldots,0)}(x)$, the coefficient of $(\frac{1}{i})^m \frac{\partial^m}{\partial x_1^m}$, vanishes at $x$.

All surfaces $x_1 = \text{const}$ are characteristics if and only if $a_{(m,0,\ldots,0)} \equiv 0$.

This remark will be used below to reduce second order operators with two independent variables to a canonical form.

It is also possible to find characteristics by solving the Hamilton-Jacobi equation $a_m(x, \varphi_x(x)) = 0$. If $\varphi$ is a solution of this equation, then all surfaces $\varphi = \text{const}$ are characteristic. Integration of the Hamilton-Jacobi equation is performed with the help of the Hamiltonian system of ordinary differential equations with the Hamiltonian $a_m(x, \xi)$ (see Chapter 12 or any textbook on classical mechanics, e.g., Arnold [1]).
1.6. Characteristics and the canonical form of 2nd order operators and 2nd order equations for \( n = 2 \)

For \( n = 2 \) characteristics are curves and are rather easy to find. For example, consider a 2nd order operator

\[
A = a \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x \partial y} + c \frac{\partial^2}{\partial y^2} + \ldots,
\]

where \( a, b, c \) are smooth functions of \( x, y \) defined in an open set \( \Omega \subset \mathbb{R}^2 \) and dots stand for terms containing at most 1st order derivatives. Let \((x(t), y(t))\) be a curve in \( \Omega \), \((dx, dy)\) its tangent vector, \((-dy, dx)\) the normal vector. The curve is a characteristic if and only if

\[
a(x, y)dy^2 - 2b(x, y)dxdy + c(x, y)dx^2 = 0
\]
on it.

If \( a(x, y) \neq 0 \), then in a neighborhood of the point \((x, y)\) we may assume that \( dx \neq 0 \); hence, \( x \) can be taken as a parameter along the characteristic which, therefore, is of the form \( y = y(x) \). Then the equation for the characteristics is of the form

\[
a(y')^2 - 2by' + c = 0.
\]

If \( b^2 - ac > 0 \), then the operator (1.14) is hyperbolic and has 2 families of real characteristics found from ordinary differential equations

\[
y' = \frac{b + \sqrt{b^2 - ac}}{a}, \tag{1.15}
\]

\[
y' = \frac{b - \sqrt{b^2 - ac}}{a}. \tag{1.16}
\]

Observe that in this case two nontangent characteristics pass through each point \((x, y)\) \( \in \Omega \). We express these families of characteristics in the form \( \varphi_1(x, y) = C_1 \) and \( \varphi_2(x, y) = C_2 \), where \( \varphi_1, \varphi_2 \in C^\infty(\Omega) \). This means that \( \varphi_1, \varphi_2 \) are first integrals of equations (1.15) and (1.16), respectively. Suppose that \( \text{grad} \varphi_1 \neq 0 \) and \( \text{grad} \varphi_2 \neq 0 \) in \( \Omega \). Then \( \text{grad} \varphi_1 \) and \( \text{grad} \varphi_2 \) are linearly independent, since characteristics of different families are not tangent. Let us introduce new coordinates: \( \xi = \varphi_1(x, y), \eta = \varphi_2(x, y) \). In these coordinates, characteristics are lines \( \xi = \text{const} \) and \( \eta = \text{const} \). But then the coefficients of \( \frac{\partial^2}{\partial \xi^2} \) and \( \frac{\partial^2}{\partial \eta^2} \) vanish identically and \( A \) takes the form

\[
A = p(\xi, \eta) \frac{\partial^2}{\partial \xi \partial \eta} + \ldots, \tag{1.17}
\]
called the canonical form. Here \( p(\xi, \eta) \neq 0 \) (recall that we have assumed \( a \neq 0 \)).

If \( c(x, y) \neq 0 \) then the reduction to the canonical form (1.17) is similar. It is not necessary to consider these two cases separately, since we have only used the existence of integrals \( \varphi_1(x, y), \varphi_2(x, y) \) with the described properties. These integrals can be also found when coefficients \( a \) and \( c \) vanish (at one point or on an open set). This case can be reduced to one of the above cases, e.g. by a rotation of coordinate axes (if \( a^2 + b^2 + c^2 \neq 0 \)).

One often considers differential equations of the form

\[
Au = f,
\]

where \( f \) is a known function, \( A \) a linear differential operator, and \( u \) an unknown function. If \( A \) is a hyperbolic 2nd order operator with two independent variables, i.e. an operator of the form (1.14 with \( b^2 - ac > 0 \)) (in this case the equation (1.18) is also called hyperbolic), then, introducing the above coordinates \( \xi, \eta \) and dividing by \( p(\xi, \eta) \), we reduce (1.18) to the canonical form

\[
\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots = 0,
\]

where dots stand for terms without 2nd order derivatives of \( u \).

Now let \( b^2 - ac \equiv 0 \) (then the operator (1.14) and the equation (1.18) with this operator are called parabolic). Let us assume that \( a \neq 0 \). Then, for characteristics we get the differential equation

\[
y' = \frac{b}{a}.
\]

Let us find characteristics and write them in the form \( \varphi(x, y) = \text{const} \), where \( \varphi \) is the first integral of (1.19) and \( \text{grad} \varphi \neq 0 \). Choose a function \( \psi \in C^\infty(\Omega) \) such that \( \text{grad} \varphi \) and \( \text{grad} \psi \) are linearly independent and introduce new coordinates \( \xi = \varphi(x, y), \eta = \psi(x, y) \). In the new coordinates the operator \( A \) does not have the term \( \frac{\partial^2}{\partial \xi^2} \), since the lines \( \varphi = \text{const} \) are characteristics. But then the coefficient of the term \( \frac{\partial^2}{\partial \xi \partial \eta} \) also vanishes, since the principal symbol should be a quadratic form of rank 1. Thus, the canonical form of a parabolic operator is

\[
A = p(\xi, \eta) \frac{\partial^2}{\partial \eta^2} + \ldots
\]

For a parabolic equation (1.18) the canonical form is

\[
\frac{\partial^2}{\partial \eta^2} + \ldots = 0.
\]
1.7. The general solution (n = 2)

Observe that if \( b^2 - ac = 0 \) and \( a^2 + b^2 + c^2 \neq 0 \), then \( a \) and \( c \) cannot vanish simultaneously, since then we would also have \( b = 0 \). Therefore, we always have either \( a \neq 0 \) or \( c \neq 0 \) and the above procedure is always applicable.

Finally, consider the case \( b^2 - ac < 0 \). The operator (1.14) and the equation (1.18) are in this case called elliptic ones. Suppose for simplicity that functions \( a, b, c \) are real analytic. Then the existence theorem for complex analytic solutions of the complex equation

\[
y' = \frac{b + \sqrt{b^2 - ac}}{a}
\]

implies the existence of a local first integral

\[
\varphi(x, y) + i\psi(x, y) = C,
\]

where \( \varphi, \psi \) are real-valued analytic functions, \( \text{grad} \varphi \) and \( \text{grad} \psi \) are linearly independent. It is easy to verify that by introducing new coordinates \( \xi = \varphi(x, y), \eta = \psi(x, y) \), we can reduce \( A \) to the canonical form

\[
A = p(\xi, \eta) \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + \ldots,
\]

where \( p(\xi, \eta) \neq 0 \). Dividing (1.18) by \( p \), we reduce it to the form

\[
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \ldots = 0
\]

1.7. The general solution of a homogeneous hyperbolic equation with constant coefficients for \( n = 2 \)

As follows from Section 1.6, the hyperbolic equation

\[
(1.20) \quad a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = 0,
\]

where \( a, b, c \in \mathbb{R} \) and \( b^2 - ac > 0 \), can be reduced by the change of variables

\[
\xi = y - \lambda_1 x, \quad \eta = y - \lambda_2 x,
\]

where \( \lambda_1, \lambda_2 \) are roots of the quadratic equation \( a\lambda^2 - 2b\lambda + c = 0 \), to the form

\[
\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.
\]

Assuming \( u \in C^2(\Omega) \), where \( \Omega \) is a convex open set in \( \mathbb{R}^2 \), we get \( \frac{\partial}{\partial \xi} (\frac{\partial u}{\partial \eta}) = 0 \) implying \( \frac{\partial u}{\partial \eta} = F(\eta) \) and

\[
u = f(\xi) + g(\eta),
\]
where \( f, g \) are arbitrary functions of class \( C^2 \). In variables \( x, y \) we have

\[
(1.21) \quad u(x, y) = f(y - \lambda_1 x) + g(y - \lambda_2 x).
\]

It is useful to consider functions \( u(x, y) \) of the form (1.21), where \( f, g \) are not necessarily of the class \( C^2 \) but from a wider class, e.g., \( f, g \in L^1_{\text{loc}}(\mathbb{R}) \), i.e., \( f, g \) are locally integrable. Such functions \( u \) are called generalized solutions of equation (1.20). For example let \( f(\xi) \) have a jump at \( \xi_0 \), i.e., has different limits as \( \xi \to \xi_0^+ \) and as \( \xi \to \xi_0^- \). Then \( u(x, y) \) has a discontinuity along the line \( y - \lambda_1 x = \xi_0 \).

Note that the lines \( y - \lambda_1 x = \text{const} \), \( y - \lambda_2 x = \text{const} \) are characteristics. Therefore, in this case discontinuities of solutions are propagating along characteristics. Similar phenomenon occurs in the case of general hyperbolic equations.

**Example.** Characteristics of the wave equation \( \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \) are \( x - at = \text{const} \), \( x + at = \text{const} \). The general solution of this equation can be expressed in the form

\[
u(t, x) = f(x - at) + g(x + at).
\]

Observe that \( f(x - at) \) is the wave running to the right with the speed \( a \) and \( g(x + at) \) is the wave running to the left with the same speed \( a \). The general solution is the sum (or superposition) of two such waves.

1.8. Appendix: Tangent and cotangent vectors

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). The tangent vector to \( \Omega \) at a point \( x \in \Omega \) is a vector \( v \) which is tangent to a parametrized curve \( x(t) \) in \( \Omega \) passing through \( x \) at \( t = 0 \). Here we assume that the function \( t \mapsto x(t) \) is a \( C^\infty \) function defined for \( t \in (-\varepsilon, +\varepsilon) \) with some \( \varepsilon > 0 \), and takes values in \( \Omega \). In coordinates \( x_1, \ldots, x_n \) we have \( x(t) = (x_1(t), \ldots, x_n(t)) \) and the tangent vector \( v \) is given as

\[
v = \dot{x}(0) = \left. \frac{dx}{dt} \right|_{t=0} = (\dot{x}_1(0), \ldots, \dot{x}_n(0))
\]

\[
\begin{pmatrix}
\left. \frac{dx_1}{dt} \right|_{t=0} \\
\vdots \\
\left. \frac{dx_n}{dt} \right|_{t=0}
\end{pmatrix}
\]

\[
= (v_1, \ldots, v_n).
\]

Interpreting \( t \) as the time variable, we can say that the tangent vector \( v = \dot{x}(0) \) is the velocity vector at the moment \( t = 0 \) for the motion given by the curve \( x(t) \).
1.8. Appendix: Tangent and cotangent vectors

The real numbers \( v_1, \ldots, v_n \) are called the coordinates of the tangent vector \( \dot{x}(0) \). They depend on the choice of the coordinate system \((x_1, \ldots, x_n)\). In fact, they are defined even for any curvilinear coordinate system.

Two tangent vectors at \( x \) can be added (by adding the corresponding coordinates of these vectors), and any tangent vector can be multiplied by any real number (by multiplying each of its coordinates by this number). These operations do not depend on the choice of coordinates in \( \Omega \). So all tangent vectors at \( x \in \Omega \) form a well defined vector space (independent of coordinates in \( \Omega \)), which will be denoted \( T_x \Omega \) and called tangent space to \( \Omega \) at \( x \).

A cotangent vector \( \xi \) at \( x \in \Omega \) is a real-valued linear function on the tangent space \( T_x \Omega \). All cotangent vectors at \( x \) form a linear space which is denoted \( T^*_x \Omega \) and called cotangent space to \( \Omega \) at \( x \).

Denote by \( T\Omega \) the union of all tangent spaces over all points \( x \in \Omega \), i.e.

\[
T\Omega = \bigcup_x T_x \Omega \cong \Omega \times \mathbb{R}^n.
\]

It is called the tangent bundle of \( \Omega \). Similarly, we will introduce the cotangent bundle

\[
T^*\Omega = \bigcup_x T^*_x \Omega \cong \Omega \times \mathbb{R}^n.
\]

For every tangent vector \( v \) at \( x \in \Omega \) and any function \( f \in C^\infty(\Omega) \) we can define the derivative of \( f \) in the direction \( v \): it is

\[
v f = \frac{df(x(t))}{dt} = \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} \dot{x}_j(0) = \sum_{j=1}^{n} v_j \frac{\partial f(x)}{\partial x_j},
\]

where \( x(t) \) is a curve in \( \Omega \) such that \( x(0) = x \) and \( \dot{x}(0) = v \). Sometimes, \( v f \) is also called directional derivative. Taking \( f \) to be a coordinate function \( x_j \), we obtain \( v x_j = v_j \), so \( v \) is uniquely defined by the corresponding directional derivative, and we can identify \( v \) with its directional derivative which is a linear function on \( C^\infty(\Omega) \).

In particular, take vectors at \( x \in \Omega \) which are tangent to the coordinate axes. Then the corresponding directional derivatives will be simply \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \), and they form a basis of \( T_x \Omega \). (For example, \( \frac{\partial}{\partial x_1} \) is tangent to the curve \( x(t) = (x_1 + t, x_2, \ldots, x_n) \), and has coordinates \((1, 0, \ldots, 0)\). For the \( j \)th partial derivative all coordinates vanish except \( j \)th one which equals 1.)
The dual basis in $T^*_x \Omega$ consists of linear functions $dx_1, \ldots, dx_n$ on $T_x \Omega$ such that $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$.

For any tangent vector $\dot{x}(0) \in T_{x(0)} \Omega$ which is the velocity vector of a $C^\infty$ curve $x(t)$ at $t = 0$, its coordinates in the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are equal to $\frac{dx_1}{dt}|_{t=0}, \ldots, \frac{dx_n}{dt}|_{t=0}$. An example of a cotangent vector is given by the differential $df$ of a function $f \in C^\infty(\Omega)$: it is a function on $T_{x(0)} \Omega$ whose value at the tangent vector $\dot{x}(0)$ is equal to $\frac{df(x(t))}{dt}|_{t=0}$ and whose coordinates in the basis $dx_1, \ldots, dx_n$ are $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$, so

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$ 

In euclidean coordinates, the differential of a smooth function $f$ can be identified with its gradient $\nabla f$, which is the tangent vector with the same coordinates.

Given a diffeomorphism $\kappa : \Omega \to \Omega_1$, we have a natural map $\kappa_* : T_x \Omega \to T_{\kappa(x)} \Omega_1$ (the tangent map or the differential of $\kappa$). Namely, for a tangent vector $\dot{x}(0)$, which is the velocity vector of a curve $x(t)$ at $t = 0$, the vector $\kappa_* \dot{x}(0)$ is the velocity vector of the curve $(\kappa \circ x)(t) = \kappa(x(t))$ at $t = 0$.

The corresponding dual map of spaces of linear functions $\kappa^* : T^*_{\kappa(x)} \Omega_1 \to T^*_x \Omega$ is defined by $(\kappa^* \xi)(\dot{x}(0)) = \xi(\kappa_* \dot{x}(0))$, where $\xi \in T_{\kappa(x)} ^* \Omega_1$.

Choose bases $\{ \frac{\partial}{\partial x_j} \}_{j=1}^n$, $\{ \frac{\partial}{\partial y_j} \}_{j=1}^n$, $\{ dy_j \}_{j=1}^n$, $\{ dx_j \}_{j=1}^n$ in spaces $T_x \Omega$, $T_{\kappa(x)} \Omega_1$, $T^*_{\kappa(x)} \Omega_1$, $T^*_x \Omega$ respectively. In these bases the matrix of $\kappa_*$ is $(\kappa_*)_{kl} = \frac{\partial y_l}{\partial x_k}$. It is called the Jacobi matrix. The matrix of $\kappa^*$ is transpose of $\kappa_*$. Since $\kappa$ is a diffeomorphism, the maps $\kappa_*$ and $\kappa^*$ are isomorphisms at each point $x \in \Omega$.

It is convenient to define a cotangent vector at $x$ by a pair $(x, \xi)$, where $\xi = (\xi_1, \ldots, \xi_n)$, $\xi \in \mathbb{R}^n$, is the set of coordinates of this vector in the basis $\{dx_j\}_{j=1}^n$ in $T^*_x \Omega$. Under the isomorphism $(\kappa^*)^{-1} : T^* \Omega \to T^* \Omega_1$ the covector $(y, \eta)$ corresponds to the covector $(x, \xi)$, where $y = \kappa(x), \eta = (\kappa^*)^{-1} \xi$.

### 1.9. Problems

1. Reduce the equations to the canonical form:

   a) $u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} + 6u_{zz} = 0$;

   b) $u_{xy} - u_{xz} + u_x + u_y - u_z = 0$. 
1.2. Reduce the equations to the canonical form:

a) \( x^2 u_{xx} + 2xyu_{xy} - 3y^2 u_{yy} - 2xu_x + 4yu_y + 16x^4 u = 0; \)

b) \( y^2 u_{xx} + 2xyu_{xy} + 2x^2 u_{yy} + yu_y = 0; \)

c) \( u_{xx} - 2u_{xy} + u_{yy} + u_x + u_y = 0. \)

1.3. Find the general solution of the equations:

a) \( x^2 u_{xx} - y^2 u_{yy} - 2yu_y = 0; \)

b) \( x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0. \)
Chapter 2

One-dimensional wave equation

2.1. Vibrating string equation

Let us derive an equation describing small vibrations of a string. Note right away that our derivation will be not mathematical but rather physical or mechanical. However its understanding is essential to grasp the physical meaning of, first, the wave equation itself, and second, but not less importantly, the initial and boundary conditions. A knowledge of the derivation and physical meaning helps also to find various mathematical tools in the study of this equation (the energy integral, standing waves, etc.). Generally, derivation of equations corresponding to different physical and mechanical problems is important for understanding mathematical physics and is essentially a part of it.

So, let us derive the equation of small vibrations of a string. We consider the vibrations of a taut string such that each point moves in direction perpendicular to the direction of the string in its equilibrium. We assume here that all the forces appearing in the string are negligible compared to the tension directed along the string (we assume the string to be perfectly flexible, i.e., nonresistant to bending).

First of all, choose the variables describing the form of the string. Suppose that in equilibrium the taut string is stretched along the $x$-axis. For the start, we will consider the inner points of the string disregarding its ends.
One-dimensional wave equation

Suppose that the vibrations keep the string in the plane \((x, y)\) so that each point of the string is only shifted parallel to \(y\)-axis and this shift at the time \(t\) equals \(u(t, x)\), see Fig. 1.

Therefore, at a fixed \(t\) the graph of \(u(t, x)\), as a function of \(x\), is the shape of the string at the moment \(t\), and for a fixed \(x\) the function \(u(t, x)\) describes the motion of one point of the string. Let us compute the length of the part of the string corresponding to the interval \((a, b)\) on the \(x\)-axis. It is equal to

\[
\int_a^b \sqrt{1 + u_x^2} \, dx.
\]

Our main assumption is the negligibility of the additional string lengthening, compared with the equilibrium length. More precisely, we assume that \(u_x^2 \ll 1\) and consider \(u_x^2\) negligible compared to 1. Note that if \(\alpha = \alpha(t, x)\) is the angle between the tangent to the string and the \(x\)-axis then \(\tan \alpha = u_x\), \(\cos \alpha = \frac{1}{\sqrt{1 + u_x^2}}\), \(\sin \alpha = \frac{u_x}{\sqrt{1 + u_x^2}}\); under our assumptions we should consider \(\cos \alpha \approx 1\) and \(\sin \alpha \approx u_x\). If \(T = T(t, x)\) is the string’s tension, then its horizontal component is \(T \cos \alpha \approx T\) and the vertical one is \(T \sin \alpha \approx Tu_x\).

By the Hooke’s law in elasticity theory (which should be considered an assumption in our model) the tension of the string at any point and at any given moment of time is proportional to the relative lengthening of a small piece of string at this point, i.e. the lengthening (the increase of the length compared with the equilibrium) divided by the equilibrium length of this piece. Since we assume the additional lengthening to be negligible, we should have \(T(t, x) = T = \text{const.}\) (Alternatively, we can argue as follows. Since all points of the string move in the vertical direction, along the \(y\)-axis, the sum of the horizontal components of all forces acting on the segment of the string over \([a, b]\) should vanish. This means that \(T(t, a) = T(t, b)\) which due to the arbitrariness of \(a\) and \(b\) yields \(T(t, x) = T(t)\), that is, \(T\) is independent of \(x\). Then, the negligibility of the lengthening implies that in fact \(T(t) = T = \text{const.}\).)
Let us write the equation of motion of the part of the string above \([a, b]\), see Fig. 2. Let \(\rho(x)\) be the linear density of the string at \(x\) (the ratio of the mass of the infinitesimal segment to the length of this segment at \(x\)). The vertical component of the force is

\[
(2.1) \quad (Tu_x)_{x=b} - (Tu_x)_{x=a} = \int_a^b Tu_{xx}(t, x)\,dx,
\]

assuming the absence of external forces. In the presence of vertical external forces distributed with density \(g(t, x)\) (per unit mass of the string), we should add to (2.1) the following term

\[
(2.2) \quad \int_a^b \rho(x)g(t, x)\,dx.
\]

The vertical component of the momentum of this segment of the string is equal to

\[
(2.3) \quad \int_a^b \rho(x)u_t(t, x)\,dx.
\]

Let us use the well-known corollary of the Newton’s Second and Third Laws which states that the rate of momentum’s change in time is equal to the sum of all external forces. (See Appendix to this Chapter.) Then (2.1) – (2.3) give

\[
\int_a^b [\rho(x)u_{tt}(t, x) - Tu_{xx}(t, x) - \rho(x)g(t, x)]\,dx = 0,
\]

or, since \(a\) and \(b\) are arbitrary,

\[
(2.4) \quad \rho(x)u_{tt} - Tu_{xx} - \rho(x)g(t, x) = 0.
\]

In particular, for \(\rho(x) = \rho = \text{const}\) and \(g(t, x) \equiv 0\) we get

\[
(2.5) \quad u_{tt} - a^2 u_{xx} = 0, \quad \text{where} \quad a = \sqrt{T/\rho}.
\]
We would have arrived at the same equation (2.5) if we had applied d’Alembert’s principle equating the sum of all the external and inertial forces to zero.

One may arrive at (2.4) via another route by using Lagrange’s equations. (See Appendix to this Chapter for an elementary introduction to the Calculus of variations and Lagrange’s equations.) Suppose the external forces are absent. Clearly, the kinetic energy of the interval \((a, b)\) of the string is

\[ K = \frac{1}{2} \int_a^b \rho(x) u_t^2(t, x) dx. \]

To calculate the potential energy of the string whose form is that of the graph of \(v(x)\), \(x \in (a, b)\), we should calculate the work necessary to move the string from the equilibrium position to \(v(x)\). Suppose this move is defined by the “curve” \(v(x, \alpha)\), \(\alpha \in [0, 1]\), so that \(v(x, 0) = 0\), \(v(x, 1) = v(x)\). The force

\[ T v_x(x + \Delta x, \alpha) - T v_x(x, \alpha) = \int_x^{x+\Delta x} T v_{xx}(x, \alpha) dx, \]

acts on the part of the string corresponding to the interval \((x, x + \Delta x)\) on the \(x\)-axis.

As \(\alpha\) varies from \(\alpha\) to \(\alpha + \Delta \alpha\), the displacement of the point with the abscissa \(x\) equals

\[ v(x, \alpha + \Delta \alpha) - v(x, \alpha) = \int_\alpha^{\alpha+\Delta \alpha} v_{\alpha}(x, \alpha) d\alpha. \]

Therefore, to move the part of the string \((x, x + \Delta x)\) from \(\alpha\) to \(\alpha + \Delta \alpha\), the work

\[ -Tv_{xx}(x, \alpha)v_{\alpha}(x, \alpha)\Delta x \cdot \Delta \alpha + o(\Delta x \cdot \Delta \alpha) \]

should be performed. Integrating with respect to \(x\) and \(\alpha\) we see that the total work performed over the segment \((a, b)\) is

\[ A = -\int_0^1 \int_a^b T v_{xx}v_{\alpha} dx d\alpha. \]

Integrating by parts, we get

\[ -\int_a^b T v_{xx}v_{\alpha} dx = -Tv_xv_{\alpha}|_a^b + \int_a^b T v_x v_{x\alpha} dx = -Tv_xv_{\alpha}|_a^b + \frac{d}{d\alpha} \int_a^b T v_x^2 dx \]

and integrating with respect to \(\alpha\) from 0 to 1, we get

\[ A = -\int_0^1 T v_{x}(x, \alpha)v_{\alpha}(x, \alpha)|_{\alpha=0}^{\alpha=1} dx + \frac{1}{2} \int_a^b T v_x^2(x) dx. \]
2.1. Vibrating string equation

Suppose that the ends of the string are fixed at the points 0 and \(l\), i.e., their displacements are equal to zero during the whole process of motion. Then we may assume that \(v(0, \alpha) = v(l, \alpha) = 0\) for \(\alpha \in [0, 1]\), and write

\[
A = \frac{1}{2} \int_0^l T v_x^2(x) dx
\]

instead of (2.6) for \(a = 0, b = l\). This clearly implies that the potential energy of the string with fixed ends 0, \(l\) at time \(t\) is equal to

\[
U = \frac{1}{2} \int_0^l T u_x^2(t, x) dx.
\]

Now we can write the Lagrangian of the string:

\[
L = K - U = \frac{1}{2} \int_0^l \rho(x) u_t^2(t, x) dx - \frac{1}{2} \int_0^l T u_x^2(t, x) dx
\]

which is a functional in \(u\) and \(u_t\) (here \(u(t, x)\) plays the role of the set of coordinates at time \(t\) and \(u_t(t, x)\) the role of the set of velocities). Let us write the action \(\int_{t_0}^{t_1} L dt\) and equate the variation of the action with respect to \(u\) to 0. The usual integration by parts under the assumption that \(\delta u|_{t=t_0} = \delta u|_{t=t_1} = 0\) gives:

\[
\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \int_0^l \rho(x) u_t(t, x) \delta u_t(t, x) dx dt - \int_{t_0}^{t_1} \int_0^l T u_x(t, x) \delta u_x(t, x) dx dt =
\]

\[
= - \int_{t_0}^{t_1} \int_0^l [\rho(x) u_{tt}(t, x) - T u_{xx}(t, x)] \delta u(t, x) dx dt,
\]

which, since \(\delta u\) is arbitrary, implies

\[
(2.7) \quad \rho(x) u_{tt}(t, x) - T u_{xx}(t, x) = 0.
\]

This means that we obtained equation (2.4) (with \(g(t, x) \equiv 0\)).

Note an important circumstance here: the total energy of the string is equal to

\[
H = K + U = \frac{1}{2} \int_0^l \rho(x) u_t^2(t, x) dx + \frac{1}{2} \int_0^l T u_x^2(t, x) dx.
\]

Let us prove the energy conservation law: the energy of an oscillating string with fixed ends is conserved. We will verify this law by using (2.7). We have

\[
\frac{dH}{dt} = \int_0^l \rho(x) u_t(t, x) u_{tt}(t, x) dx + \int_0^l T u_x(t, x) u_{tx}(t, x) dx.
\]
Integrating by parts in the second integral, we get

\[ (2.8) \quad \frac{dH}{dt} = \int_0^l u_t(t, x) [\rho(x) u_{tt}(t, x) - Tu_{xx}(t, x)] \, dx + Tu_x(t, x) u_t(t, x)|_{x=0}^l. \]

The last summand in (2.8) vanishes due to the boundary conditions \( u|_{x=0} = u|_{x=l} = 0 \), since then \( u_t|_{x=0} = \frac{d}{dt} (u|_{x=0}) = 0 \) and similarly \( u_t|_{x=l} = 0 \). The first summand vanishes due to (2.7). Thus, \( \frac{dH}{dt} = 0 \), yielding the energy conservation law: \( H(t) = \text{const.} \)

The conservation of energy implies the uniqueness of the solution for the string equation (2.4) provided \( \rho(x) > 0 \) everywhere and the law of motion for string’s ends is given:

\[ (2.9) \quad u|_{x=0} = \alpha(t), \quad u|_{x=l} = \beta(t), \]

and the initial values (the position and the velocity of the string) are fixed:

\[ (2.10) \quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) \quad \text{for } x \in [0, l]. \]

Indeed, if \( u_1, u_2 \) are two solutions of the equation (2.4) satisfying (2.9) and (2.10) then their difference \( v = u_1 - u_2 \) satisfies the homogeneous equation (2.7) and the homogeneous boundary and initial conditions

\[ v|_{x=0} = v|_{x=l} = 0, \]

\[ (2.11) \quad v|_{t=0} = v_t|_{t=0} = 0 \quad \text{for } x \in [0, l]. \]

But then the energy conservation law for \( v \) implies

\[ \int_0^l [\rho(x) v_t^2(t, x) + Tu_x^2(t, x)] \, dx \equiv 0, \]

yielding \( v_t \equiv 0 \) and \( v_x \equiv 0 \), i.e., \( v = \text{const.} \). By (2.11) we now have \( v \equiv 0 \), i.e., \( u_1 \equiv u_2 \), as required.

Finally, note that the derivation of the string equation could be, of course, carried out without the simplifying assumption \( u_x^2 \ll 1 \). As a result, we would have obtained a nonlinear equation that hardly admits investigation by simple methods. The equation (2.4) is in fact the linearization (principal linear part) of this nonlinear equation. Properties of the solution of (2.4) give some idea of the behavior of solutions of the nonlinear equation. Observe, however, that for large deformations this nonlinear equation is also inadequate to describe the physical problem, since the resistance to the bending and other effects which were unaccounted for, will arise.
2.2. Unbounded string. The Cauchy problem. D’Alembert’s formula

From the physical viewpoint, an unbounded string is an idealization meaning that we consider an inner part of the string assuming the ends to be sufficiently far away, so that during the observation time they do not affect the given part of the string. As we will see, the consideration of an unbounded string helps in the investigation of a half-bounded string (which is a similar idealization) and of a bounded string.

Passing to the mathematical discussion, consider the one-dimensional wave equation (2.5) for $x \in \mathbb{R}, t \geq 0$. A natural problem here is the Cauchy problem, i.e., an initial value problem for (2.5) with the initial conditions

\[(2.12) \quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x).\]

From the physical viewpoint (2.12) means that the initial position and the initial velocity of the string are given. We might expect that, like finite-dimensional mechanical problems, this Cauchy problem is well posed, i.e., there exists a unique solution continuously depending on the initial data $\varphi$ and $\psi$. As we will see shortly, this is really so.

We use the general solution of the equation (2.5):

\[u(t, x) = f(x - at) + g(x + at)\]

found in Section 1.7. From (2.12) we get the system of two equations to determine two arbitrary functions $f$ and $g$:

\[(2.13) \quad f(x) + g(x) = \varphi(x),\]

\[(2.14) \quad -af'(x) + ag'(x) = \psi(x).\]

The integration of the second equation yields

\[(2.15) \quad -f(x) + g(x) = \frac{1}{a} \int_{x_0}^{x} \psi(\xi)d\xi + C.\]

Now, from (2.13) and (2.15), we get

\[f(x) = \frac{1}{2} \varphi(x) - \frac{1}{2a} \int_{x_0}^{x} \psi(\xi)d\xi - \frac{C}{2},\]

\[g(x) = \frac{1}{2} \varphi(x) + \frac{1}{2a} \int_{x_0}^{x} \psi(\xi)d\xi + \frac{C}{2}.\]
Therefore,
\[
u(t, x) = \frac{1}{2} \varphi(x - at) - \frac{1}{2a} \int_{x_0}^{x-at} \psi(\xi) d\xi + \frac{1}{2} \varphi(x + at) + \frac{1}{2a} \int_{x_0}^{x+at} \psi(\xi) d\xi
\]
or
\[
(2.16) \quad u(t, x) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi.
\]
Thus, the solution \( u(t, x) \) actually exists, is unique and continuously depends on the initial data \( \varphi, \psi \) under an appropriate choice of topologies in the sets of initial data \( \varphi, \psi \) and functions \( u(t, x) \). For example, it is clear that if \( u_1 \) is a solution corresponding to the initial conditions \( \varphi_1, \psi_1 \) and for a sufficiently small \( \delta > 0 \) we have
\[
(2.17) \quad \sup_{x \in \mathbb{R}} |\varphi_1(x) - \varphi(x)| < \delta, \quad \sup_{x \in \mathbb{R}} |\psi_1(x) - \psi(x)| < \delta,
\]
then for arbitrarily small \( \varepsilon > 0 \) we have
\[
(2.18) \quad \sup_{x \in \mathbb{R}, \ t \in [0, T]} |u_1(t, x) - u(t, x)| < \varepsilon.
\]
(More precisely, for any \( \varepsilon > 0 \) and \( T > 0 \) there exists \( \delta > 0 \) such that (2.17) implies (2.18).) Therefore, the Cauchy problem is well posed.

The formula (2.16) which determines the solution of the Cauchy problem is called d’Alembert’s formula. Let us make several remarks concerning its derivation and application.

First, note that this formula makes sense for any locally integrable functions \( \varphi \) and \( \psi \) and in this case it gives a generalized solution of (2.5). Here we will only consider continuous solutions. Then we can take any continuous function for \( \varphi \) and any locally integrable one for \( \psi \). It is natural to call the functions \( u(t, x) \) obtained in this way the generalized solutions of the Cauchy problem. We will consider generalized solutions together with ordinary ones on equal footing and skip the adjective “generalized”.

Second, it is clear from d’Alembert’s formula that the value of the solution at \( (t_0, x_0) \) only depends on values of \( \varphi(x) \) at \( x = x_0 \pm at_0 \) and values of \( \psi(x) \) at \( x \in (x_0-at_0, x_0+at_0) \). In any case, it suffices to know \( \varphi(x) \) and \( \psi(x) \) on \( [x_0-at_0, x_0+at_0] \). The endpoints of the closed interval \( [x_0-at_0, x_0+at_0] \) can be determined as intersection points (in \( (t, x) \)-space) of the characteristics passing through \( (t_0, x_0) \), with the \( x \)-axis, see Fig. 3.

The triangle formed by these characteristics and the \( x \)-axis is the set of points in the half-plane \( t \geq 0 \) where the value of the solution is completely
2.2. Unbounded string. D’Alembert’s formula

Figure 3. Dependence domain

Figure 4. Influence domain

determined by initial values on the segment \([x_0 - at_0, x_0 + at_0]\). This triangle is called the dependence domain for the interval \([x_0 - at_0, x_0 + at_0]\).

An elementary analysis of the derivation of d’Alembert’s formula shows that the formula is true for any solution defined in the triangle with the characteristics as its sides and an interval \([c, d]\) of the x-axis as its base (i.e., it is not necessary to require for a solution to be defined everywhere on the half-plane \(t \geq 0\)). Indeed, from (2.13) and (2.14) we find the values \(f(x)\) and \(g(x)\) at \(x \in [c, d]\) (if the initial conditions are defined on \([c, d]\)). But this determines the values of \(u(t, x)\) when \(x - at \in [c, d]\) and \(x + at \in [c, d]\), i.e., when characteristics through \((t, x)\) intersect the segment \([c, d]\). We can, of course, assume \(f(x)\) and \(g(x)\) to be defined on \([c, d]\), i.e., \(u(t, x)\) is defined in the above described triangle which is the dependence domain of the interval \([c, d]\).

The physical meaning of the dependence domain is transparent: it is the complement to the set of points \((t, x)\) such that the wave starting the motion with the velocity \(a\) at a point outside \([c, d]\) at time \(t = 0\) cannot reach the point \(x\) by time \(t\).

Further, the values of \(\phi\) and \(\psi\) on \([c, d]\) do not affect \(u(t, x)\) if \(x + at < c\) or \(x - at > d\), i.e., if the wave cannot reach the point \(x\) during the time \(t\) starting from an inside point of the interval \([c, d]\). This explains why the domain bounded by the segment \([c, d]\) and the rays of the lines \(x + at = c\),

\[\begin{align*}
\text{Figure 3. Dependence domain} \\
\text{Figure 4. Influence domain}
\end{align*}\]
2. One-dimensional wave equation

Figure 5. Cartoon to Example 2.1

$x - at = d$ belonging to the half-plane $t \geq 0$ is called the influence domain of the interval $[c, d]$. This domain is shaded on Fig. 4. This domain is the complement to the set of points $(t, x)$ for which $u(t, x)$ does not depend on $\varphi(x)$ and $\psi(x)$ for $x \in [c, d]$.

Example 2.1. Let us draw the form of the string at different times if $\psi(x) = 0$, and $\varphi(x)$ has the form shown on the upper-most of the graphs of Fig. 5, i.e., the graph of $\varphi(x)$ has the form of an isosceles triangle with the base $[c, d]$. Let $d - c = 2l$. We draw the form of the string at the most interesting times to show all of its essentially different forms.

All these pictures together form a cartoon depicting vibrations of the string such that at the initial moment the string is pulled at the point $\frac{1}{2}(c + d)$ whereas the points $c$ and $d$ kept fixed, and then at $t = 0$ the whole
string is released. The formula

\[ u(t, x) = \frac{\varphi(x - at) + \varphi(x + at)}{2} \]

makes it clear that the initial perturbation \( \varphi(x) \) splits into two identical waves one of which runs to the left, and the other to the right with the speed \( a \).

The dotted lines on Fig. 5 depict divergent semi-waves \( \frac{1}{2} \varphi(x - at) \) and \( \frac{1}{2} \varphi(x + at) \) whose sum is \( u(t, x) \). We assume \( \varepsilon > 0 \) to be sufficiently small compared to the characteristic time interval \( \frac{1}{a} \) (actually it suffices that \( \varepsilon < \frac{1}{2a} \)).

2.3. A semi-bounded string. Reflection of waves from the end of the string

Consider the equation (2.5) for \( x \geq 0, t \geq 0 \). We need to impose a boundary condition at the end \( x = 0 \). For instance, let the motion of the string left end be given by

\[(2.19)\quad u|_{x=0} = \alpha(t), \quad t \geq 0,\]

and the initial conditions be only given for \( x \geq 0 \):

\[(2.20)\quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \geq 0.\]

The problem with initial and boundary conditions is usually called a mixed problem.

Therefore, the problem with conditions (2.19) - (2.20) for a wave equation (2.5) is an example of a mixed problem (sometimes called the first boundary value problem for the string equation). Let us verify that this boundary value problem is also well posed and solve it. ▲

▼ Let us recall an important notion of convexity.

**Definition 2.1.** Let \( A \) be a subset in \( \mathbb{R}^n \). We will say that \( A \) is convex (in \( \mathbb{R}^n \)) if for every \( a, b \in A \), the open interval connecting \( a \) and \( b \), is in fact a subset of \( A \).

For a real-valued function \( f : (a,b) \to \mathbb{R} \) with real \( a, b \), we will say that the function \( f \) is convex if its subgraph \( \{(x, y) | x \in (a, b), y < f(x)\} \) is convex in \( \mathbb{R}^2 \). ▲

**Examples.** 1. Any closed cube in \( \mathbb{R}^n \) is convex.
2. An open ball in $\mathbb{R}^n$ with any non-empty subset of the ball boundary added, is convex.

3. *(A step)* A closed cube in $\mathbb{R}^n$ with a part of a smaller cube removed, e.g.

$$[0, 1]^{n-2} \times ([0, 1]^2 \backslash (1/2, 1]^2),$$

is not convex. ▲

Returning to the boundary value problem above, we can see that the simplest way to get an analytic solution is as in the Cauchy problem. The quadrant $t \geq 0, x \geq 0$ is a convex domain, therefore, we can again represent $u$ in the form

$$u(t, x) = f(x - at) + g(x + at),$$

where $g(x)$ is now defined and needed only for $x \geq 0$ because $x + at \geq 0$ for $x \geq 0, t \geq 0$. (At the same time $f(x)$ is defined, as before, for all $x$.)

The initial conditions (2.20) define $f(x)$ and $g(x)$ for $x \geq 0$ as in the case of an unbounded string and via the same formula. Therefore, for $x - at \geq 0$ the solution is determined by d’Alembert’s formula which is, of course, already clear. But now we can make use of the boundary condition (2.19) which gives

$$f(-at) + g(at) = \alpha(t), \quad t \geq 0,$$

implying

(2.21)

$$f(z) = \alpha\left(-\frac{z}{a}\right) - g(-z), \quad z \leq 0.$$  

The summands in (2.21) for $z = x - at$ give the sum of two waves running to the right, one of them is $\alpha(t - \frac{x}{a})$ and is naturally interpreted as the wave generated by oscillations of the end of the string, the other one is $-g(-(x - at))$ and is interpreted as the reflection of the wave $g(x + at)$ running to the left from the left end (note that this reflection results in the change of the sign). If the left end is fixed, i.e., $\alpha(t) \equiv 0$, only one wave, $-g(-(x - at))$, remains and if there is no wave running to the left, then only the wave $\alpha(t - \frac{x}{a})$ induced by the oscillation $\alpha(t)$ of the end is left.

Let us indicate a geometrically more transparent method for solving the problem with a fixed end. We wish to solve the wave equation (2.5) for $t \geq 0, x \geq 0$, with the initial conditions (2.20) and the boundary condition

$$u|_{x=0} = 0, \quad t \geq 0.$$  

Let us try to find a solution in the form of the right hand side of d’Alembert’s formula (2.16), where $\varphi(x), \psi(x)$ are some extensions of functions $\varphi(x), \psi(x)$
onto the whole axis defined for \( x \geq 0 \) by initial conditions (2.20). The boundary condition yields
\[
0 = \frac{\varphi(-at) + \varphi(at)}{2} + \frac{1}{2a} \int_{-at}^{at} \psi(\xi) d\xi,
\]
implying that we will reach our goal if we extend \( \varphi \) and \( \psi \) as odd functions, i.e., if we set
\[
\varphi(z) = -\varphi(-z), \quad \psi(z) = -\psi(-z), \quad z \leq 0.
\]
Thus, we can construct a solution of our problem. The uniqueness of the solution is not clear from this method; but, luckily, the uniqueness was proved above.

**Example 2.2.** Let an end be fixed, let \( \psi(x) = 0 \), and let the graph of \( \varphi(x) \) again have the form of an isosceles triangle with the base \([l, 3l]\). Let us draw a cartoon describing the form of the string. Continuing \( \varphi(x) \) as an odd function, we get the same problem as in Example 2.1. With a dotted line we depict the graph over the points of the left half-axis (introducing it is just a mathematical trick since actually only the half-axis \( x \geq 0 \) exists). We get the cartoon described on Fig. 6.

Of interest here is the moment \( t = \frac{2l}{a} \) when near the end the string is horizontal and so looks not perturbed. At the next moment \( t = \frac{2l}{a} + \varepsilon \), however, the perturbation arises due to a non-vanishing initial velocity. For \( t > \frac{3l}{a} + \varepsilon \) we get two waves running to the right, one of which is negative and is obtained by reflection from the left end of the wave running to the left.

### 2.4. A bounded string. Standing waves. The Fourier method (separation of variables method)

Consider oscillations of a bounded string with fixed ends, i.e., solve the wave equation (2.5) with initial conditions
\[
(2.22) \quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in [0, l]
\]
and boundary conditions
\[
(2.23) \quad u|_{x=0} = u|_{x=l} = 0, \quad t \geq 0.
\]
This can be done as in the preceding section. We will, however, apply another method which also works in many other problems. Note that the
uniqueness of the solution is already proved with the help of the energy conservation law.

We will find standing waves, i.e., the solutions of the wave equation (2.5) defined for $x \in [0, l]$, satisfying (2.23) and having the form

$$u(t, x) = T(t)X(x).$$
2.4. A bounded string. Standing waves. The Fourier method

The term “standing wave” is justified since the form of the string does not actually change with the time except for a factor which is constant in \( x \) and only depends on time. Substituting \( u(t, x) \) into (2.5), we get

\[
T''(t)X(x) = a^2T(t)X''(x).
\]

Excluding the uninteresting trivial cases when \( T(t) \equiv 0 \) or \( X(x) \equiv 0 \), we may divide this equation by \( a^2T(t)X(x) \) and get

\[
\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} \tag{2.24}
\]

The left-hand side of this relation does not depend on \( x \) and the right-hand side does not depend on \( t \). Therefore, they are constants, i.e. (2.24) is equivalent to the relations

\[
T''(t) - \lambda a^2T(t) = 0, \quad X''(x) - \lambda X(x) = 0 \tag{2.25}
\]

with the same constant \( \lambda \). Further, boundary conditions (2.23) imply

\[
X(0) = X(l) = 0.
\]

The boundary value problem

\[
X'' = \lambda X, \quad X(0) = X(l) = 0 \tag{2.26}
\]

is a particular case of the so-called Sturm-Liouville problem.

We will discuss the general Sturm-Liouville problem later. Now, we solve the problem (2.26), i.e., find the values \( \lambda \) (eigenvalues) for which the problem (2.26) has nontrivial solutions (eigenfunctions), and find these eigenfunctions.

Consider the following possible cases.

a) \( \lambda = \mu^2, \mu > 0 \). Then the general solution of the equation \( X'' = \lambda X \) is

\[
X(x) = C_1 \sinh \mu x + C_2 \cosh \mu x.
\]

(Recall that \( \sinh x = \frac{1}{2}(e^x - e^{-x}) \) and \( \cosh x = \frac{1}{2}(e^x + e^{-x}) \).

From \( X(0) = 0 \) we get \( C_2 = 0 \), and from \( X(l) = 0 \) we get \( C_1 \sinh \mu l = 0 \) implying \( C_1 = 0 \), i.e. \( X(x) \equiv 0 \), hence, \( \lambda > 0 \) is not an eigenvalue.

b) \( \lambda = 0 \). Then \( X(x) = C_1 x + C_2 \) and boundary conditions imply again \( C_1 = C_2 = 0 \) and \( X(x) \equiv 0 \).

c) \( \lambda = -\mu^2, \mu > 0 \). Then

\[
X(x) = C_1 \sin \mu x + C_2 \cos \mu x;
\]
2. One-dimensional wave equation

hence, \( X(0) = 0 \) implies \( C_2 = 0 \) and \( X(l) = 0 \) implies \( C_1 \sin \mu l = 0 \). Assuming \( C_1 \neq 0 \), we get \( \sin \mu l = 0 \); therefore, the possible values of \( \mu \) are:

\[
\mu_k = \frac{k \pi}{l}, \quad k = 1, 2, \ldots,
\]

and the eigenvalues and eigenfunctions are:

\[
\lambda_k = -\left(\frac{k \pi}{l}\right)^2, \quad X_k = \sin \frac{k \pi x}{l}, \quad k = 1, 2, \ldots
\]

Let us draw graphs of several first eigenfunctions determining the form of standing waves (see pictures 7, 8, 9).

It is also easy to find the corresponding functions \( T(t) \). Namely, from (2.25) with \( \lambda = \lambda_k \) we get

\[
T_k(t) = A_k \cos \frac{k \pi a t}{l} + B_k \sin \frac{k \pi a t}{l},
\]

implying the following general form of the standing wave:

\[
u_k(t, x) = \left( A_k \cos \frac{k \pi a t}{l} + B_k \sin \frac{k \pi a t}{l} \right) \sin \frac{k \pi x}{l}, \quad k = 1, 2, \ldots
\]
The frequency of the oscillation of each point $x$ corresponding to the solution $u_k$ is equal to

$$\omega_k = \frac{k\pi a}{l}, \quad k = 1, 2, \ldots$$

These frequencies are called the \textit{eigenfrequencies} of the string.

Now let us find the general solution of (2.5) with boundary conditions (2.23) as a sum (\textit{superposition}) of stationary waves, i.e., in the form

$$u(t, x) = \sum_{k=1}^{\infty} \left( A_k \cos \frac{k\pi at}{l} + B_k \sin \frac{k\pi at}{l} \right) \sin \frac{k\pi x}{l}$$

We need to satisfy the initial conditions (2.22). Substituting (2.27) into these conditions, we get

$$\varphi(x) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{l}, \quad x \in [0, l],$$

$$\psi(x) = \sum_{k=1}^{\infty} \left( \frac{k\pi a}{l} \right) B_k \sin \frac{k\pi x}{l}, \quad x \in [0, l],$$

i.e., it is necessary to expand both $\varphi(x)$ and $\psi(x)$ into series in eigenfunctions

$$\left\{ \sin \frac{k\pi x}{l} : k = 1, 2, \ldots \right\}.$$

First of all, observe that this system is orthogonal on the segment $[0, l]$. This can be directly verified, but we may also refer to the general fact of orthogonality of eigenvectors of a symmetric operator corresponding to distinct eigenvalues. For such an operator one should take the operator $L = \frac{d^2}{dx^2}$ with the domain $D_L$ consisting of functions $v \in C^2([0, l])$ such that $v(0) = v(l) = 0$. Integrating by parts, we see that

$$(Lv, w) = (v, Lw), \quad v, w \in D_L.$$  

The parentheses denote the inner product in $L^2([0, l])$ defined via

$$(v_1, v_2) = \int_0^l v_1(x) \overline{v_2(x)} dx.$$  

Thus, the system $\{ \sin \frac{k\pi x}{l} \}_{k=1, 2, \ldots}$ is orthogonal in $L^2([0, l])$. We would like to establish its completeness.

For simplicity, take $l = \pi$ (the general case reduces to this one by introducing the variable $y = \frac{\pi x}{l}$) so that the system takes on the form $\{ \sin kx \}_{k=1}^{\infty}$ and is considered on $[0, \pi]$. Let $f \in L^2([0, \pi])$. Extend $f$ as an
odd function onto $[-\pi, \pi]$ and expand it into the Fourier series in the system
\{1, \cos kx, \sin kx : k = 1, 2, \ldots \} on the interval $[-\pi, \pi]$.

Clearly, this expansion does not contain constants and the terms with \cos kx, since the extension is an odd function. Therefore, on $[0, \pi]$, we get the expansion in the system \{\sin kx : k = 1, 2, \ldots \} only.

Note, further, that if $f$ is continuous, has a piecewise continuous derivative on $[0, \pi]$ and $f(0) = f(\pi) = 0$, then its $2\pi$-periodic odd extension is also continuous with a piecewise continuous derivative. Therefore, it can be expanded into a uniformly convergent series in terms of \{\sin kx : k = 1, 2, \ldots \}. If this $2\pi$-periodic extension is in $C^k$ and its $(k+1)$-st derivative is piecewise continuous, then this expansion can be differentiated $k$ times and the uniform convergence is preserved under this differentiation.

Thus, expansions into series (2.28) and (2.29) exist and imposing some smoothness and boundary conditions on $\varphi(x), \psi(x)$, we can achieve arbitrarily fast convergence of these series. Coefficients of these series are uniquely defined:

\[
A_k = \frac{\int_0^l \varphi(x) \sin \frac{k\pi x}{l} \, dx}{\frac{1}{l} \int_0^l \sin^2 \frac{k\pi x}{l} \, dx} = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi x}{l} \, dx,
\]

\[
B_k = \frac{l}{k\pi a} \frac{\int_0^l \psi(x) \sin \frac{k\pi x}{l} \, dx}{\frac{1}{l} \int_0^l \sin^2 \frac{k\pi x}{l} \, dx} = \frac{2}{k\pi a} \int_0^l \psi(x) \sin \frac{k\pi x}{l} \, dx.
\]

Substituting these values of $A_k$ and $B_k$ into (2.27), we get the solution of our problem.

The Fourier method can also be used to prove the uniqueness of solution. (More precisely, the completeness of the trigonometric system enables us to reduce it to the uniqueness theorem for ODE.) We will illustrate this below by an example of a more general problem.

Consider the problem of a string to which a distributed force $f(t, x)$ is applied:

\[
(2.30) \quad u_{tt} = a^2 u_{xx} + f(t, x), \quad t \geq 0, \quad x \in [0, l].
\]

The ends of the string are assumed to be fixed (i.e., conditions (2.23) are satisfied) and for $t = 0$ the initial position and velocity of the string, i.e., conditions (2.22), are defined as above.

Since the eigenfunctions $\{\sin \frac{k\pi x}{l} : k = 1, 2, \ldots \}$ constitute a complete orthonormal system on $[0, l]$, any function $g(t, x)$, defined for $x \in [0, l]$ and satisfying reasonable smoothness conditions and boundary conditions, can
be expanded in terms of this system, with coefficients depending on $t$. In particular, we can write

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t) \sin \frac{k\pi x}{l},$$

$$f(t, x) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{k\pi x}{l},$$

where $f_k(t)$ are known and $u_k(t)$ are to be determined. Substituting these decompositions into (2.30), we get

$$(2.31) \quad \sum_{k=1}^{\infty} \left[\ddot{u}_k(t) + \omega_k^2 u_k(t) - f_k(t)\right] \sin \frac{k\pi x}{l} = 0,$$

where $\omega_k = \frac{k\pi a}{l}$ are the eigenfrequencies of the string. It follows from (2.31), due to the orthogonality of the system of eigenfunctions, that

$$u_k(t) + \omega_k^2 u_k(t) = f_k(t), \quad k = 1, 2, \ldots$$

The initial conditions

$$\sum_{k=1}^{\infty} u_k(0) \sin \frac{k\pi x}{l} = \varphi(x),$$

$$\sum_{k=1}^{\infty} \dot{u}_k(0) \sin \frac{k\pi x}{l} = \psi(x),$$

determine $u_k(0)$ and $\dot{u}_k(0)$; hence, we can uniquely determine functions $u_k(t)$ and, therefore, the solution $u(t, x)$.

In particular, of interest is the case when $f(t, x)$ is a harmonic oscillation in $t$, e.g.

$$f(t, x) = g(x) \sin \omega t.$$  

Then, clearly, the right hand sides $f_k(t)$ of equations (2.32) are

$$f_k(t) = g_k \sin \omega t.$$  

For example, let $g_k \neq 0$. Then for $\omega \neq \omega_k$ (the nonresonance case) the equation (2.32) has an oscillating particular solutions of the form

$$u_k(t) = U_k \sin \omega t, \quad U_k = \frac{g_k}{\omega_k^2 - \omega^2}.$$  

Therefore, its general solution also has an oscillating form:

$$u_k(t) = U_k \sin \omega t + A_k \cos \omega kt + B_k \sin \omega kt.$$
2. One-dimensional wave equation

For $\omega = \omega_k$ (the resonance case) there is a particular solution of the form

$$u_k(t) = t(M_k \cos \omega t + N_k \sin \omega t),$$

which the reader can visualize as an oscillation with the frequency $\omega$ and an increasing amplitude. The general solution $u_k(t)$ and the function $u(t, x)$ are unbounded in this case.

Observe that to the above described problem a more general problem can be reduced, namely the one with ends that are not fixed but whose motion is given by

$$u|_{x=0} = \alpha(t), \quad u|_{x=l} = \beta(t), \quad t \geq 0.$$ (2.33)

Namely, if $u_0(t, x)$ is an arbitrary function satisfying (2.33), e.g.

$$u_0(t, x) = \frac{l - x}{l} \alpha(t) + \frac{x}{l} \beta(t),$$

then for $v(t, x) = u(t, x) - u_0(t, x)$ we get a problem with fixed ends.

Finally, note that all the above problems are well-posed which is obvious from their explicit solutions.

2.5. Appendix: Calculus of variations and classical mechanics

Variations of functionals. Euler-Lagrange equations. Calculus of variations is an analogue of differential calculus, but for (generally non-linear) functions which are defined not on a finite-dimensional space but on an infinite-dimensional space (usually a space of functions). Such functions on an infinite-dimensional space are usually called functionals to distinguish them from functions on a finite-dimensional space (which can be arguments of the functionals).

Let us consider a simplest important example of a functional where the variational calculus can be applied to finding extrema of this functional. It is defined on the space $C^2([t_0, t_1])$, consisting of real-valued continuous functions $x = x(t), \ t \in [t_0, t_1]$, such that the derivatives $\dot{x} = dx/dt, \ddot{x} = d^2x/dt^2$ exist and are continuous on $[a, b]$, and is given by the formula

$$S[x] = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt,$$

where $L = L(t, x, v)$ is a continuous function of real variables $t \in [a, b], x, v \in \mathbb{R}$, which has continuous partial derivatives $\partial L/\partial t, \partial L/\partial x, \partial L/\partial v,$
\[ \frac{\partial^2 L}{\partial t \partial v}, \frac{\partial^2 L}{\partial x \partial v}, \frac{\partial^2 L}{\partial v^2}. \]

In mechanics and physics, a functional \( S \) like this is usually called action, whereas \( L \) is called Lagrangian.

It may be not clear a priori why our requirements include the existence and continuity of the second derivative \( \ddot{x} \), as well as many partial derivatives of \( L \). Most of these requirements could be avoided or weakened, but this would complicate further exposition because \( \ddot{x} \) and the partial derivatives explicitly or implicitly enter into the extremum condition.

Now let us assume that we need to find a minimum of the action \( S \) on a set of functions \( x \in C^2([t_0, t_1]) \), whose values at the ends \( t_0, t_1 \) are fixed. (In other words, we impose boundary conditions \( x(t_0) = b_0, x(t_1) = b_1 \) where \( b_0, b_1 \) are fixed constants.) Let \( x = x(t) \) be a minimizer: a function where this minimum is attained. Then the following condition should be satisfied:

for any function \( \delta x \in C^2([t_0, t_1]) \), satisfying the boundary conditions \( \delta x(t_0) = \delta x(t_1) = 0 \),

the function \( f = f(s) \) of one variable \( s \in \mathbb{R} \), defined by

\[
 f(s) = S[x + s \delta x] = \int_{t_0}^{t_1} L(t, x(t) + s \delta x(t), \dot{x}(t) + s \dot{\delta x}(t)) dt,
\]

should attain minimum at \( s = 0 \). (In this formula \( \dot{\delta x} \) should be understood as \( (d/dt)\delta x(t) \).) Therefore, \( f'(0) = 0 \), which can be rewritten as

\[
 \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt = 0,
\]

where \( \partial L/\partial \dot{x} \) should be understood as \( \partial L(t, x(t), v)/\partial v |_{v=\dot{x}(t)} \). Integrating by parts in the second term, and taking into account the boundary conditions for \( \delta x \), we obtain

\[
 \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x dt = 0,
\]

Since \( \delta x \) can be an arbitrary function from \( C^2([t_0, t_1]) \), vanishing at the ends \( t_0, t_1 \), we easily obtain that the expression in the parentheses should vanish identically, i.e. we get the following equation

\[
 \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0,
\]

identically on the interval \([t_0, t_1]\). This is a second-order ordinary differential equation which is called Euler-Lagrange equation. Its solution satisfying the boundary conditions \( x(t_0) = b_0, x(t_1) = b_1 \), is called extremal or extremal point of the action functional \( S \). All minimizers and maximizers of \( S \) should be extremal points. Sometimes, if it is known by some reasons that, say,
the minimum is attained, and there is only one extremal, then this extremal must coincide with the minimizer.

Let us also denote by $\delta S$ the following linear functional on the space of functions $\delta x \in C^2([t_0, t_1])$, vanishing at the ends of the interval $[t_0, t_1]$:

$$\delta S[\delta x] = \frac{d}{ds}S[x + s\delta x] \bigg|_{s = 0} = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt$$

$$= \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x dt.$$  

It is the principal linear part of the functional $S$ at the point $x$, and it is called the variation of the functional $S$. Using the variation, we can rewrite the extremality condition in the form $\delta S = 0$ which means that $\delta S[\delta x] = 0$ for all variations $\delta x$, which vanish at the ends of the interval $[t_0, t_1]$. Clearly, this is equivalent to the Euler-Lagrange equation.

We can also rewrite the last equation in the form

$$\delta S = \delta S[\delta x] = \int_{t_0}^{t_1} \frac{\delta S}{\delta x} \delta x dt,$$

where

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}$$

is called the variational derivative of $S$.

There is a straightforward extension of the arguments given above, to the case when the action functional $S$ depends upon several functions $x_1(t), \ldots, x_n(t)$ belonging to $C^2([t_0, t_1])$. In fact, it can be obtained if we just use the same formulas as above but for a vector-function $x$ instead of a scalar function. (This vector-function can be interpreted as a parametrized curve in $\mathbb{R}^n$, or, alternatively, a trajectory of a point moving in $\mathbb{R}^n$.) Nevertheless, let us reproduce the formulas in detail, leaving the details of the arguments as an exercise for the readers.

The action $S$ is given by

$$S[x] = \int_{t_0}^{t_1} L(t, x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t)) dt,$$

where $L = L(t, x_1, \ldots, x_n, v_1, \ldots, v_n)$ is the Lagrangian, which is a scalar real-valued function. The variations of the given (real-valued) functions $x_1(t), \ldots, x_n(t)$ are real-valued $C^2$-functions $\delta x_1(t), \ldots, \delta x_n(t)$, vanishing at the ends $t_0, t_1$. The extremum conditions are obtained from the equations

$$\frac{\partial}{\partial s_j} S[x_1 + s_1 \delta x_1, \ldots, x_n + s_n \delta x_n] \bigg|_{s_1 = \ldots = s_n = 0} = 0, \quad j = 1, \ldots, n,$$
which lead to the Euler-Lagrange equations
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0, \quad j = 1, \ldots, n. \]
These equations are equivalent to the vanishing of the variation \( \delta S \), which is a linear functional on vector-functions \( \delta x = (\delta x_1, \ldots, \delta x_n) \); it is given by
\[
\delta S = \delta S[\delta x] = \int_{t_0}^{t_1} \sum_{j=1}^{n} \left( \frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \delta \dot{x}_j \right) dt
\]
\[ = \int_{t_0}^{t_1} \sum_{j=1}^{n} \left( \frac{\partial L}{\partial x_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} \right) \delta x_j dt. \]
This can be rewritten in the form
\[
\delta S = \delta S[\delta x] = \int_{t_0}^{t_1} \sum_{j=1}^{n} \frac{\delta S}{\delta x_j} \delta x_j dt.
\]
where
\[ \frac{\delta S}{\delta x_j} = \frac{\partial L}{\partial x_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} \]
is called the \textit{partial variational derivative} of \( S \) with respect to \( x_j \).

\textbf{Lagrangian approach in classical mechanics.} The most important equations of classical mechanics can be presented in the form of the Euler-Lagrange equations. (In mechanics they are usually called \textit{Lagrange’s equations}.) Take, for example, a point particle of mass \( m > 0 \) moving along the \( x \)-axis \( \mathbb{R} \) in a time-independent (or stationary) field of forces \( F(x) \) (which is assumed to be a real-valued continuous function on \( \mathbb{R} \)). Let \( x(t) \) denote the coordinate of the moving particle at the time \( t \). Then by Newton’s Second Law, \( (\text{mass}) \cdot \text{(acceleration)} = \text{(force)} \), or, more precisely,
\[ m \ddot{x}(t) = F(x(t)) \]
for all \( t \). It is sometimes more convenient to rewrite this equation in the form
\[ \frac{dp}{dt} = F, \]
where \( p = mx \) is called the \textit{momentum} of the moving particle.

Let us introduce a \textit{potential energy} of the particle, which is a real-valued function \( U = U(x) \) on \( \mathbb{R} \), such that \( F(x) = -U'(x) \) for all \( x \). In other words,
\[ U(x) = U(x_0) - \int_{x_0}^{x} F(s) ds, \]
which means that to move the particle from \(x_0\) to \(x\) you need to do work \(U(x) - U(x_0)\) against the force \(F\). (If \(U(x) < U(x_0)\), then in fact the force does the work.) The Newton equation rewrites then as

\[
m\ddot{x} = -U'(x).
\]

Multiplying both parts by \(\dot{x}\), we can rewrite the result as

\[
\frac{d}{dt} \frac{m\dot{x}^2}{2} = -\frac{d}{dt} U(x),
\]

where \(x = x(t)\). Taking antiderivative, we obtain

\[
\frac{m\dot{x}^2}{2} + U(x) = E = \text{const},
\]

where the constant \(E\) depends on the solution \(x = x(t)\) (but is constant along the trajectory). This is the energy conservation law for the one-dimensional motion. The quantity

\[
K = \frac{m\dot{x}^2}{2}
\]

is called kinetic energy of the moving particle, and the quantity

\[
H = K + U = H(x, \dot{x})
\]

is called energy, or full energy, or Hamiltonian. The energy conservation law can be rewritten in the form \(H(x, \dot{x}) = E = \text{const}\).

Now let us take the Lagrangian

\[
L = L(x, \dot{x}) = K - U = \frac{m\dot{x}^2}{2} - U(x),
\]

and the corresponding action \(S\). Then it is easy to see that the extremals of \(S\) will be exactly the solutions of the Newton Second Law equation, satisfying the chosen boundary conditions. In other words, the Euler-Lagrange equation for this action coincides with the Newton Second Law equation.

Now consider a motion of a particle in \(\mathbb{R}^n\). (Note that any motion of \(k\) particles, say, in \(\mathbb{R}^3\) can be equivalently presented as a motion of one particle in \(\mathbb{R}^{3k}\), the space whose coordinates consists of all coordinates of all particles.) The particle coordinates can be presented as a point \(x \in \mathbb{R}^n\), \(x = (x_1, \ldots, x_n)\), so the motion of the particle is given by functions \(x_1(t), \ldots, x_n(t)\), or by one vector-function \(x(t)\) (with values in \(\mathbb{R}^n\)). Let us assume that the force is \(F(x) = (F_1(x), \ldots, F_n(x))\), so that \(F(x) \in \mathbb{R}^n\) for every \(x \in \mathbb{R}^n\), and \(F\) is continuous. (In case when \(x\) represents a system of particles, \(F\) may include forces of interaction between these particles, which
may depend on the configuration of these particles.) Then by Newton’s Second Law, the motion satisfies the equations
\[ m_j \ddot{x}_j = F_j(x), \quad j = 1, \ldots, n. \]

Let us assume also that the field of forces is a potential field, given by a potential energy \( U = U(x) \) (which is a scalar, real-valued function on \( \mathbb{R}^n \)), so that
\[ F_j(x) = -\frac{\partial U}{\partial x_j}, \quad j = 1, \ldots, n, \]
or, in the vector analysis notation,
\[ F = -\nabla U = -\text{grad} U. \]

In this case, it is easy to see that the equations of motion coincide with the Euler-Lagrange equations for the action with the Lagrangian \( L = K - U \), where the kinetic energy \( K \) is given by
\[ K = \sum_{j=1}^n \frac{m_j \dot{x}_j^2}{2}. \]

The Hamiltonian \( H = H(x, \dot{x}) = K + U \) satisfies the energy conservation law \( H(x(t), \dot{x}(t)) = E = \text{const} \), where \( x(t) \) is the solution of the equations of motion (or, equivalently, the extremal of the action). This can be verified by a straightforward calculation:
\[
\frac{d}{dt} H(x(t), \dot{x}(t)) = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial \dot{x}} \cdot \ddot{x} \\
= \sum_{j=1}^n \left( \frac{\partial H}{\partial x_j} \dot{x}_j + \frac{\partial H}{\partial \dot{x}_j} \ddot{x}_j \right) \\
= \sum_{j=1}^n \left( \frac{\partial U}{\partial x_j} \dot{x}_j + \frac{\partial K}{\partial \dot{x}_j} \left( -\frac{1}{m_j} \frac{\partial U}{\partial x_j} \right) \right) \\
= \sum_{j=1}^n \left( \frac{\partial U}{\partial x_j} \dot{x}_j + m_j \dot{x}_j \left( -\frac{1}{m_j} \frac{\partial U}{\partial x_j} \right) \right) = 0.
\]

Now let us consider a system of \( k \) particles in \( \mathbb{R}^3 \). Assume that \( n = 3k \) and write \( x = (x_1, \ldots, x_k) \), where \( x_j \in \mathbb{R}^3 \), \( j = 1, \ldots, k \). Let us view \( x_j \) as a point in \( \mathbb{R}^3 \) where \( j \)th particle is located. For moving particles we assume that \( x_j = x_j(t) \) is a \( C^2 \) function of \( t \). Let us introduce the momentum of \( j \)th particle as \( p_j = m_j \dot{x}_j \), which is a vector in \( \mathbb{R}^3 \). The momentum of the whole system of such particles is defined as
\[ p = p_1 + \cdots + p_k, \]
which is again a vector in $\mathbb{R}^3$. Now assume that all forces acting on the particles are split into pairwise interaction forces between the particles and external forces, so that the force acting on $j$th particle is

$$F_j = \sum_{i \neq j} F_{ij} + F_{ext}^j,$$

where $F_{ij} = F_{ij}(x) \in \mathbb{R}^3$, and the summation is taken over $i$ (with $j$ fixed), $F_{ext}^j = F_{ext}^j(x) \in \mathbb{R}^3$. Here $F_{ij}$ is the force acting on the $j$th particle due to its interaction with the $i$th particle, and $F_{ext}^j$ is the external force acting upon the $j$th particle.

The equations of motion look as follows:

$$\frac{dp_j}{dt} = F_j = \sum_{i \neq j} F_{ij} + F_{ext}^j, \quad j = 1, \ldots, k.$$

Now assume that Newton’s Third Law holds for two interacting particles:

$$F_{ij}(x) = -F_{ji}(x),$$

for all $i, j$ with $i \neq j$, and for all $x$. Then we get

$$\frac{dp}{dt} = F_{ext} = \sum_{j=1}^k F_{ext}^j.$$

In words, the rate of change of the total momentum in time equals the total external force. (The pairwise interaction forces, satisfying Newton’s Third Law, do not count.) Indeed, calculating the derivative of $p$, we see that all interaction forces cancel due to Newton’s Third Law.

In particular, in the absence of external forces the conservation of momentum law holds: for any motion of these particles

$$p(t) = \text{const}.$$

Indeed, from the relations above we immediately see that $dp/dt = 0$.

**Remark.** The mechanical system, describing a particle, moving in $\mathbb{R}^n$, is said to have $n$ degrees of freedom. In the main text of this Chapter we discuss the motion of a string, which naturally has infinitely many degrees of freedom, because its position is described not by a finite set of numbers but by a function (of one variable), or by an infinite set of its Fourier coefficients.
2.6. Problems

2.1. a) Write the boundary condition on the left end of the string if a ring which can slide without friction along the vertical line is attached to this end. The ring is assumed to be massless (see Fig. 10);

   b) The same as in 2.1 a) but the ring is of mass $m$;

   c) The same as in 2.1 b) but the ring slides with a friction which is proportional to its velocity.

2.2. a) Derive the energy conservation law for a string if at the left end the boundary condition of Problem 2.1 a) is satisfied and the right end is fixed;

   b) The same in Problem 2.1 b) (take into account the energy of the ring);

   c) The same in Problem 2.1 c) (take into account the loss of energy caused by friction).

2.3. Derive the equation for longitudinal elastic vibrations of a homogeneous rod.

2.4. Write the boundary condition for the left end of the rod if this end is free.

2.5. Write the boundary condition for the left end of the rod if this end is elastically restrained; see Fig. 11.
2.6. Derive the energy conservation law for a rod with
   
a) fixed ends;
   
b) free ends;
   
c) one end is fixed, and the other restrained elastically.

2.7. Derive an equation for longitudinal vibrations of a conic rod.

2.8. Prove that the energy of an interval of a string between $c+at$ and $d-at$ is a decreasing function of time. Deduce from this the uniqueness of the solution of the Cauchy problem and an information about the dependence and influence domains for $[c,d]$.

2.9. Formulate and prove the energy conservation law for an infinite string.

2.10. Draw a cartoon describing oscillations of an infinite string with the initial values $u|_{t=0} = 0$, $u_t|_{t=0} = \psi(x)$, where $\psi(x) = \begin{cases} 1 & \text{for } x \in [c,d] \\ 0 & \text{for } x \notin [c,d] \end{cases}$.

2.11. Describe oscillations of an infinite string for $t \in (-\infty, +\infty)$ such that some interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ of the string is in equilibrium position at all times.

2.12. The same as in the above problem but the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is only fixed for $t \geq 0$.

2.13. Draw a cartoon describing oscillations of a semi-bounded string with a free end (the boundary condition $u_x|_{x=0} = 0$, see Problem 2.1 a), and the initial conditions $u|_{t=0} = \varphi(x)$, $u_t|_{t=0} = 0$, where the graph $\varphi(x)$ has the form of an isosceles triangle with the base $[l, 3l]$.

2.14. Draw a cartoon describing oscillations of a semi-bounded string with a fixed end and the initial conditions $u|_{t=0} = 0$, $u_t|_{t=0} = \psi(x)$, where $\psi(x)$ is the characteristic function of the interval $(l, 3l)$.

2.15. Derive the energy conservation law for a semi-bounded string with a fixed end.
2.16. Given the wave \( \sin \omega (t + \frac{x}{a}) \) running for \( t \leq 0 \) along the rod whose left end is elastically attached, find the reflected wave.

2.17. Given a source of harmonic oscillations with frequency \( \omega \) running along an infinite string at a speed \( v < a \), i.e., \( u|_{x=vt} = \sin \omega t \) for \( t \geq 0 \), describe the oscillations of the string to the left and to the right of the source. Find frequencies of induced oscillations of a fixed point of the string to the left and to the right of the source and give a physical interpretation of the result (Doppler’s effect).

2.18. Describe and draw standing waves for a string with free ends.

2.19. Prove the existence of an infinite series of standing waves in a rod with the fixed left end and elastically attached right one (see Problem 2.5). Prove the orthogonality of eigenfunctions. Find the shortwave asymptotics of eigenvalues.

2.20. The force \( F = F_0 \sin \omega t \), where \( \omega > 0 \), is applied to the right end of the rod for \( t \geq 0 \). Derive a formula for the solution describing forced oscillations of the rod and find the conditions for a resonance if the left end is
   a) fixed;
   b) free.
   (Comment: the forced oscillations are the oscillations which have the frequency which coincides with the frequency of the external force or other external perturbation.)

2.21. The right end of a rod is oscillating via \( A \sin \omega t \) for \( t \geq 0 \). Write a formula for the solution describing forced oscillations and find the conditions for resonance if the left end is
   a) fixed;
   b) free;
   c) elastically restrained.
Chapter 3

The Sturm-Liouville problem

3.1. Ordinary differential equations

We will briefly discuss simplest facts on Ordinary Differential Equations (ODE), especially the existence and uniqueness of solutions with appropriate initial conditions. These results are universally useful in analysis and can be found in numerous textbooks (e.g. Arnold [2], Ch. 4; Hartman [11], Ch. I; Taylor [30], vol. 1, Ch. 1.2).

A. Let us consider an ODE with an initial condition

\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \]  

Here we assume that an open interval \( I \subset \mathbb{R}^1 \) and an open subset \( V \subset \mathbb{R}^n \) are given, \( x_0 \in I, \ y_0 \in V, \ f \) is a continuous function on \( I \times V \) with the values in \( \mathbb{R}^n \). A function \( y = y(x) \) on an open interval \( J \subset I \) with values in \( V \) is called a solution of the problem (3.1) if \( y(x) \) satisfies the equation for all \( x \in J \) and the initial condition at \( x = x_0 \).

Assume also that \( f \) satisfies the Lipschitz condition in \( y \), that is

\[ |f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \]  

where \( x \in I \) and \( y_1, y_2 \in V, \) \( L \) is a non-negative constant.

**Theorem 3.1.** (A.L.Cauchy) Under the conditions above, the equation (3.1) has a unique solution \( y = y(x) \) with \( x \in J \), where \( J \) is in a sufficiently small interval \( J \subset I \), containing \( x_0 \).
We will skip the proof which can be found, e.g., in the textbooks quoted above at the beginning of this Section.

Note that a higher-order ODE problem
\[ y^{(n)}(t) = f(t, y, y', \ldots, y^{(n-1)}) \]
can be reduced to an equivalent first-order system by introducing a system of new variables \( u = (u_0, u_1, \ldots, u_{n-1}) \) with
\[ u_0 = y, \quad u_j = y^{(j)}, \quad 1 \leq j \leq n - 1, \]
which leads to
\[ \frac{du}{dt} = (u_1, \ldots, u_{n-1}, f(t, u_0, \ldots, u_{n-1})). \]

It is easy to see that the last system of equations with appropriate boundary conditions is equivalent to the original initial-value problem.

B. It is also important to extend the solution (say, \( y = y(x) \)) to the maximal interval \( J \) of the independent variable \( x \), where the solution still exists. More precisely, assume that \( J = (c, d) \subset \mathbb{R}, \quad x_0 \in J \), so that the graph of the solution \( y = y(x) \) comes to a singularity (in particular, goes to infinity), or does not have any finite limits as \( x \to c+ \) or \( x \to d- \). Then we should declare that \( y(x) \) has a singular point at this \( x = c \) or \( x = d \).

The singular points are an important topic in mathematics, because finding and investigating them may be sufficient for a complete understanding of the equations under the study.

### 3.2. Formulation of the problem

Now we will start a study of a seemingly special PDE which appears as a model for a hyperbolic second-order equation with two independent variables, \( t \) and \( x \), so that in a physical presentation \( t \) means “time” and \( x \) is a “space” variable. We will start by simplifying the equation by a change of variables.

Let us consider the equation
\[ (3.3) \quad \rho(x)u_{tt} = (p(x)u_x)_x - q(x)u, \]
which is more general than the equation derived in Chapter 2 for oscillations of a non-homogeneous string. For the future calculations to be valid, we will assume that \( \rho \in C^1([0, l]), \rho > 0 \) on \([a, b], \ p \in C^1([a, b])\)
and \( q \in C([a, b]) \). Let us try to simplify the equation by a change of variables \( u(t, x) = z(x)v(t, x) \), where \( z(x) \) is a known function. This will allow us to obtain an equation of the form (3.3) with \( \rho(x) \equiv 1 \).

Indeed, we have
\[
  u_{tt} = z v_{tt}, \quad u_x = z v_x + z_x v, \quad u_{xx} = z v_{xx} + 2z_x v_x + z_{xx} v.
\]
Substituting this into (3.3) and dividing by \( z \rho \), we get
\[
  v_{tt} = \frac{p}{\rho} v_{xx} + \left( \frac{2p}{\rho} \frac{z_x}{z} + \frac{p_x}{\rho} \right) v_x + B(x)v.
\]
This equation is of the form (3.3) with \( \rho(x) \equiv 1 \) whenever
\[
  \left( \frac{p}{\rho} \right)_x = \frac{2p}{\rho} \frac{z_x}{z} + \frac{p_x}{\rho}.
\]
Performing the differentiation in the left hand side we see that this can be equivalently rewritten in the form
\[
  \frac{z_x}{z} = -\frac{\rho_x}{2\rho}
\]
and we may take, for example,
\[
  z(x) = \rho(x)^{-1/2}.
\]
Since we always assume \( \rho(x) > 0 \), such a substitution is possible. Thus, consider the equation
\[
  u_{tt} = (p(x)u_x)_x - q(x)u.
\]
Solving it by separation of variables on the closed interval \([0, l]\), and assuming that the ends are fixed, we get the following eigenvalue problem:
\[
(3.4) \quad -(p(x)X'(x))' + q(x)X(x) = \lambda X(x),
\]
with the boundary conditions
\[
(3.5) \quad X(0) = X(l) = 0.
\]
Generalizing it, we can take the same differential operator while replacing the boundary conditions by more general conditions:
\[
\alpha X'(0) + \beta X(0) = 0, \quad \gamma X'(l) + \delta X(l) = 0,
\]
where \( \alpha, \beta, \gamma, \delta \) are real numbers, \( \alpha^2 + \beta^2 \neq 0, \gamma^2 + \delta^2 \neq 0 \). This eigenvalue problem is called the Sturm-Liouville problem. Considering this problem we usually suppose that
\[
p \in C^1([0, l]), \quad q \in C([0, l]) \quad \text{and} \quad p(x) \neq 0 \quad \text{for} \quad x \in [0, l] \quad \text{(ellipticity)}.
\]
For simplicity, we assume that \( p(x) \equiv 1 \) and the boundary conditions are as in (3.5). The general case is treated similarly. Thus, consider the problem

\[
-XX''(x) + q(x)X(x) = \lambda X(x),
\]

with the boundary conditions (3.5) and assume also that

\[
q(x) \geq 0.
\]

This is not a restriction since we can achieve (3.7) by adding the same constant to \( q \) and to \( \lambda \).

### 3.3. Basic properties of eigenvalues and eigenfunctions

We will start with the following Oscillation Theorem by C. Sturm (1836):

**Theorem 3.2.** Let \( y, z, q_1, q_2 \in C^2([a,b]) \) be real-valued functions, such that \( z(a) = z(b) = 0, z \not\equiv 0 \), \( y \) and \( z \) satisfy the equations

\[
y''(x) + q_1(x)y(x) = 0,
\]

\[
z''(x) + q_2(x)z(x) = 0.
\]

Assume also that

\[
q_1(x) \geq q_2(x), \text{ for all } x \in [a,b].
\]

Then, either there exists \( x_0 \in (a,b) \) such that \( y(x_0) = 0 \), or \( q_1(x) \equiv q_2(x) \) and \( y(x) \equiv cz(x) \) on \([a,b]\), where \( c \) is a constant.

**Proof.** (a) By a null-point of a real-valued function \( z = z(x) \in C^2([a,b]) \), satisfying (3.9), we will call a point \( x_0 \in [a,b] \), such that \( z(x_0) = 0, z \not\equiv 0 \). We will start by proving, that such null-points are simple (non-degenerate), which means that if \( z = z(\cdot) \) is a solution of (3.9), \( x_0 \in [a,b], z(x_0) = 0 \) and \( z(x) \not\equiv 0 \), then \( z'(x_0) \not= 0 \).

Indeed, if we assume that \( z'(x_0) = 0 \), then \( z(x_0) = z'(x_0) = 0 \), which implies that \( z(x) \equiv 0 \) by the existence and uniqueness theorem for the linear second order ODE in \( C^2([a,b]) \) (see Theorem 3.1 and further comments there). Moreover, the same arguments, as above, present all solutions of (3.9) as a 2-dimensional linear subspace (in \( C^2([a,b]) \)), because taking the initial conditions \( z(\cdot) \mapsto \{(z(c), z'(c))\} \) maps the space of all solutions of (3.9) to \( \mathbb{R}^2 \) by a linear isomorphism for any fixed \( c \in [a,b] \).

It follows that the set of all null-points of the solution \( z(x) \), which satisfy the condition \( z(x_0) = 0 \) with a fixed \( x_0 \in [a,b] \), is discrete, hence finite, because the interval \([a,b]\) is compact.
3.3. Eigenvalues and eigenfunctions

In particular, all null-points of $z(x)$ are isolated.

Note also that for any null-point $x_0 \in [a, b]$, the set of solutions $z(x)$ satisfying $z(x_0) = 0$, is a 1-dimensional subspace in $C^2([a, b])$.

The set $\{z\}$ of all null-points of the function $z(x)$, belonging to $[a, b]$, is discrete, hence finite, because the interval $[a, b]$ is compact.

(b) Let us consider the Wronskian

\[(3.11) \quad W(x) = y(x)z'(x) - y'(x)z(x), \quad x \in [a, b].\]

Differentiating this identity with respect to $x$, we obtain

\[(3.12) \quad W'(x) = y(x)z''(x) - y''(x)z(x) = (q_1(x) - q_2(x))y(x)z(x), \quad x \in [a, b].\]

Obviously, it suffices to consider the case when $a$ and $b$ are neighboring null-points of the function $z(\cdot)$ on $[a, b]$. Then, without loss of generality, we may assume that $z(x) > 0$ for all $x \in (a, b)$. It follows that $z'(a) > 0$ and $z'(b) < 0$.

Now, looking at $y(\cdot)$, we see two possibilities. The first one is that there exists $x_1 \in (a, b)$ such that $y(x_1) = 0$, which is enough for the validity of one of the statements of this theorem. The alternative is that $y(x) \neq 0$ for all $x \in (a, b)$. Then we can assume that $y(x) > 0$ for all $x \in (a, b)$, which we will assume from this moment to the end of the proof. In this case we also have $y(x) \geq 0$ for all $x \in [a, b]$.

Integrating (3.9) over $[a, b]$, we get $W(b) - W(a) \geq 0$. Here the equality holds if and only if $q_1(x) \equiv q_2(x)$ on $[a, b]$.

On the other hand, by the arguments above, $y(a) \geq 0$ and $y(b) \geq 0$. Then we have

\[W(b) - W(a) = y(b)z'(b) - y(a)z'(a) \leq 0,\]

with equality if and only if $y(a) = y(b) = 0$.

Comparing the obtained inequalities, we see that under our assumptions $W(b) - W(a) = 0$, implying $q_1(x) \equiv q_2(x)$ and $y(a) = y(b) = 0$. Since $z(a) = z(b) = 0$, the solutions $y(x)$ and $z(x)$ must be proportional, i.e. $y(x) = Cz(x)$, where $C$ is a constant. (Indeed, taking $C$ such that $y'(a) = Cz'(a)$, we obtain $y(x) \equiv Cz(x)$ due to the uniqueness theorem for the second order ODE.) \(\square\)

**Corollary 3.3.** Let us assume that $q_1(x) = q_2(x)$ for all $x \in [a, b]$ (that is, the equations (3.8) and (3.9) coincide). Then $W(x) = c = \text{const}$ for any pair $y, z$ of solutions (with the constant $c$ depending of the choice of the pair).
Sturm-Liouville’s operators

1. Notations. Let us introduce the Sturm-Liouville operators of the form \( L = -\frac{d^2}{dx^2} + q(x) \). Let us define \( L \) as a linear operator with the domain \( D_L \) consisting of all \( v \in C^2([0, l]) \) satisfying the boundary conditions \( v(0) = v(l) = 0 \). The Sturm-Liouville problem is to find eigenvalues and eigenfunctions of this operator.

Let us recall that an eigenvalue \( \lambda \) and an eigenvector \( v \) of the operator \( L \) are defined by the relation \( Lv = \lambda v \), where it is assumed that \( \lambda \in \mathbb{C} \), \( v \in D_L \) and \( v \neq 0 \).

2. Symmetry. The operator \( L \) is symmetric, i.e.
\[
(Lv_1, v_2) = (v_1, Lv_2)
\]
for \( v_1, v_2 \in D_L \), where \((\cdot, \cdot)\) is the scalar product (sometimes also called the inner product)
\[
(u, v) = \int_0^l u(x)v(x)dx.
\]
Indeed,
\[
(Lv_1, v_2) - (v_1, Lv_2) = \int_0^l [-v_1''\bar{v}_2 + v_1\bar{v}_2'']dx =
\]
\[
= \int_0^l \frac{d}{dx}[-v_1\bar{v}_2 + v_1\bar{v}_2']dx = [-v_1\bar{v}_2 + v_1\bar{v}_2]'|_0^l = 0.
\]
This implies that all the eigenvalues are real and the eigenfunctions corresponding to different eigenvalues are orthogonal.

3. Simplicity. All eigenvalues are simple, i.e. all eigenspaces are 1-dimensional. This was actually established above in the proof of Theorem 3.2. In short, any two solutions of the equation \( y'' + p(x)y' + q(x)y = 0 \), that vanish at some point \( x_0 \), are proportional to each other (and each of them is proportional, due to the uniqueness theorem, to the solution with the initial conditions \( y(x_0) = 0, y'(x_0) = 1 \)).

4. Enumeration of the eigenvalues. For the convenience of notations we will use a complex parameter \( k = \sqrt{\lambda} \), so that the main equation takes the form
\[
(3.13) \quad -X''(x) + q(x)X(x) = k^2X(x),
\]
and we will use the Dirichlet boundary conditions \( X(0) = X(l) = 0 \) when indicated. Denote by \( \psi = \psi(x, k) \) the solution of (3.13) with the initial conditions
\[
\psi(0, k) = 0, \quad \psi'(0, k) = k
\]
(if \( q(x) \equiv 0 \), then \( \psi(x, k) = \sin kx \)). Since the eigenvalues of \( L \) are real, we will be mostly interested in the case when \( k \geq 0 \) if \( k^2 \geq 0 \), and we can specify \( k = \pm i|k| \) when necessary. The positive eigenvalues of the Sturm-Liouville operator \( L \) have the form \( \lambda = k^2 \), where \( k \) is such that \( \psi(l, k) = 0 \).

The Sturm theorem implies that the number of the null-points of \( \psi(x, k) \) belonging to any fixed segment \([0, a]\), where \( a \leq l \), is an increasing function of \( k \). Indeed, if \( 0 = x_1(k) < x_2(k) < \cdots < x_n(k) \) are all such zeros of \( \psi(\cdot, k) \) \((x_n(k) \leq a)\), and \( k' > k \), then every open interval \((x_j(k), x_{j+1}(k))\) contains at least one zero of \( \psi(\cdot, k') \). Recalling that \( x_1(k) = 0 \) for all \( k \), we see that the total number of zeros of \( \psi(\cdot, k') \) on \([0, a]\) is greater or equal than \( n \), which proves the claim.

Therefore, as \( k \) grows, all zeros of \( \psi(x, k) \) move to the left. In fact, it is easy to see that this motion is continuous, even smooth. Indeed, the zeros \( x_j(k) \) satisfy the equation \( \psi(x_j(k), k) = 0 \). Note that the function \( \psi = \psi(x, k) \) has continuous partial derivatives in \((x, k)\) and \( \frac{\partial \psi}{\partial x}(x_j(k), k) \neq 0 \), as observed above. By the Implicit Function Theorem, the function \( x_j = x_j(k) \) is continuous and has continuous derivative \( x'_j(k) \), which can be obtained if we differentiate the equation \( \psi(x_j(k), k) = 0 \) with respect to \( k \), arriving to

\[
x'_j(k) = - \frac{\partial \psi}{\partial k} \bigg|_{x=x_j(k)}.
\]

The eigenvalues correspond to the values \( k \) for which a new zero appears at \( l \). Since the number of these zeros is finite for any \( k \), this implies that the eigenvalues form a discrete sequence

\[
\lambda_1 < \lambda_2 < \lambda_3 < \ldots,
\]

which is either finite or tends to infinity. The eigenfunction \( X_n(x) \) corresponding to the eigenvalue \( \lambda_n \), has exactly \( n - 1 \) zeros on the open interval \((0, l)\).

It is easy to see that the number of eigenvalues is actually infinite. Indeed, Sturm’s theorem implies that the number of zeros of \( \psi(x, k) \) on \((0, l)\) is not less that the number of zeros on \((0, l)\) for the corresponding solution of the equation

\[
-y'' + My = k^2 y, \text{ where } M = \sup_{x \in [0, l]} q(x).
\]

But this solution is \( c \sin \sqrt{k^2 - M} x \) with \( c \neq 0 \), and the number of its zeros on \((0, l)\) tends to infinity as \( k \rightarrow +\infty \). Thus, we have proved
3. The Sturm-Liouville problem

**Theorem 3.4.** The eigenvalues are simple (non-multiple) and they form an increasing sequence
\[ \lambda_1 < \lambda_2 < \lambda_3 < \ldots \]
tending to \( +\infty \). If \( q_1 \geq 0 \), then \( \lambda_1 > 0 \). The eigenfunctions \( X_n(x) \), corresponding to different eigenvalues \( \lambda_n \), are orthogonal. The eigenfunction \( X_n(x) \) has exactly \( n - 1 \) zeros on the open interval \((0, l)\).

**Remark 3.5.** To prove the simplicity of any eigenvalue \( \lambda \) of the Sturm-Liouville operator \( L = -\frac{d^2}{dx^2} + q(x) \), with \( q \in C([0, l]) \) and the boundary conditions \( y(0) = y(l) = 0 \), it suffices to consider the case \( \lambda = 0 \). Otherwise we can replace \( q \) and \( \lambda \) by \( q - \lambda \) and 0 respectively, which shifts every eigenvalue by subtracting \( \lambda \) and preserves the multiplicities and eigenfunctions of every eigenvalue. So, let us assume \( \lambda = 0 \).

Let us take the linear space \( M \) consisting of all functions \( y \in C^2([0, l]) \), which satisfy the equation \(-y'' = qy\). (Not all of them are the eigenfunctions because we ignored the boundary conditions.) Clearly, taking the initial values of the solutions (or the map \( y \mapsto By = \{y(0), y'(0)\} \)), we obtain a linear isomorphism \( B : M \to \mathbb{R}^2 \). In particular, \( \dim M = 2 \).

3.4. The short-wave asymptotics

Let us describe the asymptotic behavior of large eigenvalues and the corresponding eigenfunctions. The asymptotic formulas obtained are often called the high-frequency asymptotics or short-wave asymptotics, meaning that they correspond to high frequencies and therefore to short wavelengths in the nonstationary (i.e., time-dependent) problem.

The behavior of eigenvalues is easily described with the help of oscillation theorems. Namely, the eigenvalues of the operator \( L = -\frac{d^2}{dx^2} + q(x) \) are contained between the eigenvalues of \( L_1 = -\frac{d^2}{dx^2} \) and the eigenvalues of \( L_2 = -\frac{d^2}{dx^2} + M \), where \( M = \max_{x \in [0, l]} q(x) \). Since the eigenvalues of \( L_1 \) and \( L_2 \) are equal to \((\pi n/l)^2\) and \((\pi n/l)^2 + M\) respectively (with \( n = 1, 2, \ldots \)), we get
\[
(\frac{\pi n}{l})^2 \leq \lambda_n \leq (\frac{\pi n}{l})^2 + M
\]

In particular, this implies the asymptotic formula
\[
\lambda_n = (\frac{\pi n}{l})^2 \left( 1 + O\left(\frac{1}{n^2}\right) \right),
\]
or, putting \( k_n = \sqrt{\lambda_n} \):

\[
(3.14) \quad k_n = \frac{\pi n}{l} \left( 1 + O \left( \frac{1}{n^2} \right) \right) = \frac{\pi n}{l} + O \left( \frac{1}{n} \right).
\]

Now, let us find the asymptotics of the eigenfunctions \( X_n(x) \). The idea is that for large \( k \) the term \( k^2 X \) in (3.13) plays a greater role than \( q(x)X \). Therefore, let us solve the equation

\[
(3.15) \quad \psi'' + k^2 \psi = q(x)\psi
\]

with the initial conditions

\[
(3.16) \quad \psi(0) = 0, \quad \psi'(0) = k,
\]

taking \( q(x)\psi \) as the right hand side. We get an integral equation for \( \psi = \psi(x) = \psi(x, k) \) which can be solved by successive approximations. Let us write

\[
\psi(x) = C_1(x) \cos kx + C_2(x) \sin kx,
\]

then the equations for \( C_1(x) \) and \( C_2(x) \), obtained by variation of parameters, are

\[
(3.17) \quad C_1'(x) \cos kx + C_2'(x) \sin kx = 0,
\]
\[
(3.18) \quad -kC_1'(x) \sin kx + kC_2'(x) \cos kx = q(x)\psi.
\]

Solving these equations, we find \( C_1'(x) \) and \( C_2'(x) \). After integration this determines \( C_1(x) \) and \( C_2(x) \) up to arbitrary additive constants which are fixed by the initial values

\[
(3.19) \quad C_1(0) = 0, \quad C_2(0) = 1,
\]

obtained from (3.16). Then

\[
C_1(x) = - \int_0^x q(\tau)\psi(\tau) \frac{\sin k\tau}{k} d\tau,
\]
\[
C_2(x) = 1 + \int_0^x q(\tau)\psi(\tau) \frac{\cos k\tau}{k} d\tau,
\]

implying an integral equation (of so called Volterra type):

\[
(3.20) \quad \psi(x) = \sin kx + \frac{1}{k} \int_0^x \sin k(x - \tau)q(\tau)\psi(\tau)d\tau.
\]
Clearly, the solution $\psi$ of this equation satisfies (3.15) and (3.16) so it is equivalent to (3.15) with initial conditions (3.16). To solve (3.20) by successive approximations method, consider the integral operator $A$ defined by the formula

$$A\psi(x) = \int_0^x \sin k(x - \tau) q(\tau) \psi(\tau) d\tau.$$ 

The equation (3.20) becomes

$$\left( I - \frac{1}{k} A \right) \psi = \sin kx,$$

and its solution may be expressed in the form

$$(3.21) \quad \psi = \left( I - \frac{1}{k} A \right)^{-1} \sin kx = \sum_{n=0}^{\infty} \frac{1}{k^n} A^n (\sin kx),$$

provided the series in the right hand side of (3.21) converges and $A$ can be applied termwise.

It can be proved that the series (3.21) indeed converges uniformly in $x \in [0, l]$ for all $k$ but, since we are only interested in large $k$, we can restrict ourselves to an obvious remark that it converges uniformly on $[0, l]$ for large $k$ because $A$ is norm bounded uniformly in $k$ as an operator in the Banach space $C([0, l])$ (with sup norm), and the norm of $\sin kx$ in $C([0, l])$ is not greater than 1. In particular, we get

$$(3.22) \quad \psi(x, k) = \sin kx + O\left(\frac{1}{k}\right) \text{ as } k \to +\infty.$$ 

This gives the short-wave asymptotics of the eigenfunctions $X_n$. Indeed, it is clear from (3.14) and (3.22) that

$$X_n(x) = \psi(x, k_n) = \sin(k_n x) + O\left(\frac{1}{k_n}\right) = \sin \left(\frac{\pi n}{l} + O\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right),$$

hence

$$X_n(x) = \sin \frac{\pi nx}{l} + O\left(\frac{1}{n}\right) \text{ as } n \to +\infty.$$ 

### 3.5. The Green function and completeness of the system of eigenfunctions

Consider the Schrödinger operator $L = -\frac{d^2}{dx^2} + q(x)$ as a linear operator mapping its domain

$$D_L = \{v(x) \in C^2([0, l]) : v(0) = v(l) = 0\}$$
3.5. The Green function and eigenfunctions

into \( C([0,l]) \). We will see that for \( q \geq 0 \) this operator is invertible and the inverse operator \( L^{-1} \) can be expressed as an integral operator

\[
L^{-1}f(x) = \int_0^l G(x,y)f(y)dy,
\]

where \( G \in C([0,l] \times [0,l]) \). The function \( G(x,y) \) is called the \textit{Green function} of \( L \). In general, if an operator is expressed in the form of the right hand side on (3.23), then \( G(x,y) \) is called its \textit{kernel} or \textit{Schwartz’ kernel} (in honor of L. Schwartz). Therefore, \( L^{-1} \) has a continuous Schwartz kernel equal to \( G(x,y) \). Clearly, the function \( G(x,y) \) is uniquely defined by (3.23).

To find \( L^{-1}f \), we have to solve the equation

\[
-v''(x) + q(x)v(x) = f(x)
\]

with the boundary conditions

\[
v(0) = v(l) = 0.
\]

This is performed by the variation of parameters. It is convenient to use two solutions \( y_1(x), y_2(x) \) of the homogeneous equation

\[-y''(x) + q(x)y(x) = 0\]

satisfying the boundary conditions

\[
y_1(0) = 0, \quad y_1'(0) = 1;
\]

and

\[
y_2(l) = 0, \quad y_2'(l) = -1.
\]

Clearly, the solutions \( y_1(x) \) and \( y_2(x) \) are linearly independent since 0 is not an eigenvalue of \( L \) because \( q(x) \geq 0 \). Now, by the variation of parameters, writing

\[
v(x) = C_1(x)y_1(x) + C_2(x)y_2(x),
\]

we arrive to the equations

\[
\begin{align*}
C_1'(x)y_1(x) + C_2'(x)y_2(x) & = 0 \\
C_1'(x)y_1'(x) + C_2'(x)y_2'(x) & = -f(x).
\end{align*}
\]

The determinant of this system of linear equations for \( C_1'(x) \) and \( C_2'(x) \) is the Wronskian

\[
W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).
\]
It is well-known (and is easy to verify by differentiation of $W(x)$ – see calculation in the proof of Theorem 3.2) that $W(x)=\text{const}$ and $W(x) = 0$ if and only if $y_1$ and $y_2$ are linearly dependent. Therefore, in our case

$$W(x) = W = \text{const} \neq 0.$$ 

Solving the system (3.28), (3.29) by Cramer’s rule, we get

$$C'_1(x) = \frac{1}{W}f(x)y_2(x), \quad C'_2(x) = -\frac{1}{W}f(x)y_1(x).$$

Taking into account the boundary condition (3.25) for $v$ and the boundary conditions (3.26), (3.27) for $y_1, y_2$, we see that $C_1(x)$ and $C_2(x)$ should be chosen so that

$$C_1(l) = 0, \quad C_2(0) = 0.$$ 

This implies

$$C_1(x) = -\frac{1}{W} \int_x^l f(\xi)y_2(\xi)d\xi, \quad C_2(x) = -\frac{1}{W} \int_0^x f(\xi)y_1(\xi)d\xi.$$ 

Therefore, $v$ exists, is unique and is determined by the formula

$$v(x) = -\frac{1}{W} \int_0^x y_1(\xi)y_2(x)f(\xi)d\xi - \frac{1}{W} \int_x^l y_1(x)y_2(\xi)f(\xi)d\xi,$$

which can be expressed in the form

$$v(x) = \int_0^l G(x, \xi)f(\xi)d\xi, \quad (3.30)$$

where

$$G(x, \xi) = -\frac{1}{W}[\theta(x - \xi)y_1(\xi)y_2(x) + \theta(\xi - x)y_1(x)y_2(\xi)], \quad (3.31)$$

where $\theta(z)$ is the Heaviside function ($\theta(z) = 1$ for $z \geq 0$ and $\theta(z) = 0$ for $z < 0$).

Note that $G(x, \xi)$ is continuous and symmetric, i.e.,

$$G(x, \xi) = G(\xi, x). \quad (3.32)$$

(The latter property is clear without explicit computation. Indeed, $L^{-1}$ should be symmetric due to symmetry of $L$, and the symmetry of $L^{-1}$ is equivalent to that of $G(x, \xi)$.)

Consider now the operator $G$ in $L^2([0, l])$ with the kernel $G(x, \xi)$

$$Gf(x) = \int_0^l G(x, \xi)f(\xi)d\xi.$$
Since the kernel is continuous, this is a symmetric compact operator. By the Hilbert-Schmidt theorem it has a complete orthogonal system of eigenfunctions $X_n(x)$ with real eigenvalues $\mu_n$, where $n = 1, 2, \ldots$, and $\mu_n \to 0$ as $n \to \infty$. (For the proofs of these facts from functional analysis see e.g. Sect. VI.5, especially Theorem VI.16, in Reed and Simon [22].) We have

\begin{equation}
GX_n = \mu_n X_n.
\end{equation}

If $X_n$ are continuous, then, applying $L$ to (3.33), we see that

\begin{equation}
X_n = \mu_n LX_n.
\end{equation}

This implies that $\mu_n \neq 0$ and $\lambda_n = 1/\mu_n$ is an eigenvalue of $L$. The continuity of $X_n$ follows easily from (3.33), provided $\mu_n \neq 0$. In general, if $f \in L^2([0, l])$, then $Gf \in C([0, l])$ because by the Cauchy–Schwarz inequality

\begin{equation}
| (Gf)(x') - (Gf)(x'') | = \left| \int_0^l [G(x', \xi) - G(x'', \xi)] f(\xi) d\xi \right| \leq \\
\leq \sup_{\xi \in [0, l]} |G(x', \xi) - G(x'', \xi)| \cdot \int_0^l |f(\xi)| d\xi \leq \\
\leq \sup_{\xi \in [0, l]} |G(x', \xi) - G(x'', \xi)| \cdot \sqrt{l} \left( \int_0^l |f(\xi)|^2 d\xi \right)^{1/2},
\end{equation}

and $G(x, \xi)$ is uniformly continuous on $[0, l] \times [0, l]$. It remains to prove that $G$ cannot have zero as an eigenvalue on $L^2([0, l])$. But if $u \in L^2([0, l])$ and $Gu = 0$, then $u$ is orthogonal to the image of $G$, since then

\begin{equation}
(u, Gf) = (Gu, f) = 0.
\end{equation}

But all the functions $v \in D_L$ can be represented in the form $Gf$ since then $v = G(Lv)$ by construction of $G$. Since $D_L$ is dense in $L^2([0, l])$, we see that (3.34) implies $u = 0$.

Thus, the set of eigenfunctions of $G$ coincides exactly with the set of eigenfunctions of $L$. In particular, we have proved the comleteness of the system of eigenfunctions of $L$ in $L^2([0, l])$.

The following properties of the Green function are easily deducible with the help of (3.31):

a) $G(x, \xi)$ has continuous derivatives up to the second order for $x \neq \xi$ and satisfies (also for $x \neq \xi$) the equation $L_x G(x, \xi) = 0$ in $x$, where $L_x = -\frac{d^2}{dx^2} + q(x)$. 
b) $G(x, \xi)$ is continuous everywhere, whereas its derivative $G'_x$ has left and right limits at $x = \xi$ and the jump is equal to $-1$:

$$G'_x(\xi + 0, \xi) - G'_x(\xi - 0, \xi) = -1.$$ 

c) the boundary conditions $G|_{x=0} = G|_{x=l} = 0$ are satisfied.

These properties may be used in order to find $G$ without variation of parameters. It is easy to verify that $G(x, \xi)$ is uniquely defined by these conditions.

Conditions a) and b) can be easily expressed with the help of the Dirac $\delta$-function. Later on we will do it more carefully, but now we just give some heuristic arguments (on the “physical” level of rigor). It is convenient to apply formula (3.23) which expresses $L^{-1}$ in terms of its kernel from the very beginning. Let us formally apply $L$ to (3.23) and denote

$$\delta(x, \xi) = L_x G(x, \xi).$$

Then

$$f(x) = \int_0^l \delta(x, \xi) f(\xi) d\xi. \tag{3.35}$$

It is clear that $\delta(x, \xi)$ should vanish for $\xi \neq x$ and at the same time

$$\int_0^l \delta(x, \xi) d\xi = 1.$$ 

Taking $f$ with a small support and shifting the support we easily obtain from (3.35) that $\delta(x + z, \xi + z) = \delta(x, \xi)$. Therefore, $\delta(x, \xi)$ should only depend on $\xi - x$, i.e., $\delta(x, \xi) = \delta(\xi - x)$. We see that $\delta(x)$ vanishes for $x \neq 0$ and at the same time

$$\int \delta(x) dx = 1.$$ 

There is no locally integrable function with this property; nevertheless, it is convenient to make use of the symbol $\delta(x)$ if it is a part of an integrand, keeping in mind that

$$\int \delta(x) f(x) dx = f(0).$$

The “function” $\delta(x)$ is called Dirac’s $\delta$-function or the Dirac measure. In the theory of distributions (or generalized functions) $\delta(x)$ is understood as a linear functional on smooth functions whose value at a function $f(x)$ is equal to $f(0)$.

The $\delta$-function can be interpreted as a point mass, or a point charge, or a point load. For example, let a force $f$ be applied to a nonhomogeneous
string. Then when oscillations subside (e.g. due to friction), we see that the stationary form of the string $v(x)$ should satisfy (3.24), (3.25) and, therefore, is expressed by formula (3.30). If the load is concentrated near a point $\xi$ and the total load equals 1, i.e. $\int f(x)dx = 1$, then the form of the string is precisely the graph of $G(x,\xi)$. This statement can easily be made into a more precise one by passing to the limit when a so-called $\delta$-like sequence of loads $f_n$ is chosen, e.g. such that $f_n(x) \geq 0; f_n(x) = 0, \quad |x - \xi| \geq 1/n$ and $\int f_n(x)dx = 1$. We skip the details leaving them as an easy exercise for the reader.

3.6. Problems

3.1. Derive an integral equation for a function $\Phi(x)$ satisfying

$$-\Phi'' + q(x)\Phi = k^2\Phi, \quad \Phi(0) = 1, \quad \Phi'(0) = 0,$$

with $q \in G([0,l])$ and deduce from this integral equation the asymptotics of $\Phi(x)$ and $\Phi'(x)$ as $k \to +\infty$.

3.2. Using the result of Problem 3.1, prove the existence of an infinite number of eigenvalues for the following Sturm-Liouville problem:

$$-X'' + q(x)X = \lambda X, \quad X'(0) = X(l) = 0.$$

Prove the symmetry of the corresponding operator and the orthogonality of eigenfunctions.

3.3. Write explicitly the Green function of the operator $L = -\frac{d^2}{dx^2}$ with the boundary conditions $v(0) = v(l) = 0$. Give a physical interpretation.

3.4. Write explicitly the Green function of the operator $L = -\frac{d^2}{dx^2} + 1$ with the boundary conditions $v'(0) = v'(l) = 0$. Give a physical interpretation.

3.5. With the help of the Green function, prove the completeness of the system of eigenfunctions for the operator $L = -\frac{d^2}{dx^2} + q(x)$ with the boundary conditions $v'(0) = v'(l) = 0$ corresponding to free ends.

3.6. Prove that if $q(x) \geq 0$, then the Green function $G(x,\xi)$ of $L = -\frac{d^2}{dx^2} + q(x)$ with the boundary conditions $v(0) = v(l) = 0$ is positive for $x \neq 0, l$ and $\xi \neq 0, l$. 
3.7. Prove that under the conditions of the previous problem, the Green function $G(x, \xi)$ is a positive kernel, i.e. the matrix $(G(x_i, x_j))_{i,j=1}^n$ is positive definite for any finite set of distinct points $x_1, \ldots, x_n \in (0, l)$.

3.8. Expand the Green function $G(x, \xi)$ of the Sturm-Liouville problem into the series with respect to eigenfunctions $X_n(x)$ and express in terms of the Green function the following sums:

a) $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$,

b) $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$, where $\lambda_n$ ($n = 1, 2, \ldots$) is the set of eigenvalues of this problem. In particular, calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$. 
Chapter 4

Distributions

4.1. Motivation of the definition. Spaces of test functions.

▼ In Analysis and Mathematical Physics one often meets technical difficulties caused by non-differentiability of certain functions. The theory of distributions (generalized functions) makes it possible to overcome these difficulties. The Dirac \( \delta \)-function, which naturally appears in many situations (one of them was mentioned above, in Section 3.5), is an example of a distribution. In the theory of distributions a number of notions and theorems of analysis acquire simpler formulations and are free of unnatural and irrelevant restrictions.

The origin of the notion of distribution or generalized function may be explained as follows. Let \( f(x) \) be a physical quantity (such as temperature, pressure, etc.) which is a function of \( x \in \mathbb{R}^n \). If we measure this quantity at a point \( x_0 \) with the help of some device (thermometer, pressure gauge, etc.), then we actually measure a certain average of \( f(x) \) over a neighborhood of \( x_0 \), that is an integral of the form \( \int f(x)\varphi(x)dx \), where \( \varphi(x) \) is a function characterizing the measuring gadget and smeared somehow over a neighborhood of \( x_0 \).

The idea is to completely abandon the function \( f(x) \) and consider instead a linear functional by assigning to each test function \( \varphi \) the number

\[
\langle f, \varphi \rangle = \int f(x)\varphi(x)dx.
\]

(4.1)

Considering arbitrary linear functionals (not necessarily of the form (4.1)) we come to the notion of a distribution or generalized function. ▲
Let us introduce now the needed spaces of test functions. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $E(\Omega) = C^\infty(\Omega)$ be the space of smooth (infinitely differentiable) functions on $\Omega$; let $D(\Omega) = C^\infty_0(\Omega)$ be the space of smooth functions with compact support in $\Omega$, i.e., functions $\varphi \in C^\infty(\Omega)$ such that there exists a compact $K \subset \Omega$ with the property $\varphi|_{\Omega \setminus K} = 0$.

In general, the support of $\varphi \in C(\Omega)$ (denoted by $\text{supp} \varphi$) is the closure (in $\Omega$) of the set $\{x \in \Omega : \varphi(x) \neq 0\}$. Therefore, $\text{supp} \varphi$ is the minimal closed set $F \subset \Omega$ such that $\varphi|_{\Omega \setminus F} = 0$ or, equivalently, the complement to the maximal open set $G \subset \Omega$ such that $\varphi|_G = 0$. Therefore, $D(\Omega)$ consists of $\varphi \in C^\infty(\Omega)$ such that $\text{supp} \varphi$ is compact in $\Omega$.

For a compact subset $K \subset \mathbb{R}^n$ denote by $D(K) = C^\infty_0(K)$ the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\text{supp} \varphi \subset K$.

Finally, as a space of test functions we will also use $S(\mathbb{R}^n)$, L. Schwartz’s space, consisting of functions $\varphi(x) \in C^\infty(\mathbb{R}^n)$ such that $\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < +\infty$ for any multiindices $\alpha$ and $\beta$.

Theory of distributions makes use of topology in the spaces of test functions. We will introduce it with the help of a system of seminorms. First, we will give general definitions.

A seminorm in a linear space $E$ is a map $p : E \to [0, +\infty)$ such that

- $p(x + y) \leq p(x) + p(y)$, $x, y \in E$;
- $p(\lambda x) = |\lambda| p(x)$, $x \in E$, $\lambda \in \mathbb{C}$.

The property $p(x) = 0$ does not necessarily imply $x = 0$ (if this implication holds, then the seminorm is called a norm).

Let a system of seminorms $\{p_j\}_{j \in J}$, where $J$ is a set of indices, be defined on $E$. For any $\varepsilon > 0$ and $j \in J$, set

$$U_{j, \varepsilon} = \{x : x \in E, p_j(x) < \varepsilon\}$$

and introduce the topology in $E$ whose basis of neighborhoods of zero consists of all finite intersections of the sets $U_{j, \varepsilon}$ (a basis of neighborhoods of any other point $x_0$ is obtained by the shift by $x_0$, i.e., consists of finite intersections of the sets $x_0 + U_{j, \varepsilon}$). Therefore, $G \subset E$ is open if and only if together with each point $x_0$ it also contains a finite intersection of sets $x_0 + U_{j, \varepsilon}$. Clearly, $\lim_{n \to +\infty} x_n = x$ if and only if $\lim_{n \to +\infty} p_j(x_n - x) = 0$ for any $j \in J$. ▲
To avoid perpetual reference to finite intersections, we may assume that for any \( j_1, j_2 \in J \) there exists \( j_3 \in J \) such that \( p_{j_1}(\varphi) \leq p_{j_3}(\varphi) \) and \( p_{j_2}(\varphi) \leq p_{j_3}(\varphi) \) for any \( \varphi \in E \) (in this case \( U_{j_3, \varepsilon} \subset (U_{j_1, \varepsilon} \cap U_{j_2, \varepsilon}) \)). If this does not hold, we can add (to the existing system of seminorms) additional seminorms of the form

\[
p_{j_1 \ldots j_k}(\varphi) = \max\{p_{j_1}(\varphi), \ldots, p_{j_k}(\varphi)\}.
\]

This does not affect the topology defined by seminorms in \( E \) and we will always assume that this is already done. In other words, we will assume that the basis of neighborhoods of zero in \( E \) consists of all sets \( U_{j, \varepsilon} \).

The same topology in \( E \) can be defined by a different set of seminorms. Namely, assume that we have another system of seminorms \( \{p'_{j'}\}_{j' \in J'} \), with the corresponding neighborhoods of 0 denoted \( U'_{j', \varepsilon} \). Clearly, the systems of seminorms \( \{p_j\}_{j \in J}, \{p'_{j'}\}_{j' \in J'} \) define the same topology if and only if the following two conditions are satisfied:

- for every \( j' \in J' \) and \( \varepsilon' > 0 \), there exist \( j \in J \) and \( \varepsilon > 0 \) such that \( U_{j, \varepsilon} \subset U'_{j', \varepsilon'} \);
- for every \( j \in J \) and \( \varepsilon > 0 \), there exist \( j' \in J' \) and \( \varepsilon' > 0 \) such that \( U'_{j', \varepsilon'} \subset U_{j, \varepsilon} \).

It is easy to see that these conditions are satisfied if and only if

- for every \( j' \in J' \) there exist \( j \in J \) and \( C = C_{j', j} > 0 \), such that \( p'_{j'}(x) \leq C p_j(x), \ x \in E \);
- for every \( j \in J \) there exist \( j' \in J' \) and \( C' = C_{j, j'} > 0 \), such that \( p_j(x) \leq C' p'_{j'}(x), \ x \in E \).

If these conditions are satisfied for two systems of seminorms, \( \{p_j\}, \{p'_{j'}\} \), then we will say that the systems are equivalent. Since we are only interested in translation invariant topologies on vector spaces \( E \), we can freely replace a system of seminorms by any equivalent system.

If \( f \) is a linear functional on \( E \), then it is continuous if and only if there exist \( j \in J \) and \( C > 0 \) such that

\[
|\langle f, \varphi \rangle| \leq C p_j(\varphi), \quad \varphi \in E,
\]

where \( \langle f, \varphi \rangle \) denotes the value of \( f \) on \( \varphi \).

Indeed, the continuity of \( f \) is equivalent to the existence of a set \( U_{j, \varepsilon} \) such that \( |\langle f, \varphi \rangle| \leq 1 \) for \( \varphi \in U_{j, \varepsilon} \) and the latter is equivalent to (4.2) with \( C = 1/\varepsilon \).
Note that the seminorm $p_j$ itself, as a function on $E$, is continuous on $E$. Indeed, this is obvious at the point $\varphi = 0$ because the set $\{\varphi : p_j(\varphi) < \varepsilon\}$ is a neighborhood of 0 for any $\varepsilon > 0$. Now the continuity of $p_j$ at an arbitrary point $\varphi_0 \in E$ follows, because the triangle inequality a) above implies

$$p(\varphi_0) - p(\varphi) \leq p_j(\varphi_0 + \varphi) \leq p(\varphi_0) + p(\varphi),$$

so the continuity at $\varphi_0$ follows from the continuity at 0.

We will usually suppose that the following separability condition is satisfied:

(4.3) if $p_j(\varphi) = 0$ for all $j \in J$, then $\varphi = 0$.

This implies that any two distinct points $x, y \in E$ have nonintersecting neighborhoods (the Hausdorff property of the topology). Indeed, there exists $j \in J$ such that $p_j(x - y) > 0$. But then the neighborhoods $x + U_{j, \varepsilon}$, $y + U_{j, \varepsilon}$ do not intersect for $\varepsilon < \frac{1}{2} p_j(x - y)$, because if

$$z = x + t = y + s \in (x + U_{j, \varepsilon}) \cap (y + U_{j, \varepsilon}),$$

then

$$p_j(x - y) = p_j(s - t) \leq p_j(s) + p_j(t) \leq 2\varepsilon < p_j(x - y).$$

Now, consider the case which is most important for us when $J$ is a countable (infinite) set. We may assume that $J = \mathbb{N}$. Introduce a metric on $E$ by the formula

(4.4) $\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x - y)}{1 + p_k(x - y)}$.

The symmetry and the triangle inequality are easily checked. Due to separability condition (4.3), $\rho(x, y) = 0$ implies $x = y$.

Let us prove that the topology defined by the metric coincides with the topology defined above by seminorms. Since the metric (4.4) is invariant with respect to the translations (i.e., $\rho(x + z, y + z) = \rho(x, y)$), it is sufficient to verify that each neighborhood $U_{j, \varepsilon}$ contains a ball $B_\delta(0) = \{x : \rho(x, 0) < \delta\}$ of radius $\delta > 0$ with the center at the origin, and, conversely, each ball $B_\delta(0)$ contains a set of the form $U_{j, \varepsilon}$. (Note that in the future we will also write $B(\delta, 0)$ instead of $B_\delta(0)$.)

First, take an arbitrary $\delta > 0$. Let us prove that there exist $j, \varepsilon$ such that $U_{j, \varepsilon} \subset B_\delta(0)$. If $j$ is such that $p_1(x) \leq p_j(x), \ldots, p_N(x) \leq p_j(x)$, then

$$\rho(x, 0) \leq \sum_{k=1}^{N} \frac{p_k(x)}{2^k} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)} \leq p_j(x) + \frac{1}{2^N}.$$
4.1. Motivation. Spaces of test functions

Therefore, if \( \varepsilon < \delta/2 \) and \( 1/2^N < \delta/2 \), then \( U_{j,\varepsilon} \subset B_\delta(0) \).

Conversely, given \( j \) and \( \varepsilon \), we deduce the existence of \( \delta > 0 \) such that \( B_\delta(0) \subset U_{j,\varepsilon} \) from the obvious inequality

\[
\frac{1}{2^j} \frac{p_j(x)}{1 + p_j(x)} \leq \rho(x, 0).
\]

Namely, take \( \delta \) so that \( \delta < \frac{\varepsilon}{2^j(1 + \varepsilon)} \). Then, since \( f(t) = \frac{1}{2^j} \frac{t}{1 + t} \) is strictly increasing for \( t > 0 \), the inequality \( \rho(x, 0) < \delta \) implies \( p_j(x) < \varepsilon \). ▲

Definition 4.1. A countably normed space is a vector space endowed with a countable number of seminorms satisfying the separability condition.

Acting by induction, we can always replace a countable system of seminorms in \( E \) by an increasing system of seminorms \( \{p_1 \leq p_2 \leq p_3 \leq \ldots\} \) without changing topology. (Here the inequality \( p \leq p' \) between two seminorms means that \( p(x) \leq p'(x) \) for every \( x \in E \).) Indeed, we can always replace \( p_2(x) \) by \( p'_2(x) = \max\{p_1(x), p_2(x)\}, \ x \in E \), then \( p_3(x) \) by \( p'_3(x) = \max\{p_1(x), p_2(x), p_3(x)\} \), etc.

Since the topology in a countably normed space \( E \) is defined by a metric, the notion of completeness of \( E \) is well defined with respect to this metric (which is chosen as in (4.4)). Note, however, that completeness for general metric spaces is not determined by the topology (that is, for two metrics defining the same topology, one may be complete, while another is not). So for the moment, we only allow choosing the metric as in (4.4), and try to formulate the notion of completeness in the language of topology. Let us recall that a metric space \( E \) with a metric \( \rho \) is called complete if every Cauchy sequence \( \{x_j | j = 1, 2, \ldots\} \subset E \) converges in \( E \) (with respect to the metric \( \rho \)). Here a sequence \( \{x_j | j = 1, 2, \ldots\} \) is called a Cauchy sequence (with respect to \( \rho \)) if \( \rho(x_j, x_k) \to 0 \), as \( j, k \to +\infty \). But due to the translation invariance of the metric (and topology), this condition can be also formulated as follows:

\[
x_j - x_k \to 0 \quad \text{as} \quad j, k \to +\infty,
\]

or, in other words, for every neighborhood \( U \) of 0 in \( E \) there exists \( m = m(U) > 0 \), such that for every \( j, k \), which are both greater than \( m \), we have \( x_j - x_k \in U \). Clearly, the last condition only depends upon the topology in \( E \). But the notion of convergence also depends on the topology only. Therefore, the completeness property of a countably normed space is independent of the choice of a system of seminorms in the same equivalence class.
Now we can also formulate the notion of completeness in terms of seminorms. Using (4.5) and the definition of topology in terms of seminorms, we see that a sequence \( \{x_j\}_{j=1,2,...} \) is a Cauchy sequence (in terms of topology or metric) if and only if
\[
p_l(x_j - x_k) \to 0, \text{ as } j,k \to +\infty,
\]
for every seminorm \( p_l \). So to check the completeness of a countably normed space it suffices to check convergence of the sequences satisfying (4.6), which is sometimes more convenient.

**Definition 4.2.** Fréchet space is a topological vector space whose topology is defined by a structure of a complete countably normed topological vector space. In other words, it is a complete topological vector space with a countable system of seminorms defined up to the equivalence explained above. Equivalently, we can also say that a Fréchet space is a topological vector space with a selected equivalence class of countable systems of seminorms.

Since the topology in a countably normed space \( E \) can be determined by a metric \( \rho(x,y) \), the continuity of functions on \( E \) and, in general, of any maps of \( E \) into a metric space, can be determined in terms of sequences. For instance, a linear functional \( f \) on \( E \) is continuous if and only if \( \lim_{k \to +\infty} \varphi_k = 0 \) implies \( \lim_{k \to +\infty} \langle f, \varphi_k \rangle = 0 \).

**Definition 4.3.** The space of continuous linear functionals in \( E \) is called the dual space to \( E \) and is denoted by \( E' \).

Now, let us turn \( \mathcal{D}(K) \), \( \mathcal{E}(\Omega) \) and \( S(\mathbb{R}^n) \) into countably normed spaces.

1) **Space** \( \mathcal{D}(K) \). Denote
\[
p_m(\varphi) = \sum_{|\alpha|\leq m} \sup_{x \in K} |\partial^\alpha \varphi(x)|, \quad \varphi \in \mathcal{D}(K).
\]
The seminorms \( p_m(\varphi), m = 1,2,\ldots \), define a structure of a countably normed space on \( \mathcal{D}(K) \). Clearly, the convergence \( \varphi_p \to \varphi \) in the topology of \( \mathcal{D}(K) \) means that if \( \alpha \) is any multiindex, then \( \partial^\alpha \varphi_p(x) \to \partial^\alpha \varphi(x) \) uniformly on \( K \).

Let us show that \( \mathcal{D}(K) \) is in fact a Fréchet space. The only non-obvious thing to check is its completeness. Let \( \{\varphi_j\}_{j=1,2,\ldots} \) be a Cauchy sequence in \( \mathcal{D}(K) \). This means that for any multiindex \( \alpha \) the sequence \( \{\partial^\alpha \varphi_j\} \) is a Cauchy sequence in \( C_0(K) \), the space of all continuous functions
on \( \mathbb{R}^n \) supported in \( K \) (i.e. vanishing in \( \mathbb{R}^n \setminus K \)), with the sup norm \( p_0 \). Then there exists \( \psi_\alpha \in C_0(K) \) such that \( \partial^\alpha \varphi_j \to \psi_\alpha \) in \( C_0(K) \) as \( j \to +\infty \) (i.e. converges uniformly on \( \mathbb{R}^n \)). It remains to prove that in fact \( \psi_\alpha = \partial^\alpha \psi_0 \) for all \( \alpha \). But this follows from the standard analysis results on possibility to differentiate a uniformly convergent sequence, if the sequence of derivatives also converges uniformly.

2) Space \( \mathcal{E}(\Omega) \). Let \( K_l \subset \Omega, l = 1, 2, \ldots \), be a sequence of compacts such that \( K_1 \subset K_2 \subset K_3 \subset \ldots \) and for every \( x \in \Omega \) there exists \( l \) such that \( x \) belongs to the interior of \( K_l \) (i.e. \( K \) contains a neighborhood of \( x \)). For instance, we may set

\[
K_l = \{ x : x \in \Omega, |x| \leq l, \rho(x, \partial \Omega) \geq 1/l \},
\]

where \( \partial \Omega \) is the boundary of \( \Omega \) (i.e., \( \partial \Omega = \bar{\Omega} \setminus \Omega \)) and \( \rho \) is the usual Euclidean distance in \( \mathbb{R}^n \). Set

\[
p_l(\varphi) = \sum_{|\alpha| \leq l} \sup_{x \in K_l} |\partial^\alpha \varphi(x)|, \quad \varphi \in \mathcal{E}(\Omega).
\]

These seminorms turn \( \mathcal{E}(\Omega) \) into a countably normed space. Clearly, the convergence \( \varphi_p \to \varphi \) in the topology of \( \mathcal{E}(\Omega) \) means that \( \partial^\alpha \varphi_p(x) \to \partial^\alpha \varphi(x) \) uniformly on any compact \( K \subset \Omega \) and for any fixed multiindex \( \alpha \). In particular, this implies that the topology defined in this way does not depend on the choice of a system of compacts \( K_l \) described above.

3) Space \( S(\mathbb{R}^n) \). Define the system of seminorms

\[
p_k(\varphi) = \sum_{|\alpha + \beta| \leq k} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|,
\]

and the convergence \( \varphi_p \to \varphi \) in the topology of \( S(\mathbb{R}^n) \) means that if \( A \) is a differential operator with polynomial coefficients on \( \mathbb{R}^n \), then \( A\varphi_p(x) \to A\varphi(x) \) uniformly on \( \mathbb{R}^n \).

Both \( \mathcal{E}(\Omega) \) and \( S(\mathbb{R}^n) \) are also Fréchet spaces. Their completeness can be established by the same arguments as the ones used for the space \( D(K) \) above.

4.2. Spaces of distributions

The above procedure of transition from \( E \) to its dual \( E' \) makes it possible to define \( \mathcal{E}'(\Omega) \) and \( S'(\mathbb{R}^n) \) as dual spaces of \( \mathcal{E}(\Omega) \) and \( S(\mathbb{R}^n) \), respectively. We have not yet introduced topology in \( D(\Omega) \) (the natural topology in \( D(\Omega) \) is not easy to define) but we will not need it. Instead we define \( D'(\Omega) \) as
the space of linear functionals $f$ on $\mathcal{D}(\Omega)$ such that $f|_{\mathcal{D}(K)}$ is continuous on $\mathcal{D}(K)$ for any compact $K \subset \Omega$. This continuity can be also defined in terms of sequences (rather than seminorms), because the topology in $\mathcal{D}(K)$ can be defined by a metric.

It is also convenient to introduce the \textit{convergent sequences} in $\mathcal{D}(\Omega)$, saying that a sequence $\{\varphi_k\}_{k=1,2,\ldots}$, $\varphi_k \in \mathcal{D}(\Omega)$, converges to $\varphi \in \mathcal{D}(\Omega)$ if the following two conditions are satisfied:

(a) there exists a compact $K \subset \Omega$ such that $\varphi \in \mathcal{D}(K)$ and $\varphi_k \in \mathcal{D}(K)$ for all $k$;

(b) $\varphi_k \to \varphi$ in $\mathcal{D}(K)$, or, in other words, for every fixed multiindex $\alpha$, $\partial^\alpha \varphi_k \to \partial^\alpha \varphi$ uniformly on $K$ (or on $\Omega$).

Using this definition, we can easily see that a linear functional $f$ on $\mathcal{D}(\Omega)$ is a distribution if and only if $\langle f, \varphi_k \rangle \to 0$ for every sequence $\{\varphi_k\}$, such that $\varphi_k \to 0$ in $\mathcal{D}(\Omega)$.

\textbf{Definition 4.4.}

(1) The elements of $\mathcal{D}'(\Omega)$ are called \textit{distributions} in $\Omega$.

(2) The elements of $\mathcal{E}'(\Omega)$ are called \textit{distributions with compact support} in $\Omega$. (The notion of support of a distribution that justifies the above term will be defined later.)

(3) The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called \textit{tempered distributions} on $\mathbb{R}^n$.

\textbf{Example 4.1.} The “regular” distributions on $\Omega$. Let $L^1_{\text{loc}}(\Omega)$ be the space of (complex-valued) functions on $\Omega$ which are Lebesgue integrable with respect to the Lebesgue measure on any compact $K \subset \Omega$. To any $f \in L^1_{\text{loc}}(\Omega)$, assign the functional on $\mathcal{D}(\Omega)$ (denoted by the same letter), setting

\begin{equation}
\langle f, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)dx,
\end{equation}

where $\varphi \in \mathcal{D}(\Omega)$. Clearly, we get a distribution on $\Omega$ (called regular). An important fact is that if two functions $f_1, f_2 \in L^1_{\text{loc}}(\Omega)$ determine the same distribution, they coincide almost everywhere. This is a corollary of the following lemma.

\textbf{Lemma 4.5.} Let $f \in L^1_{\text{loc}}(\Omega)$ and $\int_{\Omega} f(x)\varphi(x)dx = 0$ for any $\varphi \in \mathcal{D}(\Omega)$. Then $f(x) = 0$ for almost all $x$. 
Proof of this lemma requires the existence of an ample set of functions in $\mathcal{D}(\Omega)$. We do not know yet whether there exist such nontrivial (not identically vanishing) functions. Let us construct a supply of functions belonging to $\mathcal{D}(\mathbb{R}^n)$.

First, let $\psi(t) = \theta(t)e^{-1/t}$, where $t \in \mathbb{R}$ and $\theta(t)$ is the Heaviside function. Clearly, $\psi(t) \in C^\infty(\mathbb{R})$. Therefore, if we consider the function $\varphi(x) = \psi(1 - |x|^2)$ in $\mathbb{R}^n$, we get a function $\varphi(x) \in C^\infty_0(\mathbb{R}^n)$ such that $\varphi(x) = 0$ for $|x| \geq 1$ and $\varphi(x) > 0$ for $|x| < 1$. It is convenient to normalize $\varphi$ considering instead the function $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\varphi_\varepsilon(x) = 0$ for $|x| \geq \varepsilon$ and $\varphi_\varepsilon(x) > 0$ for $|x| < \varepsilon$. Besides, $\int \varphi_\varepsilon(x)dx = 1$.

Now, introduce an important averaging (or mollifying) operation: for any $f(x) \in L^1_{\text{loc}}(\Omega)$ take a family of convolutions

$$f_\varepsilon(x) = \int f(x - y)\varphi_\varepsilon(y)dy = \int f(y)\varphi_\varepsilon(x - y)dy$$

defined for $x \in \Omega_\varepsilon = \{x : x \in \Omega, \rho(x, \partial \Omega) > \varepsilon\}$. It is clear from the dominated convergence theorem that the last integral in (4.8) is a $C^\infty$-function and

$$\partial^\alpha f_\varepsilon(x) = \int f(y)(\partial^\alpha \varphi_\varepsilon)(x - y)dy,$$

hence $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Note also the following properties of averaging:

a) If $f(x) = 0$ outside of a compact $K \subset \Omega$, then $f_\varepsilon(x) = 0$ outside of the $\varepsilon$-neighborhood of $K$. In particular, in this case $f_\varepsilon \in \mathcal{D}(\Omega)$ for a sufficiently small $\varepsilon > 0$.

b) If $f(x) = 1$ for $x \in \Omega_{2\varepsilon}$, then $f_\varepsilon(x) = 1$ for $x \in \Omega_{3\varepsilon}$. In particular, if $f$ is the characteristic function of $\Omega_{2\varepsilon}$, then $f_\varepsilon \in \mathcal{D}(\Omega)$ and $f_\varepsilon(x) = 1$ for $x \in \Omega_{3\varepsilon}$.

c) If $f \in C(\Omega)$, then $f_\varepsilon(x) \to f(x)$ uniformly on any fixed compact $K \subset \Omega$ as $\varepsilon \to 0+$.

Indeed,

$$f_\varepsilon(x) - f(x) = \int [f(x - y) - f(x)]\varphi_\varepsilon(y)dy,$$

implying

$$|f_\varepsilon(x) - f(x)| \leq \sup_{|y| \leq \varepsilon} |f(x - y) - f(x)|$$

so that the uniform continuity of $f$ on the $\varepsilon$-neighborhood of $K$ implies our statement.
d) If \( f \in L^p_{\text{loc}}(\Omega) \), where \( p \geq 1 \), then \( f_\varepsilon \to f \) with respect to the norm in \( L^p(K) \) for any fixed compact \( K \subset \Omega \) as \( \varepsilon \to 0^+ \).

Since the values \( f_\varepsilon(x) \) at \( x \in K \) only depend on the values of \( f \) on the \( \varepsilon \)-neighborhood of the compact \( K \), we may assume that \( f(x) = 0 \) for \( x \in \Omega \setminus K_1 \), where \( K_1 \) is a compact in \( \Omega \). Let the norm in \( L^p(\Omega) \) be denoted by \( \|u\|_p \), i.e.

\[
\|u\|_p = \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p}.
\]

Due to Minkowski’s inequality (see Appendix to this chapter: Sect. 4.7)

\[
\|f_\varepsilon\|_p = \left\| \int f(\cdot - y)\varphi_\varepsilon(y) dy \right\|_p \leq \int \|f(\cdot - y)\|_p |\varphi_\varepsilon(y)| dy = \|f\|_p.
\]

Due to c), the property d) holds for \( f \in C(\Omega) \). But if \( f \in L^p(\Omega) \) and \( f = 0 \) outside \( K_1 \), then we can approximate \( f \) with respect to the norm in \( L^p(\Omega) \) first, by step functions, and then by continuous functions on \( \Omega \) which vanish outside a compact \( K_2 \subset \Omega \). Let \( g \) be a continuous function on \( \Omega \) which vanishes outside \( K_2 \) and such that \( \|f - g\|_p < \delta \). Then

\[
\limsup_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_p \leq \limsup_{\varepsilon \to 0^+} \|f_\varepsilon - g_\varepsilon\|_p + \limsup_{\varepsilon \to 0^+} \|g_\varepsilon - g\|_p + \limsup_{\varepsilon \to 0^+} \|g - f\|_p = \limsup_{\varepsilon \to 0^+} \|(f - g)\varepsilon\|_p + \|g - f\|_p \leq 2\|f - g\|_p \leq 2\delta,
\]

which due to the arbitrariness of \( \delta > 0 \) implies \( \limsup_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_p = 0 \), as required.

**Proof of Lemma 4.5.** Let \( f \in L^1_{\text{loc}}(\Omega) \) and \( \int_{\Omega} f(x)\varphi(x)dx = 0 \) for every \( \varphi \in \mathcal{D}(\Omega) \). This implies \( f_\varepsilon(x) = 0 \) for all \( x \in \Omega_\varepsilon \). Therefore, the statement of the lemma follows from the property d). \( \square \)

Lemma 4.5 allows us to identify the functions \( f \in L^1_{\text{loc}}(\Omega) \) with the distributions that they define by formula (4.7). Note that the expression

\[
\langle f, \varphi \rangle = \int f(x)\varphi(x)dx
\]

is also often used for distributions \( f \) (in this case the left-hand side of (4.9) is the definition for the right hand side; when the right-hand side makes sense, this definition is consistent).
Example 4.2. “Regular” distributions in $\mathcal{E}'(\Omega)$ and $S'(\mathbb{R}^n)$. If $f \in L^1_{\text{comp}}(\Omega)$, i.e., $f \in L^1(\mathbb{R}^n)$ and $f(x) = 0$ outside a compact $K \subset \Omega$, then we may construct via the standard formula (4.9) a functional on $\mathcal{E}(\Omega)$ which is a distribution with compact support. We will identify $f$ with the corresponding element of $\mathcal{E}'(\Omega)$.

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$
\int |f(x)|(1 + |x|)^{-N} < +\infty
$$

for some $N > 0$, then (4.9) defines a functional on $S(\mathbb{R}^n)$ which is a tempered distribution. In particular, (4.10) holds if $f$ is measurable and

$$
|f(x)| \leq C(1 + |x|)^{N-n-1}.
$$

Example 4.3. Dirac’s $\delta$-function. This is the functional defined as follows:

$$
\langle \delta(x-x_0), \varphi(x) \rangle = \varphi(x_0).
$$

If $x_0 \in \Omega$, then $\delta(x-x_0) \in \mathcal{D}'(\Omega)$ and, moreover, $\delta(x-x_0) \in \mathcal{E}'(\Omega)$. Obviously, $\delta(x-x_0) \in S'(\mathbb{R}^n)$ for any $x_0 \in \mathbb{R}^n$. Instead of (4.11) one often writes, in accordance with the above-mentioned convention,

$$
\int \delta(x-x_0)\varphi(x)dx = \varphi(x_0),
$$

though, clearly, $\delta(x-x_0)$ is not a regular distribution. Indeed, if we assume that $\delta(x-x_0)$ is regular, then Lemma 4.5 would imply that $\delta(x-x_0)$ vanishes almost everywhere in $\mathbb{R}^n \setminus \{x_0\}$, hence, it vanishes almost everywhere in $\mathbb{R}^n$, and so vanishes as a distribution in $\mathbb{R}^n$.

Example 4.4. Let $L$ be a linear differential operator in $\mathbb{R}^n$, $\Gamma$ a smooth compact hypersurface in $\mathbb{R}^n$, $dS$ the area element of $\Gamma$. Then the formula

$$
\varphi \mapsto \int_{\Gamma} (L\varphi)|_{\Gamma}dS
$$

defines a nonregular distribution with compact support in $\mathbb{R}^n$. As $\Gamma$ we may take a compact submanifold (perhaps with a boundary) of any codimension in $\mathbb{R}^n$. In this case as $dS$ we may take any density on $\Gamma$ (or a differential form of the maximal order if the orientation of $\Gamma$ is fixed). In particular, the Dirac $\delta$-function appears as a particular case when $\Gamma$ is a point.
4.3. Topology and convergence in the spaces of distributions

Let \( F \) be one of the spaces of test functions \( \mathcal{D}(\Omega), \mathcal{E}(\Omega) \) or \( S(\mathbb{R}^n) \), and let \( F' \) be the corresponding dual space of distributions (continuous linear functionals on \( F \)). We will consider the space \( F' \) with so-called weak topology, i.e., the topology defined by the seminorms

\[
p_\varphi(f) = |\langle f, \varphi \rangle|, \quad \varphi \in F, \ f \in F'.
\]

In particular, the convergence \( f_k \rightarrow f \) in this topology means that

\[
\langle f_k, \varphi \rangle \rightarrow \langle f, \varphi \rangle \quad \text{for any } \varphi \in F.
\]

In this case we will also use the usual notation \( \lim_{k \rightarrow \infty} f_k = f \).

Up to now we assumed that the limit \( f \) is itself a distribution. Now let us simply assume that \( \{f_k | k = 1, 2, \ldots \} \) is a sequence of distributions (from \( F' \)), such that for any \( \varphi \in F \) there exists a limit

\[
\langle f, \varphi \rangle := \lim_{k \rightarrow +\infty} \langle f_k, \varphi \rangle.
\]

Will the limit \( f \) be always a distribution too? It is clear that the limit is a linear functional with respect to \( \varphi \). But will it be continuous? The answer is affirmative, and the corresponding property is called completeness (or weak completeness) of the distribution space. It is a particular case of a general result which holds for any Fréchet space (see Appendix to this chapter: Sect. 4.8).

**Example 4.5. \( \delta \)-like sequence.** Consider the above constructed functions \( \varphi_\varepsilon(x) \in \mathcal{D}(\mathbb{R}^n) \). Recall that they satisfy:

\begin{itemize}
  \item[a)] \( \varphi_\varepsilon(x) \geq 0 \) for all \( x \);
  \item[b)] \( \int \varphi_\varepsilon(x)dx = 1 \);
  \item[c)] \( \varphi_\varepsilon(x) = 0 \) for \( |x| \geq \varepsilon \).
\end{itemize}

Let us prove that the properties a)–c) imply

\[
\lim_{\varepsilon \rightarrow 0^+} \varphi_\varepsilon(x) = \delta(x)
\]

in \( \mathcal{D}'(\mathbb{R}^n) \). Indeed, this means that

\[
\lim_{\varepsilon \rightarrow 0^+} \int \varphi_\varepsilon(x)\psi(x)dx = \psi(0) \quad \text{for any } \psi \in \mathcal{D}(\mathbb{R}^n).
\]

With the help of b) we may rewrite (4.14) in the form of the relation

\[
\lim_{\varepsilon \rightarrow 0^+} \int \varphi_\varepsilon(x)[\psi(x) - \psi(0)]dx = 0,
\]
4.3. Topology and convergence

which is obviously true due to the estimate

\[ \left| \int \varphi_{\varepsilon}(x)[\psi(x) - \psi(0)]dx \right| \leq \sup_{|x| \leq \varepsilon} |\psi(x) - \psi(0)|. \]

Note that (4.13) holds not only in \( \mathcal{D}'(\mathbb{R}^n) \) but also in \( S'(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n) \) and even in \( \mathcal{E}'(\Omega) \) if \( 0 \in \Omega \).

Sequences of “regular” functions converging to the \( \delta \)-function are called \( \delta \)-like sequences. Properties a)–c) can be considerably weakened retaining \( \delta \)-likeness of the sequence. Thus, in the theory of Fourier series one proves the \( \delta \)-likeness (e.g. in \( \mathcal{D}'((−\pi,\pi)) \)) of the sequence of Dirichlet kernels

\[ D_k(t) = \frac{1}{2\pi} \frac{\sin(k + \frac{1}{2})t}{\sin \frac{t}{2}}, \]

defined by the condition that \( \langle D_k, \varphi \rangle \) is the sum of the \( k \) first terms of the Fourier series of \( \varphi \) at \( t = 0 \). Similarly, a \( \delta \)-like sequence in \( \mathcal{D}'((−\pi,\pi)) \) is constituted by the Fejer kernel

\[ F_k(t) = \frac{1}{2\pi k} \frac{\sin^2 \frac{kt}{2}}{\sin^2 \frac{t}{2}}, \]

defined from the condition that \( \langle F_k, \varphi \rangle \) is the arithmetic mean of the \( k \) first partial sums of the Fourier series. Later on we will encounter a number of other important examples of \( \delta \)-like sequences. Note that the \( \delta \)-likeness is essentially always proved in the same way as for \( \varphi_{\varepsilon}(x) \).

**Example 4.6.** Distributions \( \frac{1}{x + i0} \) and v.p. \( \frac{1}{x} \). The “regular” functions \( \frac{1}{x + i\varepsilon} \), and \( \frac{1}{x + i\varepsilon} \) (here \( i = \sqrt{-1} \)) of \( x \in \mathbb{R}^1 \) for \( \varepsilon > 0 \) determine tempered distributions (elements of \( S'((\mathbb{R}^1)) \)). It turns out that there exist limits in \( S'((\mathbb{R}^1)) \)

\[ \frac{1}{x + i0} = \lim_{\varepsilon \to 0+} \frac{1}{x + i\varepsilon}, \quad \frac{1}{x - i0} = \lim_{\varepsilon \to 0+} \frac{1}{x - i\varepsilon} \]

The left hand sides here are, by definition, the limits on the right hand sides.

Let us prove, e.g. the existence of the limit \( \frac{1}{x + i0} \) in \( S'((\mathbb{R}^1)) \).

We must prove that if \( \varphi \in S((\mathbb{R}^1)) \) then

\[ \lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x + i\varepsilon} dx \]

exists and is a continuous linear functional of \( \varphi \in S((\mathbb{R}^1)) \). The simplest way to do this is to integrate by parts. Namely, we have

\[ \frac{1}{x + i\varepsilon} = \frac{d}{dx} \ln(x + i\varepsilon), \]
where we take an arbitrary branch of \( \ln(x + i\varepsilon) \) (but keeping continuity for \( x \in \mathbb{R}, \varepsilon > 0 \)). Integrating by parts, we get

\[
\int_{-\infty}^{\infty} \frac{\varphi(x)}{x + i\varepsilon} \, dx = -\int_{-\infty}^{\infty} \varphi'(x) \ln(x + i\varepsilon) \, dx. \tag{4.16}
\]

Since \( \ln(x + i\varepsilon) = \ln |x + i\varepsilon| + i \arg(x + i\varepsilon) \), it is clear that, by the dominated convergence theorem, the limit of the right-hand side of (4.16), as \( \varepsilon \to 0^+ \), is equal to

\[
-\int_{-\infty}^{\infty} (\ln |x|) \varphi'(x) \, dx - i\pi \int_{-\infty}^{0} \varphi'(x) \, dx = -\int_{-\infty}^{\infty} (\ln |x|) \varphi'(x) \, dx - \pi i \varphi(0). \tag{4.17}
\]

Since \( \ln |x| \in L^1_{\text{loc}}(\mathbb{R}) \) and \( \ln |x| \) grows at infinity not faster than \( |x|^\varepsilon \) for any \( \varepsilon > 0 \), it is obvious that the right hand side of (4.17) is finite and defines a continuous functional of \( \varphi \in S(\mathbb{R}^n) \). This functional is denoted by \( \frac{1}{x + i0} \).

Let us study in detail the first summand in (4.17). We have

\[
-\int_{-\infty}^{\infty} (\ln |x|) \varphi'(x) \, dx = \lim_{\varepsilon \to 0^+} \left[ -\int_{|x| \geq \varepsilon} \ln |x| \cdot \varphi'(x) \, dx \right] = \lim_{\varepsilon \to 0^+} \left[ \ln |x| \cdot \varphi(x) |_{x=\varepsilon}^{x=\varepsilon} + \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx \right] = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx,
\]

since \( \lim_{\varepsilon \to 0^+} \ln \varepsilon [\varphi(\varepsilon) - \varphi(-\varepsilon)] = \lim_{\varepsilon \to 0^+} O(\varepsilon) \ln \varepsilon = 0 \). In particular, we have proved that the last limit in (4.18) exists and defines a functional belonging to \( S'(\mathbb{R}^1) \). This functional is denoted v.p. \( \frac{1}{x} \) (the letters v.p. stand for the first letters of French words “valeur principale” meaning the principal value). Therefore, by definition,

\[
\langle \text{v.p.} \frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx. \tag{4.19}
\]

Moreover, we may now rewrite (4.17) in the form

\[
\frac{1}{x + i0} = \text{v.p.} \frac{1}{x} - \pi i \delta(x). \tag{4.20}
\]

Similarly, we get

\[
\frac{1}{x - i0} = \text{v.p.} \frac{1}{x} + \pi i \delta(x). \tag{4.21}
\]
4.4. The support of a distribution

Formulas (4.20) and (4.21) are called Sokhotsky’s formulas. They, in particular, imply

$$\frac{1}{x + i0} - \frac{1}{x - i0} = -2\pi i \delta(x).$$

The existence of limits in (4.15) and (4.19) may be also proved by expanding $\varphi(x)$ in the sum of a function equal to $\varphi(0)$ in a neighborhood of 0 and a function which vanishes at 0. For each summand, the existence of limits is easy to verify. In the same way, one gets the Sokhotsky formulas, too.

Distributions $\frac{1}{x \pm i0}$ and v.p. $\frac{1}{x}$ are distinct “regularizations” of the nonintegrable function $\frac{1}{x}$, i.e., they allow us to make sense of the divergent integral $\int_{-\infty}^{\infty} \frac{1}{x} \varphi(x) dx$. We see that there is an ambiguity: various distributions correspond to the nonintegrable function $\frac{1}{x}$. The regularization procedure is important if we wish to use $\frac{1}{x}$ as a distribution (e.g. to differentiate it).

This procedure (or a similar one) is applicable to many other nonintegrable functions.

4.4. The support of a distribution

Let $\Omega_1, \Omega_2$ be two open subsets of $\mathbb{R}^n$ and $\Omega_1 \subset \Omega_2$. Then $\mathcal{D}(\Omega_1) \subset \mathcal{D}(\Omega_2)$. Therefore, if $f \in \mathcal{D}'(\Omega_2)$ we may restrict $f$ to $\mathcal{D}(\Omega_1)$ and get a distribution on $\Omega_1$ denoted by $f|_{\Omega_1}$. It is important that the operation of restriction has the following properties:

a) Let $\{\Omega_j : j \in J\}$ be a cover of an open set $\Omega$ by open sets $\Omega_j, j \in J$, i.e., $\Omega = \bigcup_{j \in J} \Omega_j$. If $f \in \mathcal{D}'(\Omega)$ and $f|_{\Omega_j} = 0$ for $j \in J$, then $f = 0$.

b) Let again $\Omega = \bigcup_{j \in J} \Omega_j$ and let $f_j \in \mathcal{D}'(\Omega_j)$ a set of distributions such that $f_k|_{\Omega_k \cap \Omega_l} = f_l|_{\Omega_k \cap \Omega_l}$ for any $k, l \in J$. Then there exists a distribution $f \in \mathcal{D}'(\Omega)$ such that $f|_{\Omega_j} = f_j$ for any $j \in J$.

Properties a) and b) mean that distributions constitute a sheaf.

Let us prove the property a), which is important for us. To do this, let us use a partition of unity, i.e. a family of functions $\{\varphi_j\}_{j \in J}$, such that

1) $\varphi_j \in C^\infty(\Omega)$, supp $\varphi_j \subset \Omega_j$;

2) The family $\{\varphi_j\}_{j \in J}$ is locally finite, i.e., any $x_0 \in \Omega$ has a neighborhood in which only a finite number of functions of this family, $\varphi_{j_1}, \ldots, \varphi_{j_k}$, do not vanish;

3) $\sum_{j \in J} \varphi_j \equiv 1.$
The existence of a partition of unity is proved in courses of geometry and analysis, see e.g. Theorem 10.8 in Rudin [23] or Theorem 1.11 in Warner [32].

If \( \varphi \in \mathcal{D}(\Omega) \), then property 1) implies \( \varphi_j \varphi \in \mathcal{D}(\Omega_j) \) and properties 2) and 3) imply

\[
\varphi = \sum_{j \in J} \varphi_j \varphi,
\]

where the sum locally contains only a finite number of summands which are not identically zero, since the compact \( \text{supp} \varphi \) intersects, due to property 2), only a finite number of supports \( \text{supp} \varphi_j \). If \( f|_{\Omega_j} = 0 \) for all \( j \), then

\[
\langle f, \varphi \rangle = \sum_{j \in J} \langle f, \varphi_j \varphi \rangle = 0,
\]

implying \( f = 0 \), due to the arbitrariness of \( \varphi \). This proves property a). Property b) may be proved by constructing \( f \in \mathcal{D}'(\Omega) \) from distributions \( f_j \in \mathcal{D}'(\Omega_j) \) with the help of the formula

\[
\langle f, \varphi \rangle = \sum_{j \in J} \langle f_j, \varphi_j \varphi \rangle.
\]

If \( \varphi \in \mathcal{D}(\Omega_k) \), where \( k \) is fixed, then

\[
\langle f, \varphi \rangle = \sum_{j \in J} \langle f_j, \varphi_j \varphi \rangle = \sum_{j \in J} \langle f_k, \varphi_j \varphi \rangle = \langle f_k, \sum_{j \in J} \varphi_j \varphi \rangle = \langle f_k, \varphi \rangle
\]

since \( \text{supp}(\varphi_j \varphi) \subset \text{supp} \varphi_j \cap \text{supp} \varphi \subset \Omega_j \cap \Omega_k \). We have therefore verified that \( f|_{\Omega_k} = f_k \), proving property b).

Property a) allows us to introduce for \( f \in \mathcal{D}'(\Omega) \) a well-defined maximal open subset \( \Omega' \subset \Omega \) such that \( f|_{\Omega'} = 0 \) (here \( \Omega' \) is the union of all open sets \( \Omega_1 \subset \Omega \) such that \( f|_{\Omega_1} = 0 \)). The closed subset \( F = \Omega \setminus \Omega' \) is called the support of the distribution \( f \) (denoted by \( \text{supp} f \)). Therefore, \( \text{supp} f \) is the smallest closed subset \( F \subset \Omega \) such that \( f|_{\Omega \setminus F} = 0 \).

**Example 4.7.** \( \text{supp} \delta(x) = \{0\} \). The support of the distribution from Example 4.4 belongs to \( \Gamma \). In general, if \( \text{supp} f \subset F \), then \( f \) is said to be supported on \( F \). Therefore, a distribution of the form (4.12) is supported on \( \Gamma' \).

Now, let us define supports of distributions from \( \mathcal{E}'(\Omega) \) and \( \mathcal{S}'(\mathbb{R}^n) \). This is easy if we note that there are canonical embeddings

\[
\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega),
\]
4.4. The support of a distribution

\[ S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n). \]

They are constructed with the help of embeddings

\[ D(\Omega) \subset E(\Omega), \]
\[ D(\mathbb{R}^n) \subset S(\mathbb{R}^n), \]

which enable us to define maps

\[
\begin{align*}
E'(\Omega) & \longrightarrow D'(\Omega) \\
S'(\mathbb{R}^n) & \longrightarrow D'(\mathbb{R}^n)
\end{align*}
\]

by restricting functionals onto the smaller subspace.

The continuity of these functionals on \( D(\Omega) \) and \( D(\mathbb{R}^n) \) follows from the continuity of embeddings

\[ D(K) \subset E(\Omega) \text{ for a compact } K \text{ in } \Omega \]
\[ D(K) \subset S(\mathbb{R}^n) \text{ for a compact } K \text{ in } \mathbb{R}^n. \]

Finally, the dual maps (4.22), (4.23) are embeddings, because \( D(\Omega) \) is dense in \( E(\Omega) \) and \( D(\mathbb{R}^n) \) is dense in \( S(\mathbb{R}^n) \), therefore any continuous linear functional on \( E(\Omega) \) (resp. \( S(\mathbb{R}^n) \)) is uniquely determined by its values on a dense subset \( D(\Omega) \) (resp. \( D(\mathbb{R}^n) \)).

In what follows, we will identify distributions from \( E'(\Omega) \) and \( S'(\mathbb{R}^n) \) with their images in \( D'(\Omega) \) and \( D'(\mathbb{R}^n) \), respectively.

**Proposition 4.6.** Let \( f \in D'(\Omega) \). Then \( f \in E'(\Omega) \) if and only if \( \text{supp } f \) is a compact subset of \( \Omega \), or, in other words, if \( f \) is supported on a compact subset of \( \Omega \).

**Proof.** Let \( f \in E'(\Omega) \). Then \( f \) is continuous with respect to a seminorm in \( E(\Omega) \), i.e.,

\[
|\langle f, \varphi \rangle| \leq C \sum_{|\alpha| \leq l} \sup_{x \in K} |\partial^\alpha \varphi(x)|,
\]

where the compact \( K \subset \Omega \) and numbers \( C, l \) do not depend on \( \varphi \). But this implies that \( \langle f, \varphi \rangle = 0 \) for any \( \varphi \in D(\Omega \setminus K) \), so \( f|_{\Omega \setminus K} = 0 \), i.e., \( \text{supp } f \subset K \).

Conversely, let \( f \in D'(\Omega) \) and \( \text{supp } f \subset K_1 \), where \( K_1 \) is a compact in \( \Omega \). Clearly, if \( K \subset \Omega \) contains a neighborhood of \( K_1 \), then \( f \) is determined by its restriction onto \( D(K) \) and (4.24) holds for \( \varphi \in D(K) \); hence, for \( \varphi \in D(\Omega) \) (since \( \langle f, \varphi \rangle \) does not depend on \( \varphi|_{\Omega \setminus K} \)).
Therefore, $f$ can be continuously extended to a functional $f \in \mathcal{E}'(\Omega)$. Note that this extension essentially reduces to the following two steps: 1) decompose $\varphi \in \mathcal{E}(\Omega)$ into a sum $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in \mathcal{D}(\Omega)$ and $\varphi_2 = 0$ in a neighborhood of $\text{supp} \, f$, and 2) set $\langle f, \varphi \rangle = \langle f, \varphi_1 \rangle$. Proposition 4.6 is proved.

The following important theorem describes distributions supported at a point.

**Theorem 4.7.** Let $f \in \mathcal{E}'(\mathbb{R}^n)$, supp $f = \{0\}$. Then $f$ has the form

\[ \langle f, \varphi \rangle = \sum_{|\alpha| \leq l} c_\alpha (\partial^\alpha \varphi)(0), \]

where $l$ and constants $c_\alpha$ do not depend on $\varphi$.

**Proof.** To any $\varphi \in \mathcal{E}(\mathbb{R}^n)$ assign its $l$-jet at 0, i.e., the set of derivatives

\[ j^l_0(\varphi) = \{(\partial^\alpha \varphi)(0)\}_{|\alpha| \leq l}. \]

Choose $l$ so as to satisfy (4.24), where $K$ is a compact which is a neighborhood of 0. Let us verify that $\langle f, \varphi \rangle$ actually depends only on $j^l_0(\varphi)$. Then the statement of the theorem becomes obvious since it reduces to the description of linear functionals on a finite dimensional vector space of $l$-jets at 0 and these functionals are defined as in the right-hand side of (4.25). Thus, it remains to verify that $j^l_0(\varphi) = 0$ implies $\langle f, \varphi \rangle = 0$.

Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ is chosen so that $\chi(x) = 1$ if $|x| \leq 1/2$ and $\chi(x) = 0$ if $|x| \geq 1$. Set $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon}), \varepsilon > 0$. Clearly, supp $f = \{0\}$ implies $\langle f, \varphi \rangle = \langle f, \chi_\varepsilon \varphi \rangle$ for any $\varepsilon > 0$. We will see below that $j^l_0(\varphi) = 0$ implies

\[ \sup_{x \in \mathbb{R}^n} |\partial^\alpha [\chi_\varepsilon(x) \varphi(x)]| \to 0 \quad \text{for } |\alpha| \leq l \text{ as } \varepsilon \to 0+ . \]

Therefore

\[ \langle f, \varphi \rangle = \langle f, \chi_\varepsilon \varphi \rangle = \lim_{\varepsilon \to 0+} \langle f, \chi_\varepsilon \varphi \rangle = 0 \]

due to (4.24). It remains to prove (4.26). But by the Leibniz rule

\[ \partial^\alpha [\chi_\varepsilon(x) \varphi(x)] = \sum_{\alpha' + \alpha'' = \alpha} c_{\alpha' \alpha''} [\partial^{\alpha'} \chi_\varepsilon(x)] [\partial^{\alpha''} \varphi(x)] = \]

\[ = \sum_{\alpha' + \alpha'' = \alpha} c_{\alpha' \alpha''} \varepsilon^{-|\alpha'|} \left[ (\partial^{\alpha'} \chi) \left( \frac{x}{\varepsilon} \right) \right] [\partial^{\alpha''} \varphi(x)]. \]

It follows from Taylor’s formula that

\[ \partial^{\alpha''} \varphi(x) = O(|x|^{1+1-|\alpha''|}) \text{ as } x \to 0. \]
Note that $|x| \leq \varepsilon$ on $\text{supp} \chi(\frac{x}{\varepsilon})$. Therefore, as $\varepsilon \to 0^+$,

$$\sup_{x \in \mathbb{R}^n} \varepsilon^{-|\alpha'|} \left| \left( \partial^{\alpha'} \chi \left( \frac{x}{\varepsilon} \right) \right) \left| \partial^{\alpha''} \varphi(x) \right| = O(\varepsilon^{-|\alpha'|+l+1-|\alpha''|}) = O(\varepsilon^{l+1-|\alpha|}) \to 0,$$

implying (4.26).

**4.5. Differentiation of distributions and multiplication by a smooth function**

Differentiation of distributions should be a natural extension of differentiation of smooth functions. To define it note that if $u \in C^1(\Omega)$ and $\varphi \in D(\Omega)$, then

$$\int \frac{\partial u}{\partial x_j} \varphi dx = - \int u \frac{\partial \varphi}{\partial x_j} dx \quad \text{(4.27)}$$

or

$$\left\langle \frac{\partial u}{\partial x_j}, \varphi \right\rangle = - \left\langle u, \frac{\partial \varphi}{\partial x_j} \right\rangle. \quad \text{(4.28)}$$

Therefore, it is natural to define $\partial_{x_j}u$ for $u \in \mathcal{D}'(\Omega)$ by the formula (4.28) (which we will sometimes write in the form (4.27) in accordance with the above-mentioned convention). Namely, define the value of the functional $\frac{\partial u}{\partial x_j}$ at $\varphi \in \mathcal{D}(\Omega)$ (i.e., the left-hand side of (4.27)) as the right-hand side which makes sense because $\frac{\partial \varphi}{\partial x_j} \in \mathcal{D}(\Omega)$. Since $\frac{\partial}{\partial x_j}$ continuously maps $\mathcal{D}(K)$ to $\mathcal{D}(K)$ for any compact $K \subset \Omega$, we see that $\frac{\partial u}{\partial x_j} \in \mathcal{D}'(\Omega)$. We can now define higher derivatives

$$\partial^\alpha : \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, as products of operators $\frac{\partial}{\partial x_j}$, namely, by setting $\partial^\alpha = (\frac{\partial}{\partial x_j})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Another (equivalent) way to do this is to write

$$\left\langle \partial^\alpha u, \varphi \right\rangle = (-1)^{|\alpha|} \left\langle u, \partial^\alpha \varphi \right\rangle, \quad \text{(4.29)}$$

which enables us to immediately compute derivatives of an arbitrary order. The successive application of (4.28) shows that the result obtained from (4.29) is equal to that obtained by applying $\partial^\alpha$ considered as the composition of operators $\frac{\partial}{\partial x_j}$. Since $\partial^\alpha$ continuously maps $\mathcal{E}(\Omega)$ to $\mathcal{E}(\Omega)$ and $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$, it is clear that $\partial^\alpha$ maps $\mathcal{E}'(\Omega)$ to $\mathcal{E}'(\Omega)$ and $S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.

It is also clear from (4.29) that $\partial^\alpha$ is a continuous operator in $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$ and $S'(\mathbb{R}^n)$. Therefore, we may say that $\partial^\alpha$ is the extension by continuity of the usual differentiation onto the space of distributions. Such an
extension is unique, because, as we will see in the next section, $D(\Omega)$ is dense in $D'(\Omega)$.

Finally, it is clear that $\text{supp}(\partial^\alpha u) \subset \text{supp} u$, for all $u \in D'(\Omega)$.

**Example 4.8.** The derivative of the Heaviside function $\theta(x)$. The distributional derivative of $\theta(x)$ can be calculated as follows:

$$\langle \theta', \varphi \rangle = -\langle \theta, \varphi' \rangle = -\int_0^\infty \varphi'(x)dx = \varphi(0), \quad \varphi \in D(\mathbb{R}^1),$$

i.e.,

$$\theta'(x) = \delta(x).$$

**Example 4.9.** The derivative $\partial^\alpha \delta(x)$ of $\delta(x) \in D'(\mathbb{R}^n)$. By definition,

$$\langle \partial^\alpha \delta(x), \varphi(x) \rangle = (-1)^{|\alpha|} \langle \delta(x), \partial^\alpha \varphi(x) \rangle = (-1)^{|\alpha|} \varphi(\alpha)(0).$$

Therefore, $\partial^\alpha \delta(x)$ assigns the number $(-1)^{|\alpha|} \varphi(\alpha)(0)$ to the function $\varphi(x)$. Theorem 4.7 may now be formulated as follows: any distribution supported at 0 is a finite linear combination of derivatives of the $\delta$-function.

**Example 4.10.** The fundamental solution of the Laplace operator. Consider the Laplace operator $\Delta$ in $\mathbb{R}^n$ and try to find a distribution $u \in D'(\mathbb{R}^n)$ such that

$$\Delta u = \delta(x).$$

Such $u$ is called a fundamental solution for the operator $\Delta$. It is not unique: we can add to it any distributional solution of the Laplace equation $\Delta v = 0$ (in this case $v$ will be actually a $C^\infty$ and even real-analytic function, as we will see later). Fundamental solutions play an important role in what follows. We will try to select a special (and the most important) fundamental solution, using symmetry considerations. (It will be later referred to as the fundamental solution.) The operator $\Delta$ commutes with rotations of $\mathbb{R}^n$ (or, equivalently, preserves its form under an orthogonal transformation of coordinates). Therefore, by rotating a solution $u$ we again get a solution of the same equation (it is easy to justify the meaning of the process, but, anyway, we now reason heuristically). But then we may average with respect to all rotations. It is also natural to suppose that $u$ has no singularities at $x \neq 0$. Therefore, we will seek a distribution $u(x)$ of the form $f(r)$, where $r = |x|$, for $x \neq 0$. Let us calculate $\Delta f(r)$. We have

$$\frac{\partial}{\partial x_j} f(r) = f'(r) \frac{\partial r}{\partial x_j} = f'(r) \frac{\partial}{\partial x_j} \left( \sqrt{x_1^2 + \ldots + x_n^2} \right) = f'(r) \frac{x_j}{r}$$
implying
\[ \frac{\partial^2}{\partial x_j^2} f(r) = f''(r) \frac{x_j^2}{r^2} + f'(r) \left[ \frac{1}{r^2} - \frac{x_j^2}{r^3} \right]. \]

Summing over \( j = 1, \ldots, n \) we get
\[ \Delta f(r) = f''(r) + \frac{n-1}{r} f'(r). \]

Let us solve the equation
\[ f''(r) + \frac{n-1}{r} f'(r) = 0. \]

Denoting \( f'(r) = g(r) \), we get for \( g(r) \) a separable ODE
\[ g'(r) + \frac{n-1}{r} g(r) = 0, \]

which yields \( g(r) = Cr^{-(n-1)} \), where \( C \) is a constant. Integrating, we get
\[
\begin{cases} 
  f(r) = C_1 r^{-(n-2)} + C_2 & \text{for } n \neq 2; \\
  f(r) = C_1 \ln r + C_2 & \text{for } n = 2.
\end{cases}
\]

Since constants \( C_2 \) are solutions of the homogeneous equation \( \Delta u = 0 \), we may take
\[
\begin{align*}
  f(r) &= Cr^{2-n} & \text{for } n \neq 2, \\
  f(r) &= C \ln r & \text{for } n = 2.
\end{align*}
\]

We will only consider the case \( n \geq 2 \) (the case \( n = 1 \) is elementary), and will be later analyzed in a far more general form). For convenience of the future calculations introduce
\[
\begin{align*}
  (4.30) & \quad u_n(x) = \frac{1}{2-n} r^{2-n} & \text{for } n \geq 3, \\
  (4.31) & \quad u_2(x) = \ln r.
\end{align*}
\]

Then \( u_n(x) \) is locally integrable in \( \mathbb{R}^n \), and hence may be considered as a distribution. The factor \( \frac{1}{2-n} \) is introduced so that \( u_n \) satisfies \( \frac{\partial u_n}{\partial r} = r^{1-n} \) for all \( n \geq 2 \).

Now it is convenient to make a break in our exposition to include some integral formulas for the fundamental solutions (4.30), (4.31) and related functions.
Some Integral Formulas

We will use several formulas relating integrals over a bounded domain (open set) $\Omega$ with some integrals over its boundary. The first formula of this kind is the Fundamental Theorem of Calculus

$$\int_a^b f'(x)dx = f(b) - f(a), \quad f \in C^1([a,b]),$$

which is due to Newton and Leibniz. All other formulas of the described type (due to Gauss, Ostrogradsky, Green, Stokes and others) follow from this one, most often through integration by parts.

We will proceed through the following Green formula:

$$\int_\Omega (u \Delta v - v \Delta u) dx = \int_{\partial \Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,$$

where the notations and conditions will be explained now.

Notations and Assumptions

- $\Omega$ is a bounded domain (bounded open subset) in $\mathbb{R}^n$.
- The closure of $\Omega$ is the smallest closed set $\bar{\Omega}$ such that $\Omega \subset \bar{\Omega} \subset \mathbb{R}^n$.
- The boundary $\partial \Omega$ of $\Omega$ in $\mathbb{R}^n$ is $\bar{\Omega} \setminus \Omega$.

It seems natural that $\partial \Omega$ should be called the boundary of $\Omega$. However, this would allow too much freedom in the global structure of $\bar{\Omega}$ and $\partial \Omega$. To avoid these problems we will later introduce some additional “smoothness” restrictions.

- For the closure $\bar{\Omega}$ of $\Omega$ in $\mathbb{R}^n$ we have $\bar{\Omega} = \Omega \cup \partial \Omega$, but it is possible that $\Omega \cap \partial \Omega \neq \emptyset$. (For example, consider a non-empty closed straight line interval $J \subset \Omega$ and then replace $\Omega$ by $\Omega \setminus J$.)
- $dS$ is the area element on the boundary.
- Let $k$ be fixed, $k \in \{1, 2, \ldots\}$, or $k = \infty$. (Sometimes it is also convenient to use $k = 0$, which means that only continuity of the integrand is required.) The space $C^k(\Omega)$ is defined as the space of all functions $u \in C(\Omega)$ such that any derivative $\partial^a u$ with $|a| \leq k$ exists and is continuous in $\Omega$. The space $C^\infty(\Omega)$ can be defined as the intersection of all spaces $C^k(\Omega)$. This intersection is a Fréchet space.
- $\bar{n} = \bar{n}(x)$ is an outgoing unit vector field which is based at every point $x \in \partial \Omega$, tangent to $\mathbb{R}^n$ and orthogonal to $\partial \Omega$ at all points $x \in \partial \Omega$.
- Let us also assume that $\bar{n}(x)$ is continuous with respect to $x$. It is easy to see from the Implicit Function Theorem that locally the problem has
at most one solution. This means that the solution exists and is unique in a small neighborhood of any outgoing unit vector field which is orthogonal to \( \partial \Omega \) and \( \mathbb{R}^n \) (at the points \( x \) where a local continuous outgoing normal vector exists).

- Now we will formulate conditions which guarantee that the closure \( \bar{\Omega} \) of \( \Omega \) in \( \mathbb{R}^n \) and the boundary \( \partial \Omega \subset \mathbb{R}^n \) are sufficiently regular at a point \( \bar{x} \). For simplicity it suffices that \( \bar{\Omega} \) is a finite union of simplest cells such as formulated above.

**Definition 4.8** (Smoothness of the boundary). Assume that \( U \subset \mathbb{R}^n \) is open and bounded in \( \mathbb{R}^n \), \( k \in \{1, 2, \ldots \} \). We will say that the boundary \( \partial \Omega \) is \( C^k \) (or \( \partial \Omega \in C^\infty \)) if for every \( \bar{x} \in \partial \Omega \), there exist \( r > 0 \) and a \( C^k \) function \( h : \mathbb{R}^{n-1} \to \mathbb{R} \) such that after relabeling and reorienting the coordinates axes, if necessary, we have

\[
U \cap B(\bar{x}, r) = \{ x \in B(\bar{x}, r) \mid x_n > h(x_1, \ldots, x_{n-1}) \}.
\]

(See Fig. on page 626 in Evans.)

**The outward unit normal vector field**

**Definition 4.9.** (i) If \( \partial U \) is \( C^1 \), then along \( \partial U \) we can define the outward unit normal vector field

\[
\nu = (\nu_1, \ldots, \nu_n).
\]

This field is normal at any point \( \bar{x} \in \partial U \) if \( \nu(\bar{x}) = \nu = (\nu_1, \ldots, \nu_n) \).

(ii) Let \( u \in C^1(\partial \Omega) \). We define

\[
\frac{\partial u}{\partial \nu} = \nu \cdot Du,
\]

which is called the (outward) normal derivative of \( u \). (Here \( D = \partial/\partial x \).)

We will sometimes need to change coordinates near a point \( \bar{x} \in \partial \Omega \) so as to “flatten out” the boundary. To be more specific, fix \( \bar{x} \in \partial \Omega \), and choose \( r, h, \) etc. as above. Define then \( \Phi^1(x), \ldots, \Phi^n(x) \) by the formulas

\[
\begin{align*}
y_i &= x_i = \Phi^i(x) \\
y_n &= x_n - h(x_1, \ldots, x_{n-1}) = \Phi^n(x)
\end{align*}
\]

and write

\[
y = \Phi(x).
\]
Similarly, we define $\Psi^1(x), \ldots, \Psi^n(x)$ from the formulas

\[
\begin{align*}
y_i &= x_i = \Psi^i(x) \quad (i = 1, \ldots, n-1) \\
y_n &= x_n + h(x_1, \ldots, x_{n-1}) = \Psi^n(x),
\end{align*}
\]

and write

\[ y = \Psi(x). \]

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ “flattens out $\partial \Omega$” near $\bar{x}$. Observe also that $\det \Phi = \det \Psi = 1$.

**Remark 4.10.** In particular, considering the geometric features of the curves in the picture (4.33), [page 626 in EVANS] it is not difficult to conclude that the curves can not blow up in a finite time or in any bounded domain.

**Remark 4.11.** A perturbation of liquid in a picture like [EVANS] may contain waves, bulges, ripples, swirls and many other forms of the flow which can hardly be reproduced in computer pictures (especially for 3 or more space coordinates). This makes the problem of describing such flows extraordinary challenging.

▼ Let us discuss what kind of regularity we need to require for the boundary to deserve to be called “sufficiently regular”. Unfortunately, there is no easy answer to this question: the answer is dictated by the goal. More specifically, suppose that we need to make sense of the formula (4.32), so as to give sufficient conditions for it to hold. Obviously, it is reasonable to request that we have a continuous unit normal vector and the area element (at least almost everywhere). To make sense of the left hand side, it is natural to require that $u, v \in C^2(\bar{\Omega})$ and $\partial \Omega \in C^1$, which will guarantee that the integrals on the both sides make sense. In fact, if $u, v \in C^2(\bar{\Omega})$ and $\partial \Omega \in C^1$, then Green’s formula (4.32) can be indeed verified.

Additionally, if we need a $C^2$ change of variables, which moves the boundary (say, if we would like to straighten the $C^1$ boundary locally), then the second order derivatives of the equation of the boundary (i.e. of the function $h$ in Definition 4.8) will show up in the transformed operator $\Delta$ and will be out of control. Therefore, to guarantee the continuity of the coefficients of the transformed equation, it is natural to assume $\partial \Omega \in C^2$. ▲

**Other integral formulas**

Here we will establish more elementary integral formulas (Gauss, Green, Ostrogradsky, Stokes, \ldots). We will start with the deduction of Green’s
4.5. Differentiation and multiplication by a smooth function

formula (4.32), from simpler integral identities, because the intermediate formulas have at least an equal importance.

The divergence formula, alternatively called Gauss-Ostrogradsky formula, holds:

If $\partial \Omega \in C^1$ and $F$ is a $C^1$ vector field in a neighborhood of $\overline{\Omega}$ in $\mathbb{R}^n$, then

$$
\int_\Omega \text{div} F \, dx = \int_{\partial \Omega} F \cdot \vec{n} \, dS,
$$

where

$$
\text{div} F = \nabla \cdot F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.
$$

Here $F_1, \ldots, F_n$ are the coordinates of the vector $F$. (For the proof see, for example, Rudin [23, Theorem 10.51].)

For our purposes it suffices to have this formula for the bounded domains $\Omega$ with smooth boundary $\partial \Omega$ (say, class $C^k$, with $k \geq 1$), that is, the boundary which is a compact closed hypersurface (of class $C^k$) in $\mathbb{R}^n$. More generally, we can allow the domains with a piecewise $C^k$ boundary. But the analysis of “piecewise smooth” situations is very difficult compared with the “smooth” case. So we will not discuss the piecewise smooth objects in this book.

Let us return to the deduction of Green’s formula (4.32). Taking $F = u \nabla v$, we find for $u \in C^1(\overline{\Omega})$ and $v \in C^2(\overline{\Omega})$, we easily see that

$$
\text{div}(u \nabla v) = u \Delta v + \nabla u \cdot \nabla v,
$$

hence (4.34) takes the form

$$
\int_{\Omega} (u \Delta v + \nabla u \cdot \nabla v) \, dx = \int_{\partial \Omega} u \frac{\partial v}{\partial \vec{n}} \, dS.
$$

Assuming now that $u, v \in C^2(\overline{\Omega})$ and subtracting from this formula the one obtained by transposition $u$ and $v$, we obtain (4.32).

The Green formula (4.32) is also a particular case of the Stokes formula for general differential forms (see e.g. [23], Theorem 10.33):

$$
\int_{\Omega} d\omega = \int_{\partial \Omega} \omega,
$$

where $\omega$ is an $(n-1)$-form on $\Omega$, $\omega \in C^1$, and $d\omega$ is its exterior differential. Take the differential form

$$
\omega = \sum_{j=1}^{n} (-1)^{j-1} \left( v \frac{\partial v}{\partial x_j} - u \frac{\partial u}{\partial x_j} \right) dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n,
$$

Assuming now that $u, v \in C^2(\overline{\Omega})$ and subtracting from this formula the one obtained by transposition $u$ and $v$, we obtain (4.32).
where $\widehat{dx_j}$ means that $dx_j$ is omitted. Clearly,

$$d\omega = (u\Delta v - v\Delta u)dx_1 \wedge \ldots \wedge dx_n,$$

so the left-hand side of (4.36) coincides with the left-hand side of (4.32). The right-hand side of (4.36) in this case may be easily reduced to the right-hand side of (4.32) and we leave the corresponding details to the reader as an exercise. ▲

\[\n\]

Let us return to the calculation of $\Delta u_n$ in $\mathcal{D}'(\mathbb{R}^n)$. For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle \Delta u_n, \varphi \rangle = \langle u_n, \Delta \varphi \rangle = \int_{\mathbb{R}^n} u_n(x) \Delta \varphi(x) dx = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} u_n(x) \Delta \varphi(x) dx.$$  

We now use Green’s formula (4.32) with $u = u_n$, $v = \varphi$ and $\Omega = \{ x : \varepsilon \leq |x| \leq R \}$, where $R$ is so large that $\varphi(x) = 0$ for $|x| \geq R - 1$. Since $\Delta u_n(x) = 0$ for $x \neq 0$, we get

$$\int_{|x| \geq \varepsilon} u_n(x) \Delta \varphi(x) dx = \int_{|x| = \varepsilon} \left( u_n \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial u_n}{\partial n} \right) dS.$$  

Clearly,

$$\frac{\partial \varphi}{\partial n} = -\frac{\partial \varphi}{\partial r}, \quad \frac{\partial u_n}{\partial n} = -\frac{\partial u_n}{\partial r} = -r^{1-n},$$

implying

$$\int_{|x| \geq \varepsilon} u_n(x) \Delta \varphi(x) dx = -f_n(\varepsilon) \int_{|x| = \varepsilon} \frac{\partial \varphi}{\partial r} dS + \varepsilon^{1-n} \int_{|x| = \varepsilon} \varphi(x) dS,$$

where $f_n(\varepsilon) = \frac{1}{2-n} \varepsilon^{2-n}$ for $n \geq 3$ and $f_n(\varepsilon) = \ln \varepsilon$ for $n = 2$. Since the area of a sphere of radius $\varepsilon$ in $\mathbb{R}^n$ is equal to $\sigma_{n-1} \varepsilon^{n-1}$ (here $\sigma_{n-1}$ is the area of a sphere of radius 1), the first summand in the right hand side of (4.37) tends to 0 as $\varepsilon \to 0^+$, and the second summand tends to $\sigma_{n-1} \varphi(0)$. Finally, we obtain

$$\langle \Delta u_n, \varphi \rangle = \sigma_{n-1} \varphi(0),$$

or

$$\Delta u_n = \sigma_{n-1} \delta(x).$$

Now, setting

$$\mathcal{E}_n(x) = \frac{1}{(2-n)\sigma_{n-1}} r^{2-n}, \quad \text{for } n \geq 3,$$

(4.39)

$$\mathcal{E}_2(x) = \frac{1}{2\pi} \ln r,$$

we clearly get

$$\Delta \mathcal{E}_n(x) = \delta(x),$$

(4.38)
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i.e., $\mathcal{E}_n(x)$ is a fundamental solution of the Laplace operator in $\mathbb{R}^n$. It will be later referred to as the fundamental solution of the Laplace operator. It will be clear later that for $n \geq 3$ it is the only fundamental solution such that it is a function of $r = |x|$ and, besides, tends to 0 as $|x| \to \infty$. For $n = 2$ there is no obvious way to choose the additive constant.

**Remark 4.12.** The area $\sigma_{n-1}$ of the unit sphere in $\mathbb{R}^n$. Let us compute $\sigma_{n-1}$. To do this consider Gaussian integrals

\[
I_1 = \int_{-\infty}^{\infty} e^{-x^2} dx,
\]

\[
I_n = \int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_{\mathbb{R}^n} e^{-(x_1^2 + \ldots + x_n^2)} dx_1 \ldots dx_n = I_1^n.
\]

Expressing $I_n$ in polar coordinates, we get

\[
I_n = \int_{0}^{\infty} e^{-r^2} \sigma_{n-1} r^{n-1} dr.
\]

Setting $r^2 = t$, we get

\[
I_n = \sigma_{n-1} \int_{0}^{\infty} e^{-t} t^{\frac{n-1}{2}} d\sqrt{t} = \sigma_{n-1} \cdot \frac{1}{2} \int_{0}^{\infty} t^{\frac{n-1}{2}} e^{-t} dt = \sigma_{n-1} \cdot \frac{1}{2} \Gamma\left(\frac{n}{2}\right),
\]

where $\Gamma$ is the Euler Gamma-function. This implies $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$. At the same time, for $n = 2$, we have

\[
I_2 = 2\pi \int_{0}^{\infty} e^{-r^2} rdr = -\pi e^{-r^2}\big|_{0}^{+\infty} = \pi,
\]

therefore $I_1 = \sqrt{\pi}$ and $I_n = \pi^{n/2}$. Thus,

\[
(4.40) \quad \sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.
\]

Note that this formula can also be rewritten without $\Gamma$-function, since the numbers $\Gamma\left(\frac{n}{2}\right)$ are easy to compute. Indeed, the functional equation

\[
\Gamma(s + 1) = s\Gamma(s)
\]

enables us to express $\Gamma\left(\frac{n}{2}\right)$ in terms of $\Gamma(1)$ for even $n$ and in terms of $\Gamma\left(\frac{1}{2}\right)$ for odd $n$. But for $n = 1$ we deduce from (4.40) that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. (Clearly, $\sigma_0 = 2$. However, we could also use the formula $\sigma_2 = 4\pi$ taking $n = 3$ and using the relation $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$.)

For $n = 2$, we deduce from (4.40) that $\Gamma(1) = 1$, which is also easy to verify without this formula. Therefore, for $n = 2k + 2$ we get

\[
\sigma_{2k+1} = \frac{2\pi^{k+1}}{k!},
\]
and for \( n = 2k + 1 \) we get
\[
\sigma_{2k} = \frac{2 \cdot (2\pi)^k}{(2k-1)!!}, \quad \text{where} \quad (2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1).
\]

**Remark 4.13.** A physical meaning of the fundamental solution of the Laplace operator. For an appropriate choice of units, the potential \( u(x) \) of the electrostatic field of a system of charges distributed with density \( \rho(x) \) in \( \mathbb{R}^3 \) satisfies the Poisson equation
\[
\Delta u = \rho. \tag{4.41}
\]
In particular, for a point charge at zero we have \( \rho(x) = \delta(x) \); hence, \( u(x) \) is a fundamental solution for \( \Delta \). Only potentials decaying at infinity have physical meaning. In what follows, we will prove Liouville’s theorem, which implies in particular that a solution \( u(x) \) of the Laplace equation \( \Delta u = 0 \) defined on \( \mathbb{R}^n \) tending to 0 as \( |x| \to +\infty \), is identically zero. Therefore, there is a unique fundamental solution tending to 0 as \( |x| \to +\infty \), namely \( \mathcal{E}_3(x) = -\frac{1}{4\pi r} \). It defines the potential of the point unit charge situated at 0. Besides, the potential of an arbitrary distribution of charges should, clearly, by the superposition principle, be determined by the formula
\[
u(x) = \int \mathcal{E}_3(x-y)\rho(y)dy. \tag{4.42}
\]
Applying formally the Laplace operator, we get
\[
\Delta u(x) = \int \delta(x-y)\rho(y)dy = \rho(x),
\]
i.e., Poisson’s equation (4.41) holds. These arguments may serve to deduce Poisson’s equation if, from the very beginning, the potential is defined by (4.42). Its justification may be obtained with the help of the convolution of distributions introduced below.

Let us indicate a meaning of the fundamental solution \( \mathcal{E}_2(x) \) in \( \mathbb{R}^2 \). In \( \mathbb{R}^3 \) consider an infinite uniformly charged string (with the constant linear density of the charge equal to 1) placed along \( x_3 \)-axis. From symmetry considerations, it is clear that its potential \( u(x) \) does not depend on \( x_3 \) and only depends on \( r = \sqrt{x_1^2 + x_2^2} \). Let \( x = (x_1, x_2) \). Poisson’s equation for \( u \) takes the form \( \Delta u(x) = \delta(x) \) implying \( u(x) = \frac{1}{2\pi} \ln r + C \). Therefore, in this case the potential is of the form \( \mathcal{E}_2(x_1, x_2) + C \).

Note, however, that the potential of the string is impossible to compute with the help of (4.42) which, in this case, is identically equal to +\( \infty \). What is the meaning of this potential? This is easy to understand if we recall that
4.5. Differentiation and multiplication by a smooth function

the observable physical quantity is not the potential but the electrostatic field

\[ \mathbf{E} = -\nabla u. \]

This field can be computed via the Coulomb inverse square law, and the integral that defines it converges (at the points outside the string). The potential \( u \), which can be recovered from \( \mathbf{E} \) up to an additive constant, equals \( \mathcal{E}_2(x_1, x_2) + C \).

This potential can be defined by another method when we first assume that the string is of finite length 2\( l \) and subtract from the potential of finite string a constant depending on \( l \) (this does not affect \( \mathbf{E} \! ) and then pass to the limit as \( l \to +\infty \). It is easy to see that it is possible to choose the constants in such a way that the limit of the potentials of the finite strings really exists. Then this limit must be equal to \( \mathcal{E}_2(x, y) + C \) by the above reasoning. We actually subtract an infinite constant from the potential of the form (4.42), but this does not affect \( \mathbf{E} \). This procedure is called in physics renormalization of charge and has its analogues in quantum electrodynamics.

Let us prove the possibility of renormalizing the charge. Let us write the potential of a segment of the string \( x_3 \in [-l, l] \):

\[ u_l(x_1, x_2, x_3) = -\frac{1}{4\pi} \int_{-l}^{l} \frac{dt}{\sqrt{x_1^2 + x_2^2 + (t - x_3)^2}}. \]

Since \( \sqrt{x_1^2 + x_2^2 + (t - x_3)^2} \sim |t| \) as \( t \to \infty \), it is natural to consider instead of \( u_l \) the function

\[ v_l(x_1, x_2, x_3) = \frac{1}{4\pi} \int_{-l}^{l} \left[ \frac{1}{\sqrt{x_1^2 + x_2^2 + (t - x_3)^2}} - \frac{1}{\sqrt{1 + t^2}} \right] dt, \]

that differs from \( u_l \) by a constant depending on \( l \):

\[ v_l = u_l + \frac{1}{4\pi} \int_{-l}^{l} \frac{dt}{\sqrt{1 + t^2}}. \]

The integrand in the formula for \( v_l \) is

\[ \frac{1}{\sqrt{x_1^2 + x_2^2 + (t - x_3)^2}} - \frac{1}{\sqrt{1 + t^2}} = \frac{\sqrt{1 + t^2} - \sqrt{x_1^2 + x_2^2 + (t - x_3)^2}}{\sqrt{x_1^2 + x_2^2 + (t - x_3)^2} \cdot \sqrt{1 + t^2}} = \frac{1 + 2tx_3 - x_1^2 - x_2^2 - x_3^2}{\sqrt{x_1^2 + x_2^2 + (t - x_3)^2} \cdot \sqrt{1 + t^2}} \cdot \sqrt{x_1^2 + x_2^2 + (t - x_3)^2 + \sqrt{1 + t^2}}, \]

which is asymptotically \( O \left( \frac{1}{l^2} \right) \) as \( t \to +\infty \), so that there exists a limit of the integral as \( l \to +\infty \). Clearly, the integral itself and its limit may
be computed explicitly but they are of no importance to us, since we can compute the limit up to a constant by another method.

- **Multiplication by a smooth function.**

This operation is introduced similarly to differentiation. Namely, if \( f \in L^1_{\text{loc}}(\Omega), a \in C^\infty(\Omega), \varphi \in \mathcal{D}(\Omega) \), then, clearly,

\[
\langle af, \varphi \rangle = \langle f, a\varphi \rangle.
\]

(4.43) \( \langle af, \varphi \rangle = \langle f, a\varphi \rangle \).

This formula may serve as a definition of \( af \) when \( f \in \mathcal{D}'(\Omega) \), \( a \in C^\infty(\Omega) \). It is easy to see that then we again get a distribution \( af \in \mathcal{D}'(\Omega) \). If \( f \in \mathcal{E}'(\Omega) \) then \( af \in \mathcal{E}'(\Omega) \) and \( \text{supp}(af) \subset \text{supp } f \).

Now let \( f \in S'(\mathbb{R}^n) \). Then we can use (4.43) for \( \varphi \in S(\mathbb{R}^n) \) only if \( a\varphi \in S(\mathbb{R}^n) \). Moreover, if we want \( af \in S'(\mathbb{R}^n) \), it is necessary that the operator mapping \( \varphi \) to \( a\varphi \) be a continuous operator from \( S(\mathbb{R}^n) \) to \( S(\mathbb{R}^n) \).

To this end, it suffices, e.g. that \( a \in C^\infty(\mathbb{R}^n) \) satisfies

\[
|\partial^\alpha a(x)| \leq C_\alpha (1 + |x|)^{m_\alpha}, \quad x \in \mathbb{R}^n,
\]

for every multiindex \( \alpha \) with some constants \( C_\alpha \) and \( m_\alpha \).

In particular, the multiplication by a polynomial maps \( S'(\mathbb{R}^n) \) into \( S'(\mathbb{R}^n) \).

A function \( f \in L^1_{\text{loc}}(\Omega) \) can be multiplied by any continuous function \( a(x) \). For a distribution \( f \), conditions on \( a(x) \) justifying the existence of \( af \) can also be weakened. Namely, let e.g. \( f \in \mathcal{D}'(\Omega) \) be such that for some integer \( m \geq 0 \) and every compact \( K \subset \Omega \) we have

\[
|\langle f, \varphi \rangle| \leq C_K \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha \varphi(x)|, \quad \varphi \in \mathcal{D}(K).
\]

(4.44) \(|\langle f, \varphi \rangle| \leq C_K \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha \varphi(x)|, \quad \varphi \in \mathcal{D}(K)|.

Then we say that \( f \) is a **distribution of finite order (not exceeding \( m \))**. Denote by \( \mathcal{D}_m(\Omega) \) the set of such distributions. In general case the estimate (4.44) only holds for a constant \( m \) depending on \( K \), and now we require \( m \) to be independent of \( K \). Clearly, if \( f \in \mathcal{E}'(\Omega) \), then the order of \( f \) is finite. If \( f \in L^1_{\text{loc}}(\Omega) \), then \( f \in \mathcal{D}'_0(\Omega) \).

If \( f \in \mathcal{D}'_m(\Omega) \), then we may extend \( f \) by continuity to a linear functional on the space \( \mathcal{D}_m(\Omega) \) consisting of \( C^m \)-functions with compact support in \( \Omega \). Clearly, \( \mathcal{D}_m(\Omega) = \bigcup_K \mathcal{D}_m(K) \), where \( K \) is a compact in \( \Omega \), \( \mathcal{D}_m(K) \) is the subset of \( \mathcal{D}_m(\Omega) \) consisting of functions with the support belonging to \( K \). Introducing in \( \mathcal{D}_m(K) \) the norm equal to the right-hand side of (4.44) without \( C_K \), we see that \( \mathcal{D}_m(K) \) becomes a Banach space, and \( f \in \mathcal{D}_m'(\Omega) \) can be extended to a continuous linear functional on \( \mathcal{D}_m(K) \). Therefore, it
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defines a linear functional on $D_m(\Omega)$. It is continuous in the sense that its restriction on $D_m(K)$ is continuous for any compact $K \subset \Omega$.

For instance, it is obvious that $\delta(x - x_0) \in D'_0(\Omega)$ for $x_0 \in \Omega$, so that $\delta$-function defines a linear functional on $D_0(\Omega)$. It actually defines a continuous linear functional on $C(\Omega)$.

In general, if $f \in D'_m(\Omega)$ and $\text{supp } f \subset K$, then $f$ can be extended to a continuous linear functional on $C^m(\Omega)$ if the topology in $C^m(\Omega)$ is defined by seminorms given by the right-hand side of (4.44) with any compact $K \subset \Omega$.

Now, let $f \in D'_m(\Omega)$ and $a \in C^m(\Omega)$. Then (4.43) makes sense for $\varphi \in D(\Omega)$, since then $a\varphi \in D_m(\Omega)$. Therefore, $af$ is defined for $f \in D'_m(\Omega)$ and $a \in C^m(\Omega)$.

**Example 4.11.** $a(x)\delta(x) = a(0)\delta(x)$ for $a \in C(\mathbb{R}^n)$.

**Example 4.12.** Let $n = 1$. Let us compute $a(x)\delta'(x)$ for $a \in C^\infty(\mathbb{R}^1)$. We have

$$\langle a\delta', \varphi \rangle = \langle \delta', a\varphi \rangle = -\langle \delta, (a\varphi)' \rangle = -a'(0)\varphi(0) - a(0)\varphi'(0)$$

implying

$$a(x)\delta'(x) = a(0)\delta'(x) - a'(0)\delta(x).$$

Note, in particular, that $a(0) = 0$ does not necessarily imply $a(x)\delta'(x) = 0$.

**Example 4.13.** *Leibniz rule.*

Let $f \in D'(\Omega), a \in C^\infty(\Omega)$. Let us prove that

$$\left(\frac{\partial}{\partial x_j} (af) \right) = \frac{\partial a}{\partial x_j} \cdot f + a \frac{\partial f}{\partial x_j}. \tag{4.45}$$

Indeed, if $\varphi \in D(\Omega)$ then

$$\langle \frac{\partial}{\partial x_j} (af), \varphi \rangle = -\langle a f, \frac{\partial \varphi}{\partial x_j} \rangle = -\langle f, a \frac{\partial \varphi}{\partial x_j} \rangle =$$

$$= -\left\langle f, \frac{\partial}{\partial x_j} (a \varphi) \right\rangle + \left\langle f, \frac{\partial a}{\partial x_j} \varphi \right\rangle = \left\langle \frac{\partial f}{\partial x_j}, a \varphi \right\rangle + \left\langle f, \frac{\partial a}{\partial x_j} \varphi \right\rangle =$$

$$= \left\langle a \frac{\partial f}{\partial x_j} + \frac{\partial a}{\partial x_j} f, \varphi \right\rangle,$$

as required.

By (4.45) we may differentiate the product of a distribution by a smooth function as the product of the regular functions. It follows that the Leibniz rule for higher order derivatives also holds.

On $\mathbb{R}^1$ consider a differential operator

$$L = \frac{d^m}{dx^m} + a_{m-1} \frac{d^{m-1}}{dx^{m-1}} + \ldots + a_1 \frac{d}{dx} + a_0,$$

where the $a_j$ are constants. Let us find its fundamental solution $E(x)$, i.e., a solution of $Lu = \delta(x)$. Clearly, $E(x)$ is determined up to a solution of the homogeneous equation $Lu = 0$. Besides $L E(x) = 0$ should hold for $x \neq 0$.

Therefore, it is natural to seek $E(x)$ in the form

$$E(x) = \begin{cases} y_1(x), & \text{for } x < 0 \\ y_2(x), & \text{for } x > 0, \end{cases}$$

where $y_1, y_2$ are solutions of $Ly = 0$. Moreover, subtracting $y_1(x)$, we may assume that

$$E(x) = \theta(x)y(x), \quad (4.46)$$

where $y(x)$ is a solution of $Ly = 0$. Now, let us find $L E(x)$. Clearly,

$$(\theta(x)y(x))' = y(0)\delta(x) + \theta(x)y'(x),$$

and the higher order differentiations give rise to derivatives of $\delta(x)$. If we wish these derivatives to vanish and the $\delta$-function to appear only at the very last step, we should assume

$$y(0) = y'(0) = \ldots = y^{(m-2)}(0) = 0, \quad y^{(m-1)}(0) = 1.$$ 

Such a solution $y(x)$ exists and is unique. For such a choice we get

$$\begin{align*}
(\theta(x)y(x))' &= \theta(x)y'(x) \\
(\theta(x)y(x))'' &= (\theta(x)y'(x))' = \theta(x)y''(x) \\
&\quad \ldots \\
(\theta(x)y(x))^{(m-1)} &= \theta(x)y^{(m-1)}(x) \\
(\theta(x)y(x))^{(m)} &= y^{(m-1)}(0)\delta(x) + \theta(x)y^{(m)}(x) = \delta(x) + \theta(x)y^{(m)}(x).
\end{align*}$$

Therefore,

$$L(\theta(x)y(x)) = \theta(x)Ly(x) + \delta(x) = \delta(x),$$

as required.

Why should the fundamental solution be a regular $C^\infty$-function for $x \neq 0$? This follows from the following
4.5. Differentiation and multiplication by a smooth function

Theorem 4.14. Let \( u \in \mathcal{D}'((a,b)) \) be a solution of the differential equation

\[
(4.47) \quad u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_0(x)u = f(x),
\]

where \( a_j(x), f(x) \) are \( C^\infty \)-functions on \((a,b)\). Then \( u \in C^\infty((a,b)) \).

Proof. Subtracting a smooth particular solution of (4.47) which exists, as is well-known, we see that everything is reduced to the case when \( f \equiv 0 \). Further, for \( f \equiv 0 \), equation (4.47) is easily reduced to the system of the form

\[
(4.48) \quad v' = A(x)v,
\]

where \( v \) is a vector-valued distribution (i.e. a vector whose components are distributions) and \( A(x) \) a matrix with entries from \( C^\infty((a,b)) \). Let \( \Psi(x) \) be an invertible matrix of class \( C^\infty \) satisfying

\[
\Psi'(x) = A(x)\Psi(x),
\]

i.e. \( \Psi(x) \) is a matrix whose columns constitute a fundamental system of solutions of (4.48). Set \( v = \Psi(x)w \), i.e. denote \( w = \Psi^{-1}(x)v \). Then \( w \) is a vector-valued distribution on \((a,b)\). Substituting \( v = \Psi(x)w \) in (4.48), we get \( w' = 0 \). Now, it remains to prove the following

Lemma 4.15. Let \( u \in \mathcal{D}'((a,b)) \) and \( u' = 0 \). Then \( u = \text{const} \).

Proof. The condition \( u' = 0 \) means that \( \langle u, \varphi' \rangle = 0 \) for any \( \varphi \in \mathcal{D}((a,b)) \).

But, clearly, \( \psi \in \mathcal{D}((a,b)) \) can be represented as \( \psi = \varphi' \), where \( \varphi \in \mathcal{D}((a,b)) \), if and only if

\[
(4.49) \quad \langle 1, \psi \rangle = \int_a^b \psi(x)dx = 0.
\]

Indeed, if \( \psi = \varphi' \) then (4.49) holds due to the main theorem of calculus. Conversely, if (4.49) holds, then we may set \( \varphi(x) = \int_a^x \psi(t)dt \) and, clearly, \( \varphi \in \mathcal{D}((a,b)) \) and \( \varphi' = \psi \).

Consider the map \( I : \mathcal{D}((a,b)) \to \mathbb{C} \) sending \( \psi \) to \( \langle 1, \psi \rangle \). Since \( \langle u, \psi \rangle = 0 \) for \( \psi \in \text{Ker}I \), it is clear that \( \langle u, \psi \rangle \) only depends on \( I\psi \); but since this dependence is linear, then \( \langle u, \psi \rangle = C \langle 1, \psi \rangle \), where \( C \) is a constant. But this means that \( u = C \).

\( \square \)
4.6. A general notion of the transposed (adjoint) operator.

Change of variables. Homogeneous distributions

The differentiation and multiplication by a smooth function are particular cases of the following general construction. Let $\Omega_1, \Omega_2$ be two domains in $\mathbb{R}^n$ and $L : C^\infty(\Omega_2) \longrightarrow C^\infty(\Omega_1)$ a linear operator. Let $t^L : \mathcal{D}(\Omega_1) \longrightarrow \mathcal{D}(\Omega_2)$ be a linear operator such that

$$
\langle Lf, \varphi \rangle = \langle f, t^L \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega_1),
$$

for $f \in C^\infty(\Omega_2)$. The operator $t^L$ is called the transposed or adjoint or dual of $L$. Suppose that, for any compact $K_1 \subset \Omega_1$, there exists a compact $K_2 \subset \Omega_2$ such that $t^L \mathcal{D}(K_1) \subset \mathcal{D}(K_2)$ and $t^L : \mathcal{D}(K_1) \longrightarrow \mathcal{D}(K_2)$ is a continuous map. Then formula (4.50) enables us to extend $L$ to a linear operator $L : \mathcal{D}'(\Omega_2) \longrightarrow \mathcal{D}'(\Omega_1)$.

If $\Omega_1 = \Omega_2 = \mathbb{R}^n$ and the operator $t^L$ can be extended to a continuous linear operator $t^L : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$, then it is clear that $L$ determines a linear map $L : S'(\mathbb{R}^n) \longrightarrow S'(\mathbb{R}^n)$.

Note that $L$ thus constructed is always weakly continuous. This is obvious from (4.50), since $|\langle Lf, \varphi \rangle|$ is a seminorm of the general form for $Lf$ and $|\langle f, t^L \varphi \rangle|$ is a seminorm for $f$.

Examples of transposed operators.

a) If $L = \frac{\partial}{\partial x_j}$ then $t^L = -\frac{\partial}{\partial x_j}$.

b) If $L = a(x)$ (the left multiplication by $a(x) \in C^\infty(\Omega)$), then $t^L = L = a(x)$.

In particular, the operators $\frac{\partial}{\partial x_j}$ and $a(x)$ are continuous on $\mathcal{D}'(\Omega)$ with respect to the weak topology. Moreover, $\frac{\partial}{\partial x_j}$ is continuous on $S'(\mathbb{R}^n)$.

c) A diffeomorphism $\chi : \Omega_1 \longrightarrow \Omega_2$ naturally defines a linear map $\chi^* : C^\infty(\Omega_2) \longrightarrow C^\infty(\Omega_1)$ by the formula

$$
(\chi^* f)(x) = f(\chi(x)).
$$

Clearly, the same formula defines a linear map $\chi^* : L^1_{\text{loc}}(\Omega_2) \longrightarrow L^1_{\text{loc}}(\Omega_1)$. Besides, $\chi^*$ transforms $\mathcal{D}(\Omega_2)$ to $\mathcal{D}(\Omega_1)$. We wish to extend $\chi^*$ to a continuous linear operator

$$
\chi^* : \mathcal{D}'(\Omega_2) \longrightarrow \mathcal{D}'(\Omega_1)
$$

To do so, we find the transposed operator. Let $\varphi \in \mathcal{D}(\Omega_1)$. Then
4.6. Transposed operator. Change of variables

\[ \langle \chi^* f, \varphi \rangle = \int_{\Omega_1} f(\chi(x))\varphi(x)dx = \int_{\Omega_2} f(z)\varphi(\chi^{-1}(z)) \left| \frac{\partial \chi^{-1}(z)}{\partial z} \right| dz = \left\langle f(z), \left| \frac{\partial \chi^{-1}(z)}{\partial z} \right| \cdot \varphi(\chi^{-1}(z)) \right\rangle, \]

where \( \chi^{-1} \) is the inverse of \( \chi \) and \( \frac{\partial \chi^{-1}(z)}{\partial z} \) its Jacobian.

It is clear now that the transposed operator is of the form

\[ (t\chi^* \varphi)(z) = \left| \frac{\partial \chi^{-1}(z)}{\partial z} \right| \cdot \varphi(\chi^{-1}(z)). \]

Since \( \left| \frac{\partial \chi^{-1}(z)}{\partial z} \right| \in C^\infty(\Omega_2) \), it is clear that \( t\chi^* \) maps \( \mathcal{D}(\Omega_1) \) to \( \mathcal{D}(\Omega_2) \) and defines a continuous linear map of \( \mathcal{D}(K_1) \) to \( \mathcal{D}(\chi(K_1)) \). Therefore, the general scheme determines a continuous linear map \( \chi^* : \mathcal{D}'(\Omega_2) \rightarrow \mathcal{D}'(\Omega_1). \)

If \( \Omega_1 = \Omega_2 = \mathbb{R}^n \) and \( \chi \) is an affine transformation (or at least coincides with an affine transformation outside a compact set), then \( t\chi^* \) continuously maps \( S(\mathbb{R}^n) \) to \( S(\mathbb{R}^n) \), and in this case \( \left| \frac{\partial \chi^{-1}(z)}{\partial z} \right| \neq 0 \) everywhere (it is constant everywhere if \( \chi \) is an affine map, and a constant near infinity in the general case). So a continuous linear map \( \chi^* : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \) is well defined.

**Examples.**

a) *Translation operator* \( f(x) \mapsto f(x-t) \) with \( t \in \mathbb{R}^n \) can be extended to a continuous linear operator from \( \mathcal{D}'(\Omega) \) to \( \mathcal{D}'(t+\Omega) \) and from \( \mathcal{S}'(\mathbb{R}^n) \) to \( \mathcal{S}'(\mathbb{R}^n) \). In particular, the notation \( \delta(x-x_0) \) used above agrees with the definition of the translation by \( x_0 \).

b) The *homothety* \( f(x) \mapsto f(tx) \), where \( t \in \mathbb{R} \setminus 0 \), is defined on \( \mathcal{D}'(\mathbb{R}^n) \) and maps \( \mathcal{D}'(\mathbb{R}^n) \) to \( \mathcal{D}'(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) to \( \mathcal{S}'(\mathbb{R}^n) \). For instance, let us calculate \( \delta(tx) \). We have

\[ \int \delta(tx) \varphi(x)dx = \int \delta(z) \varphi \left( \frac{z}{t} \right) |t|^{-n}dz = |t|^{-n} \varphi(0) = |t|^{-n} \int \delta(z) \varphi(z)dz. \]

Therefore,

\[ \delta(tx) = |t|^{-n} \delta(x), \quad t \in \mathbb{R} \setminus 0. \]

**Definition 4.16.** A distribution \( f \in \mathcal{S}'(\mathbb{R}^n) \) is called *homogeneous of order* \( m \in \mathbb{R} \) if

\[ f(tx) = t^m f(x), \quad t > 0. \]
It is easily checked that the requirement \( f \in S'(\mathbb{R}^n) \) can be actually replaced by \( f \in \mathcal{D}'(\mathbb{R}^n) \) (then (4.51) implies \( f \in S'(\mathbb{R}^n) \)).

Sometimes, one says “homogeneous of degree \( m \)” instead of “homogeneous of order \( m \)”.

**Examples.**

a) If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and for any \( t > 0 \), (4.51) holds almost everywhere, then \( f \) is a homogeneous distribution of order \( m \).

b) \( \delta \)-function \( \delta(x) \) is homogeneous of order \( -n \).

It is easy to verify that \( \frac{\partial}{\partial x_j} f(tx) = t \left( \frac{\partial f}{\partial x_j} \right) (tx) \). Therefore, if \( f \) is homogeneous of order \( m \), then \( \frac{\partial f}{\partial x_j} \) is homogeneous of order \( m - 1 \). For instance, \( \partial^\alpha \delta(x) \) is homogeneous of order \( -n - |\alpha| \).

Homogeneity considerations allow us to examine without computing the structure of the fundamental solution for the Laplace operator and its powers. Namely, the order of homogeneity of the distribution \( r^{2-n} \) for \( n \geq 3 \) is \( 2 - n \), and for \( r \neq 0 \) this distribution satisfies the equation \( \Delta u = 0 \). Therefore, \( \Delta r^{2-n} \) is supported at 0 and its order of homogeneity is \( -n \). But, since the order of homogeneity of \( \partial^\alpha \delta(x) \) is \( -n - |\alpha| \), it is clear that \( \Delta r^{2-n} = C \delta(x) \).

For \( n = 2 \), we have \( \Delta(\ln r) = 0 \) for \( r \neq 0 \), and \( \frac{\partial}{\partial x_j}(\ln r) \) is homogeneous of order \( -1 \). Therefore, \( \Delta(\ln r) \) is supported at 0 and its order of homogeneity is \( -2 \), so \( \Delta(\ln r) = C \delta(x) \).

It is not clear a priori why the following does not hold: \( \Delta r^{2-n} = 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \). Later on we will see, however, that if \( u \in \mathcal{D}'(\Omega) \) and \( \Delta u = 0 \), then \( u \in C^\infty(\Omega) \) (and, moreover, \( u \) is real analytic in \( \Omega \)). Since \( r^{2-n} \) has a singularity at 0, it follows that \( \Delta r^{2-n} \neq 0 \) implying \( \Delta r^{2-n} = C \delta(x) \), where \( C \neq 0 \). Similarly, for \( n = 2 \) we have \( \Delta(\ln r) = C \delta(x) \) for \( C \neq 0 \).

Finally, note that a rotation operator is defined in \( \mathcal{D}'(\mathbb{R}^n) \) and in \( S'(\mathbb{R}^n) \). This enables us to give a precise meaning to our arguments on averaging on which we based our conclusion about the existence of a spherically symmetric fundamental solution for \( \Delta \).

**Example 4.15.** **Fundamental solutions for powers of the Laplace operator.**

The spherical symmetry and homogeneity considerations make it clear that in order to find the fundamental solution of \( \Delta^m \) in \( \mathbb{R}^n \), it is natural to consider the function \( r^{2m-n} \). Then we get \( \Delta^{m-1} r^{2m-n} = C_1 r^{2-n} \). Can it
be $C_1 = 0$? It turns out that if $2m - n \not\in 2\mathbb{Z}_+$ (i.e., $2m - n$ is not an even nonnegative integer) or, equivalently, $r^{2m-n} \not\in C^\infty(\mathbb{R}^n)$, then $C_1 \neq 0$ and, therefore, $\Delta^m r^{2m-n} = C\delta(x)$, where $C \neq 0$. Indeed, for $r \neq 0$ we have, for any $\alpha \in \mathbb{R}$:

$$\Delta r^\alpha = \left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr}\right) r^\alpha = [\alpha(\alpha-1) + \alpha(n-1)] r^{\alpha-2} = \alpha(\alpha+n-2) r^{\alpha-2}.$$  

Therefore,

$$\Delta r^{2k-n} = (2k-n)(2k-2)r^{2k-n-2}$$

and consecutive applications of $\Delta$ to $r^{2m-n}$ yield a factor of the form $(2m-2)(2m-n)$, then $(2m-4)(2m-n-2)$, etc., implying $C_1 \neq 0$ for $2m-n \not\in 2\mathbb{Z}_+$.

Therefore, for $2m - n \not\in 2\mathbb{Z}_+$ the fundamental solution $\mathcal{E}_n^m(x)$ of $\Delta^m$ is of the form

$$\mathcal{E}_n^m(x) = C_{m,n} r^{2m-n}.$$  

Further, let $2m-n \in 2\mathbb{Z}_+$ so that $r^{2m-n}$ is a polynomial in $x$. Then consider the function $r^{2m-n} \ln r$. We get

$$\Delta^{m-1}(r^{2m-n} \ln r) = \begin{cases} C_1 \ln r & \text{for } n = 2 \\ C_2 r^{2-n} & \text{for } n \geq 4 \end{cases}$$

($C_2$ depends on $n$), where $C_1 \neq 0$ and $C_2 \neq 0$. Consider for the sake of definiteness the case $n \geq 4$. Since

$$\Delta^{m-1} f(r) = \left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr}\right)^{m-1} f(r)$$

and $\frac{d}{dr} \ln r = \frac{1}{r}$, it follows that

$$\Delta^{m-1}(r^{2m-n} \ln r) = \Delta^{m-1}(r^{2m-n} \ln r) \cdot \ln r + C_2 r^{2-n} = C_2 r^{2-n}$$

because $r^{2m-n}$ is a polynomial in $x$ of degree $2m - n$. Here $C_2 \neq 0$ since $r^{2m-n} \ln r \not\in C^\infty(\mathbb{R}^n)$ and the equation $\Delta u = 0$ has no solutions with singularities. (Indeed, if it had turned out that $C_2 = 0$, then for some $k$ we would have had that $u = \Delta^{k-1}(r^{2m-n} \ln r)$ had a singularity at $0$ and satisfied $\Delta u = 0$). Note though that $C_2$ may be determined directly. Therefore, for $2m-n \in 2\mathbb{Z}_+$,

$$\mathcal{E}_n^m(x) = C_{m,n} r^{2m-n} \ln r.$$
4.7. Appendix: Minkowski inequality

Let \((M, \mu)\) be a space with a positive measure. Here \(M\) is a set where a \(\sigma\)-algebra of subsets is chosen. (We will call a subset \(S \subset M\) measurable if it belongs to the \(\sigma\)-algebra.) The measure \(\mu\) is assumed to be defined on the \(\sigma\)-algebra and take values in \([0, +\infty]\). We will also assume that \(\mu\) is \(\sigma\)-finite, i.e. there exist measurable sets \(M_1 \subset M_2 \subset \ldots\), such that \(\mu(M_j)\) is finite for every \(j\), and \(M\) is the union of all \(M_j\)’s.

Then the function spaces \(L^p(M, \mu), 1 \leq p < +\infty\), are well defined, with the norm

\[
\|f\|_p = \left(\int_M |f(x)|^p \mu(dx)\right)^{1/p}, \quad 1 \leq p < +\infty.
\]

The Minkowski inequality in its simplest form is the triangle inequality for the norm \(\|\cdot\|_p\):

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad f, g \in L^p(M, \mu).
\]

By induction, this easily extends to several functions:

\[
\|f_1 + f_2 + \cdots + f_k\|_p \leq \|f_1\|_p + \|f_2\|_p + \cdots + \|f_k\|_p,
\]

where \(f_j \in L^p(M, \mu), j = 1, 2, \ldots, k\). Multiplying each \(f_j\) by \(a_j > 0\), we obtain

\[
(4.52) \quad \|a_1 f_1 + a_2 f_2 + \cdots + a_k f_k\|_p \leq a_1 \|f_1\|_p + a_2 \|f_2\|_p + \cdots + a_k \|f_k\|_p.
\]

A natural generalization of this inequality can be obtained if we replace summation by integration over a positive measure. More precisely, let \((N, \nu)\) be another space with a positive measure \(\nu\), and \(f = f(x, y)\) be a measurable function on \((M \times N, \mu \times \nu)\), where \(\mu \times \nu\) is the product measure defined by the property \((\mu \times \nu)(A \times B) = \mu(A)\nu(B)\) for any measurable sets \(A \subset M, B \subset N\) with finite \(\mu(A)\) and \(\nu(B)\). Then we have

**Theorem 4.17** (Minkowski’s inequality). Under the conditions above

\[
(4.53) \quad \left\| \int_N f(\cdot, y) \nu(dy) \right\|_p \leq \int_N \|f(\cdot, y)\|_p \nu(dy),
\]

or, equivalently,

\[
(4.54) \quad \left(\int_M \left(\int_N |f(x, y)|^p \nu(dy)\right)^{1/p} \mu(dx)\right)^{1/p} \leq \int_N \left(\int_M |f(x, y)|^p \mu(dx)\right)^{1/p} \nu(dy).
\]
The inequality (4.52) is a particular case of (4.53). To see this it suffices to choose \( N \) to be a finite set of \( k \) points, and take measures of these points to be \( a_1, \ldots, a_k \).

Vice versa, if we know that (4.52) holds, then (4.53) becomes heuristically obvious. For example, if we could present the integrals in (4.53) as limits of the corresponding Riemann sums, then (4.53) would follow from (4.52). But this is not so easy to do rigorously even if the functions are sufficiently regular. The easiest way to give a rigorous proof, is by a direct use of the following very important

**Theorem 4.18** (Hölder’s inequality). Let \( f \in L^p(M, \mu) \), \( g \in L^q(M, \mu) \), where \( 1 < p, q < +\infty \), and

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

Then \( fg \in L^1(M, \mu) \) and

\[
\left| \int_M f(x)g(x)\mu(dx) \right| \leq \|f\|_p\|g\|_q.
\]

**Proof.** Let us start with the following elementary inequality:

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q},
\]

where \( a \geq 0, b \geq 0 \), and \( p, q \) are as in Lemma above. It is enough to consider the case \( a > 0, b > 0 \). Consider the curve \( t = s^{p-1} \), \( s \geq 0 \), in the positive quadrant of the real plane \( \mathbb{R}^2_+ \), and two disjoint bounded domains \( G_1 \) and \( G_2 \), so that each of them is bounded by this curve together with the coordinate axes and straight line intervals as follows (see fig. ??): the boundary of \( G_1 \) consists of two straight line intervals \( \{(s,0) : 0 \leq s \leq a\} \), \( \{(a,t) : 0 \leq t \leq a^{p-1}\} \), and the curvilinear part \( \{(s,t) : t = s^{p-1}, 0 \leq s \leq a\} \); the boundary of \( G_2 \) also consists of two straight line intervals \( \{(0,t) : 0 \leq t \leq b\} \), \( \{(s,b) : 0 \leq s \leq b^{1/(p-1)}\} \), and generally different curvilinear piece of the same curve, namely, \( \{(s,t) : s = t^{1/(p-1)}, 0 \leq t \leq b\} \).

It is easy to see that the areas of \( G_1, G_2 \) are \( a^p/p \) and \( b^q/q \) respectively, and their union contains the rectangle \([0,a] \times [0,b]\) up to a set of measure 0. Therefore, (4.57) follows by comparison of the areas.

To prove (4.56) note first that replacing \( f, g \) by \(|f|, |g|\), we may assume that \( f \geq 0 \) and \( g \geq 0 \). Dividing both parts of (4.56) by \( \|f\|_p\|g\|_q \), we see that it suffices to prove it for normalized functions \( f, g \), i.e. such that
\[ \|f\|_p = \|g\|_q = 1. \]

To this end apply (4.57) with \( a = f(x), b = g(x) \) to get

\[ f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}. \]

Integrating this over \( M \) with respect to \( \mu \) and using (4.55), we come to the desired result. \( \square \)

**Proof of Theorem 4.17.** To prove (4.54) let us make some simplifying assumptions. Without loss of generality we can assume that \( f \) is positive (otherwise, replace it by \(|f|\), which does not change the right hand side, and can only increase the left hand side).

We can assume that both sides of (4.54) are finite. If the right hand side is infinite, then the inequality is obvious. If the left hand side is infinite, we can appropriately truncate \( f \) (using \( \sigma \)-finiteness of the measures), and then deduce the proof in the general case in the limit from a monotone approximation.

Introducing a function

\[ h(x) := \int_N f(x, y) \nu(dy), \]

which is measurable on \( M \) by Fubini’s theorem, we can rewrite the \( p \)th power of the left hand side of (4.54) as

\[ \int_M h(x)^p \mu(dx) = \int_M \left( \int_N f(x, y) \nu(dy) \right) h(x)^{p-1} \mu(dx) \]

\[ = \int_N \left( \int_M f(x, y) h(x)^{p-1} \mu(dx) \right) \nu(dy) \]

Now applying Hölder’s inequality (4.56) to the internal integral in the right hand side, and taking into account that \( q = p/(p - 1) \), we obtain

\[ \int_M f(x, y) h(x)^{p-1} \mu(dx) \leq \left( \int_M f(x, y)^p \mu(dx) \right)^{1/p} \left( \int_M h(x)^p \mu(dx) \right)^{(p-1)/p} \]

Using this to estimate the right hand side of the previous identity, we obtain

\[ \int_M h(x)^p \mu(dx) \leq \int_N \left( \int_M f(x, y)^p \mu(dx) \right)^{1/p} \nu(dy) \left( \int_M h(x)^p \mu(dx) \right)^{(p-1)/p} \]

Now dividing both sides by the last factor in the right hand side, we obtain the desired inequality (4.54). This division is possible if we assume that the last factor does not vanish. But in opposite case \( f = 0 \) almost everywhere and the statement is trivially true. \( \square \)
4.8. Appendix: Completeness of distribution spaces

Let $E$ be a Fréchet space with the topology given by seminorms $\{p_j | j = 1, 2, \ldots \}$, such that
\[
p_1(x) \leq p_2(x) \leq \ldots, \quad \text{for all } x \in E.
\]
Let $E'$ denote the dual space of all linear continuous functionals $f : E \to \mathbb{R}$.

Let us say that a set $F \subset E'$ is bounded if there exist an integer $j \geq 1$ and a positive constant $C$ such that
\[
|\langle f, \varphi \rangle| \leq C p_j(\varphi), \quad \text{for all } f \in F, \varphi \in E.
\]

Note that for $F$ consisting of just one element $f$, this is always true, being equivalent to the continuity of $f$ – see (4.2). So the requirement (4.58) means equicontinuity, or uniform continuity of all functionals $f \in F$.

Let us also say that $F$ is weakly bounded if for every $\varphi \in E$ there exists a positive $C_\varphi$ such that
\[
|\langle f, \varphi \rangle| \leq C_\varphi, \quad \text{for all } f \in F.
\]

The following theorem is a simplest version of the Banach-Steinhaus theorem, which is also called the uniform boundedness principle.

**Theorem 4.19.** If under the assumptions described above, $F$ is weakly bounded, then it is bounded.

Completeness of $E$ plays important role in the proof, which relies on the following

**Theorem 4.20.** (Baire’s Theorem) A non-empty complete metric space can not be presented as at most countable union of nowhere dense sets.

Here nowhere dense set $S$ in a metric space $M$ is a set whose closure does not have interior points. So the theorem can be equivalently reformulated as follows: a non-empty complete metric space can not be presented as at most countable union of closed sets without interior points.

**Proof of Theorem 4.20.** Let $M$ be a non-empty complete metric space with the metric $\rho$, $\{S_j | j = 1, 2, \ldots \}$ be a sequence of nowhere dense subsets of $M$. We will produce a point $x \in M$ which does not belong to any $S_j$.

Passing to the closures we can assume that $S_j$ is closed for every $j$. Now, clearly $S_1 \neq M$ because otherwise $S_1$ would be open in $M$, so it would not be nowhere dense. Hence there exists a closed ball $\bar{B}_1 = \bar{B}(x_1, r_1) =$
\{x \in M | \rho(x, x_1) \leq r_1\}, such that \(\bar{B}_1 \cap S_1 = \emptyset\). By induction we can construct a sequence of closed balls \(\bar{B}_j = \bar{B}(x_j, r_j), j = 1, 2, \ldots\), such that \(\bar{B}_{j+1} \subset \bar{B}_j, r_{j+1} \leq r_j/2\) and \(\bar{B}_j \cap S_j = \emptyset\) for all \(j\). It follows that \(B_j\) does not intersect with \(S_1 \cup S_2 \cup \cdots \cup S_j\). On the other hand, the sequence of centers \(\{x_j | j = 1, 2, \ldots\}\) is a Cauchy sequence because by the triangle inequality, for \(m < n\),

\[
\rho(x_m, x_n) \leq \rho(x_m, x_{m+1}) + \cdots + \rho(x_{n-1}, x_n) \leq r_m + r_{m+1} + \cdots \leq 2r_m \leq 2^{-j+2}r_1.
\]

Therefore, due to completeness of \(M\), \(x_m \to x\) in \(M\), and \(x\) belongs to all balls \(\bar{B}_j\). It follows that \(x \not\in S_j\), for all \(j\). \(\square\)

**Proof of Theorem 4.19.** Let us take for any \(j = 1, 2, \ldots\),

\[
S_j = \{\varphi | \varphi \in E, \sup_{f \in F} |\langle f, \varphi \rangle| \leq j\rho_j(\varphi)\}
\]

Then \(S_j \subset S_{j+1}\) for all \(j\), each set \(S_j\) is closed, and \(E\) is the union of all \(S_j, j = 1, 2, \ldots\). Therefore, by Baire’s Theorem, there exists \(j\) such that \(S_j\) has a non-empty interior. Taking a fundamental system of neighborhoods of any point \(\varphi_0 \in E\), consisting of sets

\[
B_j(\varphi_0, \varepsilon) = \{\varphi_0 + \varphi | p_j(\varphi) < \varepsilon\},
\]

where \(\varepsilon > 0, j = 1, 2, \ldots\), we see that we can find \(\varphi_0, j\) and \(\varepsilon\), such that \(B_j(\varphi_0, \varepsilon) \subset S_j\). Then for any \(f \in F\) and any \(\varphi \in B_j(0, \varepsilon)\)

\[
|\langle f, \varphi \rangle| \leq |\langle f, \varphi + \varphi_0 \rangle| + |\langle f, \varphi_0 \rangle| \leq j\rho_j(\varphi + \varphi_0) + C_1 \leq j\rho_j(\varphi) + C_2 \leq j\varepsilon + C_2,
\]

where \(C_1, C_2 > 0\) are independent of \(\varphi\) and \(f\) (but may depend on \(\varphi_0, j\) and \(\varepsilon\)). We proved this estimate for \(\varphi \in E\) such that \(p_j(\varphi) < \varepsilon\). By scaling we deduce that

\[
|\langle f, \varphi \rangle| \leq (j + \varepsilon^{-1}C_2)p_j(\varphi)
\]

for all \(f \in F\) and \(\varphi \in E\), which implies the desired result. \(\square\)

**Corollary 4.21** (Completeness of the dual space). Let \(E\) be a Fréchet space, \(\{f_k \in E^* | k = 1, 2, \ldots\}\) be a sequence of continuous linear functionals such that for every \(\varphi \in E\) there exists limit

\[
\langle f, \varphi \rangle := \lim_{k \to +\infty} \langle f_k, \varphi \rangle.
\]

Then the limit functional \(f\) is also continuous, i.e. \(f \in E^*\).
4.9. Problems

Proof. Clearly, the set \( F = \{ f_k \mid k = 1, 2, \ldots, \} \) is weakly bounded. Therefore, by Theorem 4.19 there exists a seminorm \( p_j \), which is one of the seminorms defining topology in \( E \), and a constant \( C > 0 \), such that

\[
|\langle f_k, \varphi \rangle| \leq C p_j(\varphi), \quad \varphi \in E, \; k = 1, 2, \ldots.
\]

Taking limit as \( k \to +\infty \), we obtain

\[
|\langle f, \varphi \rangle| \leq C p_j(\varphi), \; \varphi \in E,
\]

with the same \( C \) and \( j \). This implies continuity of \( f \). \( \square \)

Applying this corollary Theorem 4.19 to the distribution spaces \( \mathcal{D}'(\Omega) \), \( \mathcal{E}'(\Omega) \) and \( \mathcal{S}'(\mathbb{R}^n) \), we obtain their completeness. (In case of \( \mathcal{D}'(\Omega) \) we should apply the corollary to \( E = \mathcal{D}(K) \)).

4.9. Problems

4.1. Find \( \lim_{\varepsilon \to 0^+} \chi_\varepsilon(x) \) in \( \mathcal{D}'(\mathbb{R}^1) \), where \( \chi_\varepsilon(x) = 1/\sqrt{\varepsilon} \) for \( x \in (-\varepsilon, \varepsilon) \) and \( \chi_\varepsilon(x) = 0 \) for \( x \not\in (-\varepsilon, \varepsilon) \).

4.2. Find \( \lim_{\varepsilon \to 0^+} \psi_\varepsilon(x) \) in \( \mathcal{D}'(\mathbb{R}^1) \), where \( \psi_\varepsilon(x) = 1/\varepsilon \) for \( x \in (-\varepsilon, \varepsilon) \) and \( \psi_\varepsilon(x) = 0 \) for \( x \not\in (-\varepsilon, \varepsilon) \).

4.3. Find \( \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \sin \frac{x}{\varepsilon} \) in \( \mathcal{D}'(\mathbb{R}^1) \).

4.4. Prove that if \( f \in C^\infty(\mathbb{R}) \), then \( \frac{1}{\varepsilon}(f(x) - f(0)) \in C^\infty(\mathbb{R}) \).

4.5. Consider the canonical projection \( p : \mathbb{R}^1 \to S^1 = \mathbb{R}/2\pi\mathbb{Z} \). It induces an isomorphism \( p^* : C^\infty(S^1) \to C^\infty_{2\pi}(\mathbb{R}) \), where \( C^\infty_{2\pi}(\mathbb{R}) \) is the set of all 2\( \pi \)-periodic \( C^\infty \)-functions on \( \mathbb{R} \). Introduce a natural Fréchet topology in the space \( C^\infty(S^1) \) and define the set of distributions on \( S^1 \) as the dual space \( \mathcal{D}'(S^1) = (C^\infty(S^1))' \). Identify each function \( f \in L^1(S^1) \) (or, equivalently, locally integrable 2\( \pi \)-periodic function \( f \) on \( \mathbb{R} \)) with a distribution on \( S^1 \) by the formula

\[
\langle f, \varphi \rangle = \int_0^{2\pi} f(x)\varphi(x)dx.
\]

Prove that \( p^* \) can be extended by continuity to an isomorphism

\[
\mathcal{D}'(S^1) \to \mathcal{D}'_{2\pi}(\mathbb{R}),
\]

where \( \mathcal{D}'_{2\pi}(\mathbb{R}) \) is the set of all 2\( \pi \)-periodic distributions.
4.6. Prove that the trigonometric series
\[ \sum_{k \in \mathbb{Z}} f_k e^{ikx} \]
converges in \( \mathcal{D}'(\mathbb{R}^1) \) if and only if there exist constants \( C \) and \( N \) such that \( |f_k| \leq C(1 + |k|)^N \). Prove that if \( f \) is the sum of this series, then \( f \in \mathcal{D}'_2(\mathbb{R}) \) and the series above is in fact its Fourier series, that is
\[ f_k = \frac{1}{2\pi} \langle f, e^{-ikx} \rangle, \]
where the angle brackets denote the natural pairing between \( \mathcal{D}'(S^1) \) and \( C^\infty(S^1) \).

4.7. Prove that any \( 2\pi \)-periodic distribution (or, which is the same, distribution on \( S^1 \)) is equal to the sum of its Fourier series.

4.8. Find the Fourier series expansion of the \( \delta \)-function on \( S^1 \) or, equivalently, of the distribution
\[ \sum_{k=-\infty}^{\infty} \delta(x + 2k\pi) \in \mathcal{D}'_2(\mathbb{R}^1). \]

4.9. Use the result of the above problem to prove the following Poisson summation formula:
\[ \sum_{k=-\infty}^{\infty} f(2k\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k), \]
where \( f \in S(\mathbb{R}^1) \), and \( \hat{f} \) is the Fourier transform of \( f \).

4.10. Find all \( u \in \mathcal{D}'(\mathbb{R}) \), such that:
   a) \( xu(x) = 1 \);
   b) \( xu'(x) = 1 \);
   c) \( xu'(x) = \delta(x) \).

4.11. Verify that \( -\frac{e^{ikr}}{4\pi r} \), where \( r = |x| \), is the fundamental solution of the Helmholtz operator \( \Delta + k^2 \) in \( \mathbb{R}^3 \).
Chapter 5

Convolution and Fourier transform

5.1. Convolution and direct product of regular functions

The convolution of locally integrable functions $f$ and $g$ on $\mathbb{R}^n$ is the integral

$$ (f * g)(x) = \int f(x - y)g(y)dy, $$

over $\mathbb{R}^n$. Obviously, the integral (5.1) is not always defined. Let us consider it when $f, g \in C(\mathbb{R}^n)$ and one of these functions has a compact support (in this case (5.1) makes sense and, clearly, $f * g \in C(\mathbb{R}^n)$). The change of variables $x - y = z$ turns (5.1) into

$$ (f * g)(x) = \int f(z)g(x - z)dz, $$

i.e., the convolution is commutative:

$$ f * g = g * f. $$

Further, let $f \in C^1(\mathbb{R}^n)$. Then it is clear that (5.1) may be differentiated under the integral sign, and we get

$$ \frac{\partial}{\partial x_j} (f * g) = \frac{\partial f}{\partial x_j} * g. $$

In general, if $f \in C^m(\mathbb{R}^n)$, then

$$ \partial^\alpha (f * g) = (\partial^\alpha f) * g \quad \text{for } |\alpha| \leq m. $$
The convolution is associative, i.e.,

\[ f * (g * h) = (f * g) * h \]

if \( f, g, h \in C(\mathbb{R}^n) \) and two of the three functions have compact supports. This can be verified by the change of variables but a little later we will prove this in another way.

Let \( \varphi \in C(\mathbb{R}^n) \) and \( \varphi \) have a compact support. Then

\[ \langle f \ast g, \varphi \rangle = \int f(x - y)g(y)\varphi(x)dydx \]

may be transformed by the change \( x = y + z \) to the form

\[
(5.3) \quad \langle f \ast g, \varphi \rangle = \int f(z)g(y)\varphi(z + y)dydz.
\]

By Fubini’s theorem, this makes obvious the already proved commutativity of the convolution. Further, this also implies

\[ \langle (f \ast g) \ast h, \varphi \rangle = \int f(x)g(y)h(z)\varphi(x + y + z)dxdydz = \langle f \ast (g \ast h), \varphi \rangle, \]

proving the associativity of the convolution.

Note the important role played in (5.3) by the function \( f(x)g(y) \in C(\mathbb{R}^{2n}) \) called the direct or tensor product of functions \( f \) and \( g \) and denoted by \( f \otimes g \) or just \( f(x)g(y) \) if we wish to indicate the arguments.

Suppose that \( f \) and \( g \) have compact supports. Then in (5.3) we may set \( \varphi(x) = e^{-i\xi \cdot x} \), where \( \xi \in \mathbb{R}^n \). We get

\[
\tilde{f} \ast \tilde{g}(\xi) = \tilde{f}(\xi) \cdot \tilde{g}(\xi),
\]

where the tilde denotes the Fourier transform:

\[ \tilde{f}(\xi) = \int f(x)e^{-i\xi \cdot x}dx. \]

Therefore, the Fourier transform turns the convolution into the usual product of Fourier transforms.

Let us analyze the support of a convolution. Let \( A, B \subset \mathbb{R}^n \). Set

\[ A + B = \{x + y : x \in A, y \in B\} \]

(sometimes \( A + B \) is called the arithmetic sum of subsets \( A \) and \( B \)). Assuming that \( A \) is closed and \( B \) is compact in \( \mathbb{R}^n \), we easily see that \( A + B \) is also closed in \( \mathbb{R}^n \). Taking \( A = \text{supp } f, B = \text{supp } g \), we obtain

\[
(5.4) \quad \text{supp}(f \ast g) \subset \text{supp } f + \text{supp } g.
\]
Indeed, it follows from (5.3) that if supp $\varphi \cap (\text{supp } f + \text{supp } g) = \emptyset$, then $\langle f \ast g, \varphi \rangle = 0$ yielding (5.4).

5.2. Direct product of distributions

Let $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$, $f_1 \in \mathcal{D}'(\Omega_1)$, $f_2 \in \mathcal{D}'(\Omega_2)$. Let us define the direct or tensor product $f_1 \otimes f_2 \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ of distributions $f_1$ and $f_2$. Instead of $f_1 \otimes f_2$, we will often write also $f_1(x)f_2(y)$, where $x \in \Omega_1, y \in \Omega_2$.

Let $\varphi(x, y) \in \mathcal{D}(\Omega_1 \times \Omega_2)$. For $f_j \in L_{\text{loc}}^1(\Omega_j)$ the Fubini theorem yields

$$\langle f_1 \otimes f_2, \varphi \rangle = \langle f_1(x), \langle f_2(y), \varphi(x, y) \rangle \rangle = \langle f_2(y), \langle f_1(x), \varphi(x, y) \rangle \rangle. \tag{5.5}$$

This formula should be considered as a basis for definition of $f_1 \otimes f_2$ if $f_1$ and $f_2$ are distributions. Let us show that it makes sense. First, note that any compact $K$ in $\Omega_1 \times \Omega_2$ is contained in a compact of the form $K_1 \times K_2$, where $K_j$ is a compact in $\Omega_j, j = 1, 2$. In particular, supp $\varphi \subset K_1 \times K_2$ for some $K_1, K_2$. Further, $\varphi(x, y)$ can be now considered as a $C^\infty$-function of $x$ with values in $\mathcal{D}(K_2)$ (the $C^\infty$-function with values in $\mathcal{D}(K_2)$ means that all the derivatives are limits of their difference quotients in the topology of $\mathcal{D}(K_2)$). Therefore, $\langle f_2(y), \varphi(x, y) \rangle \in \mathcal{D}(K_1)$, since $f_2$ is a continuous linear functional on $\mathcal{D}(K_2)$. This implies that the functional

$$\langle F, \varphi \rangle = \langle f_1(x), \langle f_2(y), \varphi(x, y) \rangle \rangle$$

is well defined. It is easy to verify that $F \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and supp $F \subset \text{supp } f_1 \times \text{supp } f_2$. We can also construct the functional $G \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ by setting

$$\langle G, \varphi \rangle = \langle f_2(y), \langle f_1(x), \varphi(x, y) \rangle \rangle.$$

Let us verify that $F = G$. Note that if $\varphi = \varphi_1 \otimes \varphi_2$, where $\varphi_j \in \mathcal{D}(\Omega_j)$ i.e., $\varphi(x, y) = \varphi_1(x)\varphi_2(y)$, then, clearly,

$$\langle F, \varphi \rangle = \langle G, \varphi \rangle = \langle f_1, \varphi_1 \rangle \langle f_2, \varphi_2 \rangle.$$

Therefore, obviously, to prove $F = G$ it suffices to prove the following

Lemma 5.1. In $\mathcal{D}(\Omega_1 \times \Omega_2)$, finite linear combinations of the functions $\varphi_1 \otimes \varphi_2$, where $\varphi_j \in \mathcal{D}(\Omega_j)$ for $j = 1, 2$, are dense. More precisely, if $K_j$ is a compact in $\Omega_j, j = 1, 2$, and $\hat{K}_j$ is a compact in $\Omega_j$, such that $K_j$ is contained in the interior of $\hat{K}_j$, then any function $\varphi \in \mathcal{D}(K_1 \times K_2)$ can be approximated, as close as we like, in $\mathcal{D}(\hat{K}_1 \times \hat{K}_2)$ by finite sums of functions of the form $\varphi_1 \otimes \varphi_2$, where $\varphi_j \in \mathcal{D}(\hat{K}_j)$. 
Proof. Consider the cube $Q_d$ in $\mathbb{R}^{n_1+n_2}$:

$$Q_d = \left\{ (x, y) : |x_j| \leq \frac{d}{2}, |y_l| \leq \frac{d}{2}, j = 1, \ldots, n_1; l = 1, \ldots, n_2 \right\},$$

i.e., the cube with the center at the origin with edges of length $d$ parallel to the coordinate axes. Take $d$ large enough for the compact set $K_1 \times K_2$ to belong to the interior of $Q_d$. In $L^2(Q_d)$, the exponentials

$$\exp \left( \frac{2\pi i}{d} k \cdot z \right) : k \in \mathbb{Z}^n, n = n_1 + n_2; z \in \mathbb{R}^n$$

considered as functions of $z$, constitute a complete orthonormal system. A function $\varphi = \varphi(x, y) = \varphi(z)$ can be expanded into the Fourier series

$$\varphi(z) = \sum_k \varphi_k \exp \left( \frac{2\pi i}{d} k \cdot z \right),$$

where

$$\varphi_k = \frac{1}{d^n} \int_{Q_d} \varphi(z) \exp \left( -\frac{2\pi i}{d} k \cdot z \right) dz.$$

The Fourier series (5.6) converges absolutely and uniformly in $Q_d$ to $\varphi(z)$ and it can be term-wise differentiated any number of times. Indeed, integrating by parts shows that the absolute values of

$$(1 + |k|^2)^N \varphi_k = \frac{1}{d^n} \int_{Q_d} \varphi(z) \left( 1 - \frac{d^2}{4 \pi^2 \Delta_z} \right)^N \exp \left( -\frac{2\pi i}{d} k \cdot z \right) dz =$$

$$= \frac{1}{d^n} \int_{Q_d} \left[ \left( 1 - \frac{d^2}{4 \pi^2 \Delta_z} \right)^N \varphi(z) \right] \exp \left( -\frac{2\pi i}{d} k \cdot z \right) dz$$

are bounded by a constant $C_N$ independent of $k$, i.e.,

$$|\varphi_k| \leq C_N (1 + |k|^2)^{-N}.$$

This implies the convergence of (5.6) and the possibility of its termwise differentiation.

Therefore, (5.6) converges in the topology of $C^\infty(Q_d)$. Now, let $\chi_j \in \mathcal{D}(\hat{K}_j)$, where $\chi_j = 1$ in a neighborhood of $K_j$. We get

$$\varphi(x, y) = \chi_1(x) \chi_2(y) \varphi(x, y) =$$

$$= \sum_{k_1, k_2} \varphi_k \chi_1(x) \exp \left( \frac{2\pi i}{d} k_1 \cdot x \right) \chi_2(y) \exp \left( \frac{2\pi i}{d} k_2 \cdot y \right),$$

where $k_1 \in \mathbb{Z}^{n_1}$, $k_2 \in \mathbb{Z}^{n_2}$, $k = (k_1, k_2)$ and the series converges in the topology $\mathcal{D}(\hat{K}_1 \times \hat{K}_2)$. But then the partial sums of this series give the required approximation for $\varphi(x, y)$. \qed
Now we are able to introduce the following

**Definition 5.2.** If \( f_j \in \mathcal{D}'(\Omega_j), j = 1, 2 \), then the direct or tensor product of \( f_1 \) and \( f_2 \) is the distribution on \( \Omega_1 \times \Omega_2 \) defined via (5.5) and denoted by \( f_1 \otimes f_2 \) or \( f_1(x)f_2(y) \) or \( f_1(x) \otimes f_2(y) \).

Clearly,

\[
\text{supp}(f_1 \otimes f_2) \subset \text{supp} f_1 \times \text{supp} f_2,
\]
and if \( f_j \in \mathcal{S}'(\mathbb{R}^{n_j}) \) then \( f_1 \otimes f_2 \in \mathcal{S}'(\mathbb{R}^{n_1+n_2}) \). Further, it is easy to verify that

\[
\frac{\partial}{\partial x_j}(f_1 \otimes f_2) = \frac{\partial f_1}{\partial x_j} \otimes f_2, \quad \frac{\partial}{\partial y_l}(f_1 \otimes f_2) = f_1 \otimes \frac{\partial f_2}{\partial y_l}.
\]

**Example 5.1.** \( \delta(x)\delta(y) = \delta(x,y) \).

**Example 5.2.** Let \( t \in \mathbb{R}^1, x \in \mathbb{R}^{n-1}, \alpha(x) \in C(\mathbb{R}^{n-1}) \). The distribution \( \delta(t) \otimes \alpha(x) \in \mathcal{D}'(\mathbb{R}_{t,x}) \) is called the single layer with density \( \alpha \) on the plane \( t = 0 \). Its physical meaning is as follows: it describes the charge supported on the plane \( t = 0 \) and smeared over the plane with the density \( \alpha(x) \). Clearly,

\[
(5.7) \quad \langle \delta(t) \otimes \alpha(x), \varphi(t,x) \rangle = \int_{\mathbb{R}^{n-1}} \alpha(x)\varphi(0,x)dx.
\]

For \( \beta(x) \in C(\mathbb{R}^{n-1}) \), the distribution \( \delta'(t) \otimes \beta(x) \) is called the double layer with density \( \beta \) on the plane \( t = 0 \). Its physical meaning is as follows: it describes the distribution of dipoles smeared with the density \( \beta \) over the plane \( t = 0 \) and oriented along the \( t \)-axis.

Recall that a dipole in electrostatics is a system of two very large charges of the same absolute value and of opposite signs placed very close to each other. The product of charges by the distance between them should be a definite finite number called the dipole moment. Clearly, in \( \mathcal{D}'(\mathbb{R}^n) \), the following limit relation holds

\[
\delta'(t) \otimes \beta(x) = \lim_{\varepsilon \to +0} \left[ \frac{\delta(t+\varepsilon)}{2\varepsilon} \otimes \beta(x) - \frac{\delta(t-\varepsilon)}{2\varepsilon} \otimes \beta(x) \right],
\]

implying the discussed above interpretation of the double layer. It is also clear that

\[
(5.8) \quad \langle \delta'(t) \otimes \beta(x), \varphi(t,x) \rangle = -\int_{\mathbb{R}^{n-1}} \frac{\partial\varphi}{\partial t}(0,x)\beta(x)dx.
\]

The formulas similar to (5.7), (5.8) can be used to define the single and double layers on any smooth surface \( \Gamma \) of codimension 1 in \( \mathbb{R}^n \). Namely, if
\( \alpha, \beta \in C(\Gamma) \) then we can define the distributions \( \alpha \delta \Gamma \) and \( \frac{\partial}{\partial n}(\beta \delta \Gamma) \) by the formulas
\[
\langle \alpha \delta \Gamma, \varphi \rangle = \int_{\Gamma} \alpha(x) \varphi(x) dS,
\]
\[
\left\langle \frac{\partial}{\partial n}(\beta \delta \Gamma), \varphi \right\rangle = -\int_{\Gamma} \beta(x) \frac{\partial \varphi}{\partial n} dS,
\]
where \( dS \) is the area element of \( \Gamma \), \( \bar{n} \) being the external normal to \( \Gamma \). Note, however, that, locally, a diffeomorphism reduces these distributions to the above direct products \( \delta(t) \otimes \alpha(x) \) and \( \delta'(t) \otimes \beta(x) \).

Note another important property of the direct product: it is associative i.e., if \( f_j \in \mathcal{D}'(\Omega_j), j = 1, 2, 3, \) then
\[
(f_1 \otimes f_2) \otimes f_3 = f_1 \otimes (f_2 \otimes f_3) \text{ in } \mathcal{D}'(\Omega_1 \times \Omega_2 \times \Omega_3).
\]
The proof follows from the fact that finite linear combinations of functions of the form \( \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \) with \( \varphi_j \in \mathcal{D}(\Omega_j), j = 1, 2, 3, \) are dense in \( \mathcal{D}(\Omega_1 \times \Omega_2 \times \Omega_3) \) in the same sense as in Lemma 5.1.

5.3. Convolution of distributions

It is clear from (5.3) that it is natural to try to define the convolution of distributions \( f, g \in \mathcal{D}'(\mathbb{R}^n) \) with the help of the formula
\[
(5.9) \quad \langle f * g, \varphi \rangle = \langle f(x)g(y), \varphi(x + y) \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).
\]
This formula (and the convolution itself) does not always make sense, since \( \varphi(x + y) \not\in \mathcal{D}(\mathbb{R}^{2n}) \) for \( \varphi \not= 0 \). Nevertheless, sometimes the right-hand side of (5.9) can be naturally defined. Let for example \( f \in \mathcal{E}'(\mathbb{R}^n) \). Let us set
\[
(5.10) \quad \langle f(x)g(y), \varphi(x + y) \rangle = \langle f(x), \langle g(y), \varphi(x + y) \rangle \rangle,
\]
which makes sense since \( \langle g(y), \varphi(x + y) \rangle \in \mathcal{E}(\mathbb{R}^n_y) \). We may proceed the other way around, setting
\[
(5.11) \quad \langle f(x)g(y), \varphi(x + y) \rangle = \langle g(y), \langle f(x), \varphi(x + y) \rangle \rangle,
\]
which also makes sense, since \( \langle f(x), \varphi(x + y) \rangle \in \mathcal{D}(\mathbb{R}^n_y) \).

Let us show that both methods give the same answer. To this end, take a function \( \chi \in \mathcal{D}(\mathbb{R}^n) \) such that \( \chi(x) = 1 \) in a neighborhood of \( \text{supp } f \). Then \( f = \chi f \) and we have
\[
\langle f(x), \langle g(y), \varphi(x + y) \rangle \rangle = \langle \chi(x)f(x), \langle g(y), \varphi(x + y) \rangle \rangle = \\
= \langle f(x), \chi(x) \langle g(y), \varphi(x + y) \rangle \rangle = \langle f(x), \langle g(y), \chi(x)\varphi(x + y) \rangle \rangle,
\]
and, similarly,
\[ \langle g(y), \langle f(x), \varphi(x+y) \rangle \rangle = \langle g(y), \langle f(x), \chi(x)\varphi(x+y) \rangle \rangle. \]

The right-hand sides of these equations are equal, due to the property of
the direct product (5.5) proved above, since \( \chi(x)\varphi(x+y) \in \mathcal{D}(\mathbb{R}^{2n}) \). Therefore, their left-hand sides coincide. Now we may introduce the following definition.

**Definition.** The **convolution** of two distributions \( f, g \in \mathcal{D}'(\mathbb{R}^n) \) one of which
has a compact support is the distribution \( f * g \in \mathcal{D}'(\mathbb{R}^n) \) defined via (5.9)
understood in the sense of (5.10) or (5.11).

It follows from the arguments above that the convolution is commutative,
i.e.,
\[ f * g = g * f, \quad f \in \mathcal{E}'(\mathbb{R}^n), \quad g \in \mathcal{D}'(\mathbb{R}^n); \]
besides, it is associative, i.e.
\[ (f * g) * h = f * (g * h), \]
if two of the three distributions have compact support. This is easy to verify
from the associativity of the direct product and the equation
\[ \langle (f * g) * h, \varphi \rangle = \langle f(x)g(y)h(z), \varphi(x+y+z) \rangle, \]
because the same holds if \( (f * g) * h \) is replaced by \( f * (g * h) \).

The rule (5.2) of differentiating the convolution applies also to distribu-
tions. Indeed,
\[ \langle \partial^\alpha(f * g), \varphi \rangle = (-1)^{|\alpha|}\langle f * g, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|}\langle f(x)g(y), (\partial^\alpha \varphi)(x+y) \rangle, \]
but \( (\partial^\alpha \varphi)(x+y) \) can be represented in the form \( \partial^\alpha_x \varphi(x+y) \) or \( \partial^\alpha_y \varphi(x+y) \)
and then, moving \( \partial^\alpha \) backwards, we see that (5.2) holds:
\[ \partial^\alpha(f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g). \]

**Example 5.3.** \( \delta(x) \ast f(x) = f(x), \quad f \in \mathcal{D}'(\mathbb{R}^n). \)

From the algebraic viewpoint, we see that the addition and convolution
endow \( \mathcal{E}'(\mathbb{R}^n) \) with the structure of an associative and commutative algebra
(over \( \mathbb{C} \)) with the unit \( \delta(x) \), and \( \mathcal{D}'(\mathbb{R}^n) \) is a two-sided module over this algebra.

\[ \nabla \] Now we will introduce one of the key notions in partial differential equa-
tions (with constant coefficients): fundamental solution.
Definition 5.3. Let $p(D)$ be a linear differential operator with constant coefficients in $\mathbb{R}^n$. A distribution $\mathcal{E}(x) \in \mathcal{D}'(\mathbb{R}^n)$ is its fundamental solution if

$$p(D)\mathcal{E}(x) = \delta(x).$$

Example 5.4. A solution of nonhomogeneous equation with constant coefficients.

Proposition 5.4. Let $p(D)$ be a linear differential operator with constant coefficients in $\mathbb{R}^n$, $f \in \mathcal{E}'(\mathbb{R}^n)$. Then the equation

$$p(D)u = f$$

has a particular solution

$$u = \mathcal{E} * f.$$

Proof. Indeed,

$$p(D)(\mathcal{E} * f) = [p(D)\mathcal{E}] * f = \delta * f = f,$$

as required. \qed

Example 5.5. Using the above example we can find a particular solution of a nonhomogeneous ordinary differential equation with constant coefficients without variation of parameters. Indeed, let $n = 1$ and $p(D) = L$, where

$$L = \frac{d^m}{dx^m} + a_{m-1} \frac{d^{m-1}}{dx^{m-1}} + \ldots + a_1 \frac{d}{dx} + a_0,$$

and all coefficients $a_j$ are constants. In this case, we may take $\mathcal{E}(x) = \theta(x)y(x)$, where $y$ is the solution of the equation $Ly = 0$ satisfying the initial conditions

$$y(0) = y'(0) = \ldots = y^{(m-2)}(0) = 0, \quad y^{(m-1)}(0) = 1.$$

(see (4.46)). This implies that if $f \in C(\mathbb{R}^1)$ has a compact support, then

$$u_1(x) = \int \mathcal{E}(x - t)f(t)dt = \int_{-\infty}^{x} y(x - t)f(t)dt$$

is a particular solution of the equation $Lu = f$. Note that since $y(x - t)$ satisfies, as a function of $x$, the equation $Ly(x - t) = 0$ for any $t \in \mathbb{R}$, it follows that

$$u_2(x) = \int_{-\infty}^{0} y(x - t)f(t)dt$$

satisfies the equation $Lu_2 = 0$. Therefore, together with $u_1(x)$, the function

$$u(x) = \int_{0}^{x} y(x - t)f(t)dt$$

(5.12)
is also a particular solution of $Lu = f$. Formula (5.12) gives a solution of $Lu = f$ for an arbitrary $f \in C(R^1)$ (not necessarily with a compact support) since the equality $Lu = f$ at $x$ depends only on the behavior of $f$ in a neighborhood of $x$, and the solution $u$ itself given by this formula is determined in a neighborhood of $x$ by the values of $f$ on a finite segment only.

An example: a particular solution of the equation $u'' + u = f$ is given by the formula

$$u(x) = \int_0^x \sin(x - t) f(t) dt.$$ 

**Remark 5.5.** If the potential in the right hand side of an integral like the ones taken above, is multiplied by a constant (for example, changes sign), then a particular solution can be obtained by the same procedure.

**Example 5.6.** Potentials. If $\mathcal{E}_n$ is the fundamental solution of the Laplace operator $\Delta$ in $R^n$ defined by (4.38), (4.39), then $u = -\mathcal{E}_n * f$ with $f \in \mathcal{E}'(R^n)$ is called a potential and satisfies the Poisson equation $\Delta u = -f$.

In particular, if $f = \rho$, where $\rho$ is piecewise continuous, then for $n = 3$ we get the Newtonian potential

$$u(x) = \frac{1}{4\pi} \int \frac{\rho(y)}{|x - y|} dy,$$

and for $n = 2$, the logarithmic potential

$$u(x) = -\frac{1}{2\pi} \int \rho(y) \ln |x - y| dy.$$ 

In these examples, the convolution $\mathcal{E}_n * f$ is locally integrable for any $n$. As a distribution, it coincides with a locally integrable function

$$u(x) = -\int \mathcal{E}_n(x - y) \rho(y) dy,$$

since $\mathcal{E}_n(x - y) \rho(y)$ is locally integrable in $R^{2n}_{x,y}$ and, by the Fubini theorem, we can make the change of variables in the integral $\langle u, \varphi \rangle$ which leads us to the definition of the convolution of distributions.

Let us give other examples of potentials. The convolution

(5.13) 

$$u = -\mathcal{E}_n * \alpha \delta \Gamma,$$

where $\Gamma$ is a smooth compact surface of codimension 1 in $R^n$ and $\alpha \in C(\Gamma)$, is a simple layer potential and may be interpreted as a potential of a system of charges smeared over $\Gamma$ with the density $\alpha$. 
The convolution
\[ u = -\mathcal{E}_n * \frac{\partial}{\partial n} (\beta \delta_{\Gamma}) \]
is called a double layer potential and denotes the potential of a system of dipoles smeared over \( \Gamma \) and oriented along the exterior normal with the density of the dipole moment equal to \( \beta(x) \).

Note that outside \( \Gamma \) both potentials (5.13) and (5.14) are \( C^\infty \)-functions given by the formula
\[
\begin{align*}
  u(x) &= -\int_{\Gamma} \mathcal{E}_n(x-y) \alpha(y) dS_y, \\
  u(x) &= \int_{\partial \mathcal{E}_n(x-y)} \frac{\partial \mathcal{E}_n(x-y)}{\partial n_y} \beta(y) dS_y.
\end{align*}
\]
The behavior of the potentials near \( \Gamma \) is of interest, and we will later discuss it in detail.

5.4. Other properties of convolution. Support and singular support of a convolution

First, let us study the convolution of a smooth function and a distribution.

**Proposition 5.6.** Let \( f \in \mathcal{E}'(\mathbb{R}^n) \) and \( g \in C^\infty(\mathbb{R}^n) \), or \( f \in \mathcal{D}'(\mathbb{R}^n) \) and \( g \in C^\infty_0(\mathbb{R}^n) \). Then \( f \ast g \in C^\infty(\mathbb{R}^n) \) and
\[
(f \ast g)(x) = \langle f(y), g(x-y) \rangle = \langle f(x-y), g(y) \rangle
\]

**Proof.** First, observe that \( g(x-y) \) is an infinitely differentiable function of \( x \) with values in \( C^\infty(\mathbb{R}^n) \). If, moreover, \( g \in C^\infty_0(\mathbb{R}^n) \), then the support of \( g(x-y) \) as of a function of \( y \), belongs to a compact set if \( x \) runs over a compact set. Therefore, \( \langle f(y), g(x-y) \rangle \in C^\infty(\mathbb{R}^n) \). The equation
\[
\langle f(x-y), g(y) \rangle = \langle f(y), g(x-y) \rangle
\]
follows since, by definition, the change of variables in distributions is performed as if we deal with an ordinary integral. It remains to verify that if \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), then
\[
\langle f \ast g, \varphi \rangle = \langle \langle f(y), g(x-y) \rangle, \varphi(x) \rangle
\]
or
\[
\langle f(x), \langle g(y), \varphi(x+y) \rangle \rangle = \langle \langle f(y), g(x-y) \rangle, \varphi(x) \rangle.
\]
Since \( g \in C^\infty(\mathbb{R}^n) \), this means that
\[
\left\langle f(x), \int g(y) \varphi(x+y) dy \right\rangle = \int \langle f(y), g(x-y) \rangle \varphi(x) dx,
\]
5.4. Support of a convolution

or

\[ \left\langle f(y), \int g(x - y)\varphi(x)dx \right\rangle = \int \langle f(y), g(x - y) \rangle \varphi(x)dx, \]

i.e., we are to prove that the continuous linear functional \( f \) may be inserted under the integral sign. This is clear, since the integral (Riemann) sums of the integral

\[ \int g(x - y)\varphi(x)dx, \]

considered as functions of \( y \), tend to the integral in the topology in \( C^\infty(\mathbb{R}_y^n) \), because derivatives \( \partial_y^\alpha \) of the Riemann integral sums are Riemann integral sums themselves for the integrand in which \( g(x - y) \) is replaced with \( \partial_y^\alpha g(x - y) \). Proposition 5.6 is proved. \( \square \)

**Proposition 5.7.** Let \( f_k \to f \) in \( \mathcal{D}'(\mathbb{R}^n) \) and \( g \in \mathcal{E}'(\mathbb{R}^n) \), or \( f_k \to f \) in \( \mathcal{E}'(\mathbb{R}^n) \) and \( g \in \mathcal{D}'(\mathbb{R}^n) \). Then \( f_k * g \to f * g \) in \( \mathcal{D}'(\mathbb{R}^n) \).

**Proof.** The proposition follows immediately from

\[ \langle f_k * g, \varphi \rangle = \langle f_k(x), \langle g(y), \varphi(x + y) \rangle \rangle. \]

\( \square \)

**Corollary 5.8.** \( C_0^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{D}'(\mathbb{R}^n) \) and in \( \mathcal{E}'(\mathbb{R}^n) \).

**Proof.** If \( \varphi_k \in C_0^\infty(\mathbb{R}^n) \), \( \varphi_k(x) \to \delta(x) \) in \( \mathcal{E}'(\mathbb{R}^n) \) as \( k \to +\infty \) and \( f \in \mathcal{D}'(\mathbb{R}^n) \) then \( \varphi_k * f \in C^\infty(\mathbb{R}^n) \) and \( \lim_{k \to +\infty} \varphi_k * f = \delta * f = f \) in \( \mathcal{D}'(\mathbb{R}^n) \). Therefore, \( C^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{D}'(\mathbb{R}^n) \). But, clearly, \( C_0^\infty(\mathbb{R}^n) \) is dense in \( C^\infty(\mathbb{R}^n) \). Therefore, \( C_0^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{D}'(\mathbb{R}^n) \).

It is easy to see that if \( f_l \in \mathcal{E}'(\mathbb{R}^n) \) and \( \text{supp} f_l \subset K \), where \( K \) is a compact in \( \mathbb{R}^n \) (independent of \( l \)), then \( f_l \to f \) in \( \mathcal{D}'(\mathbb{R}^n) \) if and only if \( f_l \to f \) in \( \mathcal{E}'(\mathbb{R}^n) \). But then it follows from the above argument that \( C_0^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{E}'(\mathbb{R}^n) \). \( \square \)

A similar argument shows that \( C_0^\infty(\Omega) \) is dense in \( \mathcal{E}'(\Omega) \) and \( \mathcal{D}'(\Omega) \) and, besides, \( C_0^\infty(\mathbb{R}^n) \) is dense in \( S'(\mathbb{R}^n) \).

These facts also say that in order to verify any distributional formula whose both sides continuously depend, for example, on \( f \in \mathcal{D}'(\mathbb{R}^n) \), it suffices to do so for \( f \in C_0^\infty(\mathbb{R}^n) \).

**Proposition 5.9.** Let \( f \in \mathcal{E}'(\mathbb{R}^n) \), \( g \in \mathcal{D}'(\mathbb{R}^n) \). Then

\[ \text{supp}(f * g) \subset \text{supp} f + \text{supp} g. \]
Proof. We have to prove that if \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) and \( \text{supp} \varphi \cap (\text{supp} f + \text{supp} g) = \emptyset \), then \( \langle f \ast g, \varphi \rangle = 0 \). But this is clear from (5.9) since in this case \( \text{supp} \varphi(x+y) \) does not intersect with \( \text{supp}[f(x)g(y)] \). \( \square \)

**Definition 5.10.** The singular support of a distribution \( f \in \mathcal{D}'(\Omega) \) is the smallest closed subset \( F \subset \Omega \) such that \( f|_{\Omega \setminus F} \in C^\infty(\Omega \setminus F) \), and is denoted \( \text{sing supp} f \).

**Proposition 5.11.** Let \( f \in \mathcal{E}'(\mathbb{R}^n), g \in \mathcal{D}'(\mathbb{R}^n) \). Then
\[
(5.15) \quad \text{sing supp}(f \ast g) \subset \text{sing supp} f + \text{sing supp} g.
\]

Proof. Using cut-off functions we can, for any \( \varepsilon > 0 \), represent any distribution \( h \in \mathcal{D}'(\mathbb{R}^n) \) as a sum \( h = h_\varepsilon + h_\varepsilon' \), where \( \text{supp} h_\varepsilon \) is contained in the \( \varepsilon \)-neighborhood of \( \text{sing supp} h \) and \( h_\varepsilon' \in C^\infty(\mathbb{R}^n) \). Let us choose such representation for \( f \) and \( g \):
\[
f = f_\varepsilon + f_\varepsilon', \quad g = g_\varepsilon + g_\varepsilon'.
\]
Then
\[
f \ast g = f_\varepsilon \ast g_\varepsilon + f_\varepsilon \ast g_\varepsilon' + f_\varepsilon' \ast g_\varepsilon + f_\varepsilon' \ast g_\varepsilon'.
\]
In this sum the three last summands belong to \( C^\infty(\mathbb{R}^n) \) by Proposition 5.6. Therefore,
\[
\text{sing supp}(f \ast g) \subset \text{sing supp}(f_\varepsilon \ast g_\varepsilon) \subset \text{supp}(f_\varepsilon \ast g_\varepsilon) \subset \text{supp} f_\varepsilon + \text{supp} g_\varepsilon.
\]
The latter set is contained in the \( 2\varepsilon \)-neighborhood of \( \text{sing supp} f + \text{sing supp} g \). Since \( \varepsilon \) is arbitrary, this implies (5.15), as required. \( \square \)

5.5. Relation between smoothness of a fundamental solution and that of solutions of the homogeneous equation

We will first prove the following simple but important

**Lemma 5.12.** Let \( p(D) \) be a linear differential operator with constant coefficients, \( \mathcal{E} = \mathcal{E}(x) \) its fundamental solution, \( u \in \mathcal{E}'(\mathbb{R}^n) \) and \( p(D)u = f \). Then \( u = \mathcal{E} \ast f \).

Proof. We have
\[
\mathcal{E} \ast f = \mathcal{E} \ast p(D)u = p(D)\mathcal{E} \ast u = \delta \ast u = u,
\]
as required. \( \square \)
Remark 5.13. The requirement \( u \in \mathcal{E}'(\mathbb{R}^n) \) is essential here (the assertion is clearly wrong without it). Namely, it allows us to move \( p(D) \) from \( u \) to \( \mathcal{E} \) in the calculation given above.

Theorem 5.14. Assume that \( \mathcal{E} \) is a fundamental solution of \( p(D) \), and \( \mathcal{E} \in C^\infty(\mathbb{R}^n\setminus\{0\}) \). If \( u \in \mathcal{D}'(\Omega) \) is a solution of \( p(D)u = f \) with \( f \in C^\infty(\Omega) \), then \( u \in C^\infty(\Omega) \).

Proof. Note that the statement of the theorem is local, i.e., it suffices to prove it in a neighborhood of any point \( x_0 \in \Omega \). Let \( \chi \in \mathcal{D}(\Omega), \chi(x) = 1 \) in a neighborhood of \( x_0 \). Consider the distribution \( \chi u \) and apply \( p(D) \) to it. Denoting \( p(D)(\chi u) \) by \( f_1 \), we see that \( f_1 = f \) in a neighborhood of \( x_0 \) and, in particular, \( x_0 \notin \text{sing supp } f_1 \). By Lemma 5.12, we have

\[
(5.16) \quad \chi u = \mathcal{E} \ast f_1.
\]

We now use Proposition 5.11. Since \( \text{sing supp } \mathcal{E} = \{0\} \), we get

\[
\text{sing supp}(\chi u) \subset \text{sing supp } f_1.
\]

Therefore, \( x_0 \notin \text{sing supp}(\chi u) \) and \( x_0 \notin \text{sing supp } u \), i.e., \( u \) is a \( C^\infty \)-function in a neighborhood of \( x_0 \), as required. \( \square \)

A similar fact is true for the analyticity of solutions.

Recall that a function \( u = u(x) \) defined in an open set \( \Omega \subset \mathbb{R}^n \), with values in \( \mathbb{R} \) or \( \mathbb{C} \), is called real analytic if for every \( y \in \Omega \) there exists a neighborhood \( U_y \) of \( y \) in \( \Omega \), such that \( u|_{U_y} = u|_{U_y}(x) \) can be presented in \( U_y \) as a sum of a convergent power series

\[
u(x) = \sum_{\alpha} c_\alpha (x - y)^\alpha, \quad x \in U_y,
\]

with the sum over all multiindices \( \alpha \).

Consider first the case of the homogeneous equation \( p(D)u = 0 \).

Theorem 5.15. Assume that a fundamental solution \( \mathcal{E}(x) \) of \( p(D) \) is real analytic in \( \mathbb{R}^n\setminus\{0\} \), \( u \in \mathcal{D}'(\Omega) \) and \( p(D)u = 0 \). Then \( u \) is a real analytic function in \( \Omega \).

Proof. We know already that \( u \in C^\infty(\Omega) \). Using the same argument as in the proof of Theorem 5.14, consider (5.16), where now \( f_1 \in \mathcal{D}(\Omega) \) and \( f_1(x) = 0 \) in a neighborhood of \( x_0 \). Then for \( x \) close to \( x_0 \) we have

\[
u(x) = \int_{|y-x_0| \geq \varepsilon > 0} \mathcal{E}(x - y)f_1(y)dy.
\]
Note that the analyticity of $E(x)$ for $x \neq 0$ is equivalent to the possibility to extend $E(x)$ to a holomorphic function in a complex neighborhood of the set $\mathbb{R}^n \setminus \{0\}$ in $\mathbb{C}^n$, i.e., in an open set $G \subset \mathbb{C}^n$ such that $G \cap \mathbb{R}^n = \mathbb{R}^n \setminus \{0\}$.

Recall, that a complex-valued function $\varphi = \varphi(z)$ defined for $z \in G$, where $G$ is an open subset in $\mathbb{C}^n$, is called holomorphic or complex analytic or even simply analytic, if one of the following two equivalent conditions is fulfilled:

a) $\varphi(z)$ can be expanded in a power series in $z - z_0$ in a neighborhood of each point $z_0 \in G$, i.e.,

$$\varphi(z) = \sum_{\alpha} c_\alpha (z - z_0)^\alpha,$$

where $z$ is close to $z_0$, $c_\alpha \in \mathbb{C}$ and $\alpha$ runs over all multiindices;

b) $\varphi(z)$ is continuous in $G$ and differentiable with respect to each complex variable $z_j$ (i.e., is holomorphic with respect to each $z_j$).

Proof can be found on the first pages of any textbook on functions of several complex variables (e.g. Gunning and Rossi [10], Ch.1, Theorem A2, or Hörmander [13], Ch.2).

Note that in (5.17) the variable $y$ varies over a compact $K \subset \mathbb{R}^n$. Fix $y_0 \neq x_0$. Then for $|z - x_0| < \delta$, $|w - y_0| < \delta$ with $z, w \in \mathbb{C}^n$ and $\delta$ sufficiently small, we can define $E(z - w)$ as a holomorphic function in $z$ and $w$. Then, clearly,

$$\int_{|y-y_0|\leq \delta} E(z-y)f_1(y)dy$$

is well defined for $|z-x_0| < \delta$ and holomorphic in $z$ since we can differentiate with respect to $z$ under the integral sign. But using a partition of unity, we can express the integral in (5.17) as a finite sum of integrals of this form. Therefore, we can define $u(z)$ for complex $z$, which are close to $x_0$, so that $u(z)$ is analytic in $z$. In particular, $u(x)$ can be expanded in a Taylor series in a neighborhood of $x_0$, i.e., $u(x)$ is real analytic. \[\square\]

The case of a nonhomogeneous equation reduces to the above due to the Cauchy-Kovalevsky theorem which we will give in a form convenient for us.

**Theorem 5.16. (Cauchy-Kovalevsky)** Consider the equation

$$\frac{\partial^m u}{\partial x_m^n} + \sum_{|\alpha| \leq m, \alpha_n < m} a_\alpha(x) D^\alpha u(x) = f(x),$$

(5.18)
5.5. Smoothness of solutions

where \( a_\alpha(x), f(x) \) are real analytic in a neighborhood of \( 0 \in \mathbb{R}^n \). Consider the Cauchy problem for this equation, i.e., the initial value problem with the initial conditions

\[
(5.19) \quad u|_{x_n=0} = \varphi_0(x'), \ldots, \frac{\partial^{m-1} u}{\partial x_n^{m-1}}|_{x_n=0} = \varphi_{m-1}(x'),
\]

where \( x' \in \mathbb{R}^{n-1} \), \( \varphi_j(x') \) are real analytic functions in a neighborhood of \( 0 \in \mathbb{R}^{n-1} \). The problem (5.18) – (5.19) has a unique solution, which is real analytic in a sufficiently small ball \(|x| \leq \varepsilon\) in \( \mathbb{R}^n \).

We will not prove this theorem (the reader can find the proof e.g. in Petrovskii [21], Ch.1, Hörmander [12], Theorem 5.1.2 or [14], v.1, Theorem 9.4.5). Note only that one of the ways to prove it is to seek \( u(x) \) as a power series whose coefficients are uniquely determined by (5.18), (5.19), and then estimate the coefficients, proving the convergence of this series.

Note that if \( P = p(x, D) \) is a linear differential operator of order \( \leq m \) with analytic coefficients in a neighborhood of \( 0 \) whose principal symbol \( p_m(x, \xi) \) is such that \( p_m(0, \xi) \not\equiv 0 \) (i.e., one of the highest coefficients does not vanish at \( 0 \)), then by rotating the coordinate axes, we can reduce \( Pu = f \) to the form (5.18). Namely, we should choose coordinate axes so that the plane \( x_n = 0 \) were noncharacteristic at \( 0 \). Therefore, the equation \( Pu = f \), where \( f \) is an analytic function in a neighborhood of \( 0 \), always has a solution \( u(x) \) analytic in a neighborhood of \( 0 \). In particular, this always holds for operators with constant coefficients. It is clear now that Theorem 5.15 implies

**Theorem 5.17.** Let a fundamental solution \( \mathcal{E}(x) \) of \( p(D) \) be real analytic for \( x \neq 0 \). If \( u \in \mathcal{D}'(\Omega), p(D)u = f \) and \( f \) is real analytic in \( \Omega \), then so is \( u \).

**Corollary 5.18.** If \( u \in \mathcal{D}'(\Omega), \Delta^m u = f \), where \( f \) is real analytic in \( \Omega \), then so is \( u \).

**Remark 5.19.** Any differential operator \( p(D) \) with constant coefficients has a fundamental solution. It is clear from Theorem 5.14 that the following conditions on \( p(D) \) are equivalent:

a) Operator \( p(D) \) has a fundamental solution \( \mathcal{E}(x) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \);

b) if \( u \in \mathcal{D}'(\Omega) \) and \( p(D)u = 0 \), then \( u \in C^\infty(\Omega) \).
The operators \( p(D) \) with this property are called \textit{hypoelliptic}. It can be proved that \( p(D) \) is hypoelliptic if and only if

\[
\frac{\partial p(\xi)}{\partial \xi_j} \bigg|_{\xi} \to 0 \quad \text{as} \quad |\xi| \to \infty, \quad j = 1, \ldots, n.
\]

Further, by Theorem 5.15, the following two conditions are equivalent:

c) \( p(D) \) has a fundamental solution \( E(x) \) which is real analytic for \( x \neq 0 \).

d) If \( u \in \mathcal{D}'(\Omega) \) and \( p(D)u = 0 \), then \( u \) is real analytic in \( \Omega \).

It can be proved that c) and d) are equivalent to the ellipticity of \( p(D) \) in the sense of Section 1.5.

Proofs of all statements formulated in this remark, can be found, for example, in Shilov [27], Ch.III, or Hörmander [14], Theorem 11.1.1.

5.6. Solutions with isolated singularities. A removable singularity theorem for harmonic functions

Consider a hypoelliptic operator \( p(D) \) i.e., an operator \( p(D) \) with a fundamental solution \( E(x) \in C^\infty(\mathbb{R}^n \backslash \{0\}) \). The fundamental solution \( E(x) \) itself satisfies, for \( x \neq 0 \), the equation \( p(D)u = 0 \) and at \( x = 0 \) has a singularity. The same applies to \( \partial^a E(x) \). It turns out that if the singularity of \( u \) is not worse than that of a power of \( |x|^{-1} \), this example actually exhausts all the possibilities.

\textbf{Theorem 5.20.} Let \( u(x) \in C^\infty(\Omega \backslash \{0\}) \), where \( \Omega \) is a domain in \( \mathbb{R}^n, 0 \in \Omega \), and let \( u(x) \) be a solution of the equation \( p(D)u = 0 \) in \( \Omega \backslash \{0\} \), where \( p(D) \) is a hypoelliptic operator. Let \( E(x) \) be a fundamental solution of \( p(D) \). Assume also that in a neighborhood of 0 we have

\begin{equation}
|u(x)| \leq C|x|^{-N}, \quad x \neq 0,
\end{equation}

for some \( C \) and \( N \). Then \( u(x) \) can be represented in \( \Omega \backslash \{0\} \) as

\begin{equation}
u(x) = \sum_{|a| \leq p} c_a \partial^a E(x) + u_0(x),
\end{equation}

where \( c_a \in \mathbb{C}, u_0(x) \in C^\infty(\Omega) \) and \( p(D)u_0 = 0 \).

In the proof we will need the following

\textbf{Lemma 5.21.} If \( u(x) \in C^\infty(\Omega \backslash \{0\}) \) satisfies (5.20) and (5.21) then there exists \( \hat{u} \in \mathcal{D}'(\Omega) \), such that \( \hat{u}|_{\Omega \backslash 0} = u \).
Solutions with isolated singularities

Proof. We would like to give a meaning to the integral $\int u(x)\varphi(x)dx$, where $\varphi(x) \in D(\Omega)$, in such a way that its value on $\varphi \in D(\Omega \setminus \{0\})$ would be natural (for such $\varphi$ this is the usual integral of a continuous function with a compact support). To this end subtract from $\varphi$ its Taylor polynomial at 0, and note that

$$\varphi(x) - \sum_{|\alpha| \leq N-1} \frac{\varphi^{(\alpha)}(0)}{\alpha!} x^\alpha = O(|x|^N) \text{ as } |x| \to 0.$$  

Let $\chi \in D(\Omega), \chi(x) = 1$ in a neighborhood of 0. Set

$$\langle \hat{u}, \varphi \rangle = \int_\Omega u(x)[\varphi(x) - \chi(x) \sum_{|\alpha| \leq N-1} \frac{\varphi^{(\alpha)}(0)}{\alpha!} x^\alpha] dx,$$

where the integral is already defined since the integrand is bounded. Clearly, if $\varphi \in D(\Omega \setminus \{0\})$, then $\langle \hat{u}, \varphi \rangle = \int u(x)\varphi(x)dx$, because $\varphi^{(\alpha)}(0) = 0$ for any $\alpha$. It is also easy to see that $\hat{u} \in D'(\Omega)$, thus proving the Lemma. □

Proof of Theorem 5.20. Clearly, $p(D)\hat{u}$ is a distribution with the support at 0, hence, due to Theorem 4.7 (see also Example 4.8),

$$p(D)\hat{u}(x) = \sum_{|\alpha| \leq p} c_\alpha \partial^\alpha \delta(x).$$

But setting

$$u_0(x) = \hat{u}(x) - \sum_{|\alpha| \leq p} c_\alpha \partial^\alpha \mathcal{E}(x),$$

we, clearly, get $u_0 \in D'(\Omega)$ and $p(D)u_0 = 0$ implying $u_0 \in C^\infty(\Omega)$. □

In particular, for the Laplace operator and $n \geq 3$, a solution of the equation $\Delta u = 0$ (called a harmonic function) defined in $\Omega \setminus \{0\}$ and satisfying (5.20) is a sum of homogeneous functions of homogeneity orders $2-n, 2-n-1, \ldots, 2-n-p$ and a harmonic function in $\Omega$. For $n = 2$ such a solution is the sum of $c_0 \ln |x|$, homogeneous functions of orders $-1, -2, \ldots, -p$, and a harmonic function in $\Omega$. In particular, this implies the following removable singularity theorem for harmonic functions:

**Theorem 5.22.** (On removable singularity) Let $u(x)$ be a harmonic function in $\Omega \setminus \{0\}$ such that in a neighborhood of 0

$$\lim_{|x| \to 0} |x|^{n-2} \cdot u(x) = 0,$$

for $n \geq 3$,
\[
\lim_{|x| \to 0} [(\ln |x|)^{-1}u(x)] = 0, \text{ for } n = 2.
\]

Then \(u(x)\) can be extended to a harmonic function in \(\Omega\).

5.7. Estimates of derivatives for a solution of a hypoelliptic equation

Recall that an operator \(p(D)\) with constant coefficients in \(\mathbb{R}^n\) is called hypoelliptic if it has a fundamental solution \(E(x) \in \mathcal{D}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus 0)\) or, equivalently, any solution \(u \in \mathcal{D}'(\Omega)\) of the equation \(p(D)u = 0\) actually belongs to \(C^\infty(\Omega)\).

It is clear from the calculation of the fundamental solution of the Laplacian \(\Delta\), given above, that \(\Delta\) is hypoelliptic. Another example, to be discussed below in Section 7, is the heat operator.

**Proposition 5.23.** Let \(\Omega_1, \Omega_2\) be two bounded open sets in \(\mathbb{R}^n\) such that \(\overline{\Omega}_2 \subset \Omega_1\) (here “bar” means the closure in \(\mathbb{R}^n\)), and \(p(D)\) is a hypoelliptic operator in \(\mathbb{R}^n\). Then solutions \(u \in C^\infty(\Omega_1)\) of the equation \(p(D)u = 0\) satisfy

\[
(5.22) \quad \sup_{x \in \Omega_2} |\partial^\alpha u(x)| \leq C_\alpha \int_{\Omega_1} |u(x)| dx,
\]

where \(\alpha\) is an arbitrary multiindex, and constants \(C_\alpha\) do not depend on the choice of \(u\) (but may depend on the choice of \(p, \Omega_1\) and \(\Omega_2\), as well as of \(\alpha\)). In particular, estimates

\[
(5.23) \quad \sup_{x \in \Omega_2} |\partial^\alpha u(x)| \leq C'_\alpha \sup_{x \in \Omega_1} |u(x)|
\]

hold, where \(C'_\alpha\) are other constants, also independent of \(u\).

**Proof.** It suffices to consider the case when the right hand side of (5.22) is finite (otherwise the inequality is obvious).

We will use the closed graph theorem in the following form:

If \(E_1, E_2\) are two Banach spaces and a linear operator \(A : E_1 \to E_2\) is everywhere defined and has a closed graph (i.e., the set \(\{(x, Ax) : x \in E_1\}\) is closed in \(E_1 \times E_2\) or, in other words, \(x_n \to x\) and \(Ax_n \to y\) imply \(Ax = y\), then this operator is continuous (or, which is the same, bounded). (For the proof, based on the Baire Theorem 4.20, see e.g. Reed and Simon \[22\], Theorem III.12.)

As \(E_1\), we take the subspace of \(L^1(\Omega_1)\) consisting of the solutions of the equation \(p(D)u = 0\) (the operator is applied in the distributional sense).
5.7. Estimates of derivatives of a solution

It is important that $E_1$ is a closed subspace of $L^1(\Omega_1)$. In fact, if $u_n \to u$ in $L^1(\Omega_1)$ and $p(D)u_n = 0$, then $u_n \to u$ weakly in $\mathcal{D}'(\Omega_1)$, i.e., $\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\Omega_1)$. But then $\langle u_n, p(-D)\varphi \rangle \to \langle u, p(-D)\varphi \rangle$ for any $\varphi \in \mathcal{D}(\Omega_1)$ since in this case $p(-D)\varphi \in \mathcal{D}(\Omega_1)$. The latter limit means that $p(D)u_n \to p(D)u$ weakly in $\mathcal{D}'(\Omega_1)$ implying $p(D)u = 0$, as required. Therefore, $E_1$ is a Banach space. Notice that, actually, $E_1 \subset C^\infty(\Omega_1)$ by hypoellipticity of $p(D)$.

As $E_2$, we take the Banach space of functions $u \in C^\infty(\Omega_2)$ such that $p(D)u = 0$ and all derivatives of $u$ of order no greater than $k$ (here $k \in \mathbb{Z}_+$ is arbitrary) are bounded. In $E_2$, introduce the usual norm

$$
||u||_k = \sum_{|\alpha| \leq k} \sup_{x \in \Omega_2} |\partial^\alpha u(x)|.
$$

If $u_n \to u$ with respect to $||\cdot||_k$ and $p(D)u_n = 0$, then clearly $p(D)u = 0$ (in $\mathcal{D}'(\Omega_2)$) and, therefore, $u \in C^\infty(\Omega_2)$. Therefore, $E_2$ is a Banach space (it is a closed subspace of a Banach space of functions $u \in C^k(\Omega_2)$ with bounded derivatives of order up to $k$ included).

Now, consider a linear operator $A : E_1 \to E_2$ that maps $u$ into $u|_{\Omega_2}$. To verify that the graph of $A$ is closed, we need to prove that if $u_n \to u$ in $E_1$ and $Au_n \to v$ in $E_2$, then $v = Au$, i.e., $v = u|_{\Omega_2}$. But $u_n \to u$ in $E_1$ implies that $u_n \to u$ weakly in $\mathcal{D}'(\Omega_1)$ and therefore, clearly, $u_n|_{\Omega_2} \to u|_{\Omega_2}$ weakly in $\mathcal{D}'(\Omega_2)$. But on the other hand, we know that $u_n|_{\Omega_2} \to v$ in $E_2$ (hence, weakly in $\mathcal{D}'(\Omega_2)$). Since one sequence cannot have two different weak limits in $\mathcal{D}'(\Omega_2)$, it follows that $v = u|_{\Omega_2}$, as required.

By the closed graph theorem $A$ is bounded. But this means that

$$
||u||_k \leq C_k \int_{\Omega_1} |u(x)|dx,
$$

which, since $k$ is arbitrary, implies (5.22). Applying an obvious inequality

$$
\int_{\Omega_1} |u(x)|dx \leq C \sup_{x \in \Omega_1} |u(x)|,
$$

we also get (5.23).

This proof is somewhat ineffective, in the sense that it does not allow to get any information about the constants $C_k$, because the constants which appear due to the application of the closed graph theorem, are out of our control. But it is possible to get effective constants in a weaker result, where every higher derivative $\partial^\alpha u$ is estimated through $L^1$-norms of the derivatives of order $\leq \deg p$. We will sketch now how this can be done.
We will argue as in the proof of Theorem 5.14. Let us choose a function \( \chi \in D(\Omega_1) \), such that \( \chi = 1 \) in a neighborhood of \( \bar{\Omega}_2 \). Then we obtain for any \( u \in C^\infty(\Omega_1) \),

\[
\chi u = \mathcal{E} * f, \quad \text{where} \quad f = p(D)(\chi u).
\]

Now note that if additionally \( p(D)u = 0 \) in \( \Omega_1 \), then \( f = 0 \) in a neighborhood \( U \) of \( \bar{\Omega}_2 \), which implies that for \( x \in \Omega_2 \) we can write

\[
u(x) = \int_{\Omega_1 \setminus \bar{U}} \mathcal{E}(x - y)f(y)dy,
\]

where the integrand is in fact a \( C^\infty \) function of \((x,y)\) for all \( x \in U \), and the integration is taken over a compact set which is independent of \( x \in \Omega_2 \). Therefore we can write the derivatives \( \partial^\alpha u(x) \) for \( x \in \Omega_2 \) by differentiating under the integral sign, i.e.

\[
\partial^\alpha u(x) = \int_{\Omega_1 \setminus \bar{U}} \partial^\alpha_x \mathcal{E}(x - y)f(y)dy.
\]

it follows that

\[
|\partial^\alpha u(x)| \leq C \sum_{|\alpha| \leq \deg p} \int_{\Omega_1} |\partial^\alpha u(y)|dy,
\]

for every \( x \in \Omega_2 \).

Proposition 5.23 implies

**Corollary 5.24.** If \( \{u_j| j = 1, 2, \ldots \} \) is a sequence of distributional (hence \( C^\infty \)) solutions of the equation \( p(D)u_j = 0 \) on \( \Omega_1 \), \( p(D) \) is hypoelliptic, and \( u_j \to u \), in \( L^1(\Omega_1) \), then \( u_j \to u \) in \( C^\infty(\Omega_1) \).

Making the arguments given here more precise, it is easy to see that if \( u_j \to u \) in \( D'(\Omega) \) and \( p(D)u_j = 0 \), then \( u_j \to u \) in the topology of \( C^\infty(\Omega_1) \). In particular, it follows that for solutions of \( p(D)u = 0 \), the convergence in \( L^1(\Omega_1) \) implies the uniform convergence of derivatives of any order on \( \Omega_2 \).

More precise estimates can be obtained in concrete situations (e.g. for the Laplacian in a ball).

Proposition 5.23 also makes it possible, for instance, to get information on the growth or decay of derivatives of a solution of the equation \( p(D)u = 0 \) from similar data concerning the solution itself. We will see an important example of such an argument in Section 7.6.
5.8. Fourier transform of tempered distributions

Denote by $F$ the Fourier transform given by

$$(F f)(\xi) = \tilde{f}(\xi) = \int f(x) e^{-ix \cdot \xi} dx.$$ 

It is easy to see that it is well defined on functions $f \in S(\mathbb{R}^n)$ ($f = f(x)$), because of their fast decay (faster than any polynomial) as $|x| \to \infty$. Let us describe properties of $\tilde{f}$ assuming that $f \in S(\mathbb{R}^n)$. It is obvious that $F f$ is bounded. Differentiating $\tilde{f}$ (with respect to $\xi$), we come to integrals of the same type, so it follows that $F$ maps $S(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, and every derivative of $F f$ is bounded. Multiplying $F f(\xi)$ by $\xi^\alpha$ (where $\alpha$ is a multiindex) and integrating by parts, we get

$$\xi^\alpha \tilde{f}(\xi) = \int f(x) (-D_x)^\alpha e^{-ix \cdot \xi} dx = \int (D^\alpha f)(x) e^{-ix \cdot \xi} dx.$$ 

This is also an integral of the same type (the Fourier transform of a function $D^\alpha f \in S(\mathbb{R}^n)$). So we see that $f$ remains bounded after differentiation of any order and multiplication by any polynomial. This means that $F f \in S(\mathbb{R}^n)$, so $F$ is a linear map of $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. 

It easily follows from the argument above that $F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous if the spaces $S(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are considered with their Fréchet topologies (introduced in the previous chapter). Moreover, it is known from Fourier analysis that in fact the map

$$F : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$$

is a topological isomorphism (a continuous linear operator with continuous inverse) and the inverse operator is given by the formula

$$(5.24) \quad (F^{-1} g)(x) = (2\pi)^{-n} \int g(\xi) e^{ix \cdot \xi} d\xi.$$ 

(See, for example, Hörmander [12], Corollary 1.7.1, Hörmander [14], Theorem 7.1.5, or Schwartz [25], Ch.5.)

Clearly, if $f \in S(\mathbb{R}^n)$, then

$$(5.25) \quad \langle Ff, \varphi \rangle = \langle f, F\varphi \rangle, \quad \varphi \in S(\mathbb{R}^n).$$

In other words, $^t F = F$. Therefore, using (5.25) we may extend $F$ to a continuous map $F : S'(\mathbb{R}^n) \longrightarrow S'(\mathbb{R}^n)$. Clearly, the Fourier transform $F$ gives a topological isomorphism of $S'(\mathbb{R}^n)$ onto $S'(\mathbb{R}^n)$, and the inverse operator $F^{-1}$ is also equal to $^t(F^{-1})$, where $F^{-1}$ is given by (5.24), i.e.,

$$(5.26) \quad \langle F^{-1} f, \varphi \rangle = \langle f, F^{-1} \varphi \rangle, \quad f \in S'(\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).$$
Thus we can define the Fourier transform of any function \( f(x) \) such that \( |f(x)| \leq C(1 + |x|)^N \). The Fourier transform is in general not a function but a distribution from \( S' (\mathbb{R}^n) \).

Note that since \( S(\mathbb{R}^n) \) is dense in \( S'(\mathbb{R}^n) \), then \( F \) on \( S'(\mathbb{R}^n) \) is the extension by continuity of the Fourier transform defined on \( S(\mathbb{R}^n) \). This remark makes it possible to extend by continuity certain facts from \( S(\mathbb{R}^n) \) to \( S'(\mathbb{R}^n) \) without proving them separately. For instance, we do not need to verify formula (5.26), since it holds for \( f \in S(\mathbb{R}^n), \varphi \in S(\mathbb{R}^n) \) and both sides are continuous with respect to \( f \in S'(\mathbb{R}^n) \) in the weak topology of \( S'(\mathbb{R}^n) \).

As in the case of \( f \in S(\mathbb{R}^n) \), the Fourier transform of a distribution \( f \in S'(\mathbb{R}^n) \) will be denoted by \( \hat{f}(\xi) \).

Let us find the Fourier transform of \( D^\alpha f \). If \( f \in S(\mathbb{R}^n) \), then integration by parts yields

\[
\hat{D^\alpha f}(\xi) = \int e^{-ix\cdot\xi} D^\alpha f(x) dx = \int (-D)^{\alpha} e^{-ix\cdot\xi} f(x) dx = \int \xi^\alpha e^{-ix\cdot\xi} f(x) dx = \xi^\alpha \hat{f}(\xi).
\]

Therefore

(5.27) \( \hat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi) \)

implying

(5.28) \( \hat{p(D)f}(\xi) = p(\xi) \hat{f}(\xi) \)

for any polynomial \( p(\xi) \). Thus, a Fourier transform converts \( p(D) \) into the multiplication by the symbol \( p(\xi) \).

Note that formula (5.28) holds by continuity for \( f \in S'(\mathbb{R}^n) \).

Example 5.7. Let us find the Fourier transform of the \( \delta \)-function. For \( \varphi \in S(\mathbb{R}^n) \) we have

\[
\langle \hat{\delta}(\xi), \varphi(\xi) \rangle = \langle \delta(x), \int \varphi(\xi) e^{-ix\cdot\xi} d\xi \rangle = \int \varphi(\xi) d\xi = \langle 1, \varphi(\xi) \rangle
\]

implying

(5.29) \( \hat{\delta}(\xi) = 1 \).

Let us also find the Fourier transform of 1. We have

\[
\langle F(1), \varphi(\xi) \rangle = \int \varphi(\xi) e^{-ix\cdot\xi} d\xi dx = (2\pi)^n \varphi(0)
\]
by the inversion formula (5.24). Therefore,

\[ F(1)(\xi) = (2\pi)^n \delta(\xi). \]

**Example 5.8.** Let us find the Fourier transform of the Heaviside function \( \theta(x) \). It is most convenient to represent \( \theta(x) \) as the limit of functions which decay at infinity, so that their Fourier transforms can be understood in the ordinary sense. This can be done, for instance, as follows: clearly,

\[ \theta(x) = \lim_{\varepsilon \to 0^+} \theta(x)e^{-\varepsilon x} \text{ in } S'(\mathbb{R}^1). \]

We have

\[
\int \theta(x)e^{-\varepsilon x}e^{-i\xi \cdot x}dx = \int_0^\infty e^{-x(i\xi+\varepsilon)}dx = \frac{1}{i\xi + \varepsilon} = -\frac{i}{\xi - i\varepsilon}.
\]

Passing to the limit as \( \varepsilon \to +0 \), we get

\[ \tilde{\theta}(\xi) = -\frac{i}{\xi - i0}. \]

Note that \( \frac{d}{dx}\theta(x) = \delta(x) \) and \( iD_x(\theta(x)) = \delta(x) \). This agrees with (5.27) and (5.29) due to the obvious identity \( i\xi(-\frac{i}{\xi - i0}) = 1 \).

We can try to use the last observation for an alternative way to calculate \( \tilde{\theta}(\xi) \). Namely, since \( \theta'(x) = \delta(x) \), it follows that \( i\xi \cdot \tilde{\theta}(\xi) = 1 \). A distribution \( g(\xi) \in S'(\mathbb{R}^1) \) satisfying \( \xi g(\xi) = 1 \) is not, however, uniquely defined, though, clearly, it should be equal to \( 1/\xi \) for \( \xi \neq 0 \). It is easy to see that all the ambiguity reduces to adding \( C\delta(\xi) \), where \( C \) is an arbitrary constant. This constant can be determined by applying the definition of the Fourier transform of distributions to a particular function \( \varphi(\xi) \in S(\mathbb{R}^1) \). It is simpler, however, to use the direct method explained above.

**Proposition 5.25.** Let \( f \in \mathcal{E}'(\mathbb{R}^n) \). Then \( \hat{f}(\xi) \in C^\infty(\mathbb{R}^n) \) and

\[
(5.30) \quad \hat{f}(\xi) = \langle f, e^{-i\xi \cdot x} \rangle.
\]

There exists a constant \( N > 0 \) such that

\[
(5.31) \quad |\partial_\xi^\alpha \hat{f}(\xi)| \leq C_\alpha (1 + |\xi|)^N
\]

for any multiindex \( \alpha \).

**Proof.** Let us choose constants \( N, C \) and a compact \( K \subset \mathbb{R}^n \) so that

\[
(5.32) \quad |\langle f, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial_\alpha \varphi(x)|, \quad \varphi \in C^\infty(\mathbb{R}^n).
\]
Set $g(\xi) = \langle f, e^{-i\xi \cdot x} \rangle$. Since $e^{-i\xi \cdot x}$ is an infinitely differentiable function in $\xi$ with values in $C^\infty(\mathbb{R}^n)$, then, clearly, $g(\xi) \in C^\infty(\mathbb{R}^n_\xi)$ and
\[
\partial^\alpha_\xi g(\xi) = \langle f, (-ix)^\alpha e^{-i\xi \cdot x} \rangle,
\]
implying that (5.31) holds for $g(\xi)$. It remains to verify that $g(\xi) = \tilde{f}(\xi)$, i.e., that
\[
\left< f(x), \int \varphi(\xi) e^{-i\xi \cdot x} d\xi \right> = \int \varphi(\xi) \langle f(x), e^{-i\xi \cdot x} \rangle d\xi, \quad \varphi \in S(\mathbb{R}^n_\xi).
\]
But this follows from the convergence of $\int \varphi(\xi) e^{-i\xi \cdot x} d\xi$ in the topology of $C^\infty(\mathbb{R}^n_\xi)$ which means that this integral and integrals obtained from it by taking derivatives of any order converge uniformly with respect to $x \in K$ (where $K$ is a compact set in $\mathbb{R}^n$).

**Remark 5.26.** The condition $\tilde{f} \in C^\infty(\mathbb{R}^n)$ and the estimate (5.31) are necessary but not sufficient for a distribution $f$ to have a compact support. For instance, if $f \in \mathcal{E}'(\mathbb{R}^n)$, then, clearly, $\tilde{f}(\xi)$ may be defined via (5.30) for $\xi \in \mathbb{C}^n$ and we get an entire (i.e., homomorphic everywhere in $\mathbb{C}^n$) function satisfying
\[
|\tilde{f}(\xi)| \leq C(1 + |\xi|)^N e^{A|\text{Im}\xi|}.
\]
It turns out that this condition on $\tilde{f}$ is already necessary and sufficient for $f \in \mathcal{E}'(\mathbb{R}^n)$. This fact is a variant of the Paley-Wiener theorem and its proof may be found in Hörmander [12], Theorem 1.7.7, or [14], Theorem 7.3.1.

**Proposition 5.27.** Let $f \in \mathcal{E}'(\mathbb{R}^n), g \in S'(\mathbb{R}^n)$. Then
\[
(5.33) \quad \hat{f} * \hat{g}(\xi) = \tilde{f}(\xi) \hat{g}(\xi).
\]

**Proof.** Note that the right-hand side of (5.33) makes sense due to Proposition 5.25. It is easy also to verify that $\hat{f} * \hat{g} \in S'(\mathbb{R}^n)$ so that the left-hand side of (5.33) makes sense. Indeed, if $\varphi \in S(\mathbb{R}^n)$, then $\langle f(y), \varphi(x + y) \rangle \in S(\mathbb{R}^n_\varphi)$ due to (5.32). Therefore, for $\varphi \in S(\mathbb{R}^n)$ the equality $\langle f * g, \varphi \rangle = \langle g(x), \langle f(y), \varphi(x + y) \rangle \rangle$ makes sense and we clearly get $f * g \in S'(\mathbb{R}^n)$.

Now let us verify (5.33). For $\varphi \in S(\mathbb{R}^n)$ we get
\[
\left< \hat{f} * \hat{g}(\xi), \varphi(\xi) \right> = \left< f * g, \int \varphi(\xi) e^{-i\xi \cdot x} d\xi \right> = \left< g(x), \int \varphi(\xi) e^{-i\xi \cdot (x + y)} d\xi \right> = \left< g(x), \int \varphi(\xi) \tilde{f}(\xi) e^{-i\xi \cdot x} d\xi \right> = \langle \hat{g}(\xi), \tilde{f}(\xi) \varphi(\xi) \rangle = \langle \tilde{f}(\xi) \hat{g}(\xi), \varphi(\xi) \rangle,
\]
5.9. Applying Fourier transform to find fundamental solutions

Let $p(D)$ be a differential operator with constant coefficients, $E \in S'(\mathbb{R}^n)$ its fundamental solution. The Fourier transform $\tilde{E}(\xi) \in S'(\mathbb{R}^n)$ satisfies

\begin{equation}
(5.34) \quad p(\xi)\tilde{E}(\xi) = 1.
\end{equation}

Clearly, $\tilde{E}(\xi) = 1/p(\xi)$ on the open set $\{\xi : p(\xi) \neq 0\}$. If $p(\xi) \neq 0$ everywhere in $\mathbb{R}^n$, then we should just take $\tilde{E}(\xi) = 1/p(\xi)$ and apply the inverse Fourier transform. (It can be proved, though non-trivially, that in this case $1/p(\xi)$ is estimated by $1 + |\xi|^N$ with some $N > 0$, hence $1/p(\xi) \in S'(\mathbb{R}^n)$ – see, e.g., Hörmander [14], Example A.2.7 in Appendix A2, vol. 2.) When $p(\xi)$ has zeros we should regularize $1/p(\xi)$ in neighborhoods of zeros of $p(\xi)$ so as to get a distribution $\tilde{E}(\xi)$ equal to $1/p(\xi)$ for $p(\xi) \neq 0$ and satisfying (5.34). For instance, for $n = 1$ and $P(\xi) = \xi$ we can set $\tilde{E}(\xi) = \text{v.p.}rac{1}{\xi}$ or $\tilde{E}(\xi) = \frac{1}{\xi} + i0$ or, in general, $\tilde{E}(\xi) = \text{v.p.}\frac{1}{\xi} + C\delta(\xi)$, where $C$ is an arbitrary constant.

In the general case, we must somehow give meaning to integrals of the form

$$\int_{\mathbb{R}^n} \frac{\varphi(\xi)}{p(\xi)} d\xi, \quad \varphi \in S(\mathbb{R}^n)$$

to get a distribution with needed properties. It can often be done, e.g. by entering the complex domain (the construction of the distribution $\frac{1}{\xi+0}$ is an example). Sometimes $1/p(\xi)$ is locally integrable. Then it is again natural to assume $\tilde{E}(\xi) = 1/p(\xi)$. For instance, for $p(D) = \Delta$, i.e., $p(\xi) = -\xi^2$, we can set $\tilde{E}(\xi) = -1/\xi^2$ for $n \geq 3$.

Since, as is easy to verify, the Fourier transform and its inverse preserve the spherical symmetry and transform a distribution of homogeneity degree $\alpha$ into a distribution of homogeneity degree $-\alpha-n$, then, clearly, $F^{-1}(\frac{1}{|\xi|^{2n}}) = C|x|^{2-n}$, where $C \neq 0$. Therefore, we get the fundamental solution $E_n(x)$ described above.

For $n = 2$ the function $\frac{1}{|\xi|^2}$ is not integrable in a neighborhood of 0. Here a regularization is required. For instance, we can set

$$\langle \tilde{E}_2(\xi), \varphi(\xi) \rangle = -\int_{\mathbb{R}^2} \frac{\varphi(\xi) - \chi(\xi)\varphi(0)}{|\xi|^2} d\xi,$$

where $\chi(\xi) \in C_0^\infty(\mathbb{R}^2), \chi(\xi) = 1$ in a neighborhood of 0. It can be verified that $E_2(x) = \frac{1}{2\pi} \ln |x| + C$, where $C$ depends on the choice of $\chi(\xi)$.
5.10. Liouville’s theorem

Theorem 5.28. Assume that the symbol $p(\xi)$ of an operator $p(D)$ vanishes (for $\xi \in \mathbb{R}^n$) only at $\xi = 0$. If $u \in S'(\mathbb{R}^n)$ and $p(D)u = 0$, then $u(x)$ is a polynomial in $x$. In particular, if $p(D)u = 0$, where $u(x)$ is a bounded measurable function on $\mathbb{R}^n$, then $u = \text{const.}$

Remarks.

a) The inclusion $u \in S'(\mathbb{R}^n)$ holds, for instance, if $u \in C(\mathbb{R}^n)$ and $|u(x)| \leq C(1 + |x|)^N, \, x \in \mathbb{R}^n$, for some $C$ and $N$.

b) If $p(\xi_0) = 0$ for some $\xi_0 \neq 0$, then $p(D)u = 0$ has a bounded nonconstant solution $e^{i\xi_0 \cdot x}$. Therefore, $p(\xi) \neq 0$ for $\xi \neq 0$ is a necessary condition for the validity of the theorem.

c) If $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, then $p(D)u = 0$ for $u \in S'(\mathbb{R}^n)$ implies, as we will see from the proof of the theorem, that $u \equiv 0$.

Proof of Theorem 5.28. Applying the Fourier transform we get

$$p(\xi)\tilde{u}(\xi) = 0.$$ 

Since $p(\xi) \neq 0$ for $\xi \neq 0$, then, clearly, the support of $\tilde{u}(\xi)$ is equal to $\{0\}$ (provided $u \neq 0$). In more detail: if $\varphi(\xi) \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, then $\frac{\varphi(\xi)}{p(\xi)} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ implying

$$\langle \tilde{u}(\xi), \varphi(\xi) \rangle = \left\langle \tilde{u}(\xi), p(\xi) \frac{\varphi(\xi)}{p(\xi)} \right\rangle = \left\langle p(\xi)\tilde{u}(\xi), \frac{\varphi(\xi)}{p(\xi)} \right\rangle = 0.$$ 

Thus, supp $\tilde{u}(\xi) \subset \{0\}$. Therefore,

$$\tilde{u}(\xi) = \sum_{|\alpha| \leq p} a_\alpha \delta^{(\alpha)}(\xi), \quad a_\alpha \in \mathbb{C},$$

implying

$$u(x) = \sum_{|\alpha| \leq p} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{C},$$

as required. \qed

Example. If $u(x)$ is a harmonic function everywhere on $\mathbb{R}^n$ and $|u(x)| \leq M$, then $u = \text{const}$ (this statement is often called Liouville’s theorem for harmonic functions). If $u(x)$ is a harmonic function in $\mathbb{R}^n$ and $|u(x)| \leq C(1 + |x|)^N$, then $u(x)$ is a polynomial. The same holds for solutions of $\Delta^m u = 0$.

Later we will prove a more precise statement:
If \( u \) is harmonic and bounded below (or above), e.g., \( u(x) \geq -C, x \in \mathbb{R}^n \), then \( u = \text{const} \) (see e.g. Petrovskii [21]).

This statement fails already for the equation \( \Delta^2 u = 0 \) satisfied by the nonnegative polynomial \( u(x) = |x|^2 \).

Let us give one more example of application of Liouville’s theorem. Let for \( n \geq 3 \) a function \( u(x) \) be defined and harmonic in the exterior of a ball, i.e., for \( |x| > R \) and \( u(x) \to 0 \) as \( |x| \to +\infty \). We wish to describe in detail the behavior of \( u(x) \) as \( |x| \to \infty \). To this end we may proceed, for example, as follows. Take a function \( \chi(x) \in C_0^\infty(\mathbb{R}^n) \) such that

\[
\chi(x) = \begin{cases} 
0 & \text{for } |x| \leq R + 1, \\
1 & \text{for } |x| \geq R + 2.
\end{cases}
\]

Now, consider the following function \( \hat{u}(x) \in C_0^\infty(\mathbb{R}^n) \):

\[
\hat{u}(x) = \begin{cases} 
\chi(x)u(x) & \text{for } |x| > R, \\
0 & \text{for } |x| \leq R.
\end{cases}
\]

Clearly, \( \Delta \hat{u}(x) = f(x) \in C_0^\infty(\mathbb{R}^n) \). Let us verify that

\[
(5.35) \quad \hat{u} = \mathcal{E}_n \ast f,
\]

where \( \mathcal{E}_n \) is the standard fundamental solution of the Laplace operator.

Clearly, \( (\mathcal{E}_n \ast f)(x) \to 0 \) as \( |x| \to \infty \) and, by assumption, the same holds for \( u(x) \) (and hence, for \( \hat{u}(x) \), since \( \hat{u}(x) = u(x) \) for \( |x| > R + 2 \)). Moreover,

\[
\Delta(\mathcal{E}_n \ast f) = \Delta \mathcal{E}_n \ast f = f,
\]

so \( \Delta(\hat{u} - \mathcal{E}_n \ast f) = 0 \) everywhere on \( \mathbb{R}^n \). But, by Liouville’s theorem, this implies that \( \hat{u} - \mathcal{E}_n \ast f = 0 \), as required.

Formula (5.35) can be rewritten in the form

\[
u(x) = \int_{|y| \leq R+2} \mathcal{E}_n(x - y)f(y)dy, \quad |x| > R + 2.
\]

The explicit form of \( \mathcal{E}_n \) now implies

\[
|u(x)| \leq \frac{C}{|x|^{n-2}}, \quad |x| > R + 2.
\]

The above argument is also applicable in a more general situation. For harmonic functions a more detailed information on the behavior of \( u(x) \) as \( |x| \to \infty \) can be also obtained with the help of the Kelvin transformation

\[
u(x) = |x|^{2-n}u \left( \frac{x}{|x|^2} \right).
\]
It is easy to verify that the Kelvin transformation preserves harmonicity. For \( n = 2 \) the Kelvin transformation is the change of variables induced by the inversion \( x \mapsto x/|x|^2 \).

If \( u(x) \) is defined for large \( |x| \), then \( v(x) \) is defined in a punctured neighborhood of 0, i.e., in \( \Omega \setminus \{0\} \), where \( \Omega \) is a neighborhood of 0. If \( u(x) \to 0 \) as \( |x| \to +\infty \), then \( v(x)|x|^{n-2} \to 0 \) as \( |x| \to 0 \), and by the removable singularity theorem, \( v(x) \) can be extended to a function which is harmonic everywhere in \( \Omega \). (We will denote the extension by the same letter \( v \)). Observe that Kelvin transformation is its own inverse (or, as they say, is an involution), i.e., we may express \( u \) in terms of \( v \) via the same formula

\[
u(y) = \frac{|y|^{2-n}}{|y|^{2|\alpha|}} \sum_{\alpha} c_{\alpha} y^{\alpha} \quad \text{in a neighborhood of infinity. In particular,}
\]

\[
u(y) = c_0 |y|^{2-n} + O(|y|^{1-n}) \quad \text{as } |y| \to +\infty.
\]

### 5.11. Problems

**5.1.** Find the Fourier transforms of the following distributions:

a) \( \frac{1}{x+a} \);

b) \( \delta(r - r_0), r = |x|, x \in \mathbb{R}^3; \)

c) \( \frac{\sin(r_0|x|)}{|x|}, x \in \mathbb{R}^3; \)

d) \( x^k \theta(x), k \in \mathbb{Z}_+. \)

**5.2.** Find the inverse Fourier transforms of the following distributions:

a) \( 1/|\xi|^2, \xi \in \mathbb{R}^n, n \geq 3; \)

b) \( \frac{1}{|\xi|^2 + k^2}, \xi \in \mathbb{R}^3, k > 0; \)

c) \( \frac{1}{|\xi|^{2-k^2}} \pm i^0, \xi \in \mathbb{R}^3, k > 0. \)

**5.3.** Using the result of Problem 5.2 b), find the fundamental solution for the operator \(-\Delta + k^2 \) in \( \mathbb{R}^3 \).
5.4. We will say that the convolution \( f \ast g \) of two distributions \( f, g \in \mathcal{D}'(\mathbb{R}^n) \) is defined if for any sequence \( \chi_k \in \mathcal{D}(\mathbb{R}^n) \) such that \( \chi_k(x) = 1 \) for \( |x| \leq k, k = 1, 2, \ldots \) the limit \( f \ast g = \lim_{k \to \infty} f \ast (\chi_k g) \) exists in \( \mathcal{D}'(\mathbb{R}^n) \) and does not depend on the choice of the sequence \( \chi_k \). Prove that if \( f, g \in \mathcal{D}'(\mathbb{R}) \) and the supports of \( f \) and \( g \) belong to \([0, +\infty)\), then \( f \ast g \) is defined, and addition and convolution define on the set \( \mathcal{D}'(\mathbb{R}_+) \) of distributions with supports in \([0, +\infty)\) the structure of an associative commutative ring with unit.

5.5. Prove that \( \delta^{(k)}(x) \ast f(x) = f^{(k)}(x) \) for \( f \in \mathcal{D}'(\mathbb{R}^1) \).

5.6. Find the limits \( \lim_{t \to -\infty} e^{ixt}x^+ \) and \( \lim_{t \to +\infty} e^{ixt}x^+ \) in \( \mathcal{D}'(\mathbb{R}^1) \).

5.7. For \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) define an integration operator setting \( I f = \int_0^\infty f(\xi)d\xi \). Prove that \( I f = f \ast \theta \) and \( I \) can be extended by continuity to \( \mathcal{D}'(\mathbb{R}_+) \).

5.8. Using the associativity of the convolution, prove that
\[
I^m f = \left[ \frac{1}{(m-1)!} x^{m-1} \theta(x) \right] \ast f,
\]
where \( m = 1, 2, \ldots \), for \( f \in \mathcal{D}'(\mathbb{R}_+) \).

5.9. Set \( x_+^\lambda = \theta(x)x^\lambda \) for Re \( \lambda > -1 \), so that \( x_+^\lambda \in L^1_{\text{loc}}(\mathbb{R}_+) \) and, in particular, \( x_+^1 \in \mathcal{D}'(\mathbb{R}_+) \). Define the fractional integral \( I^\lambda \) by the formula
\[
I^\lambda f = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \ast f, \quad \text{Re} \ \lambda > 0.
\]
Prove that \( I^\lambda I^\mu = I^{\lambda+\mu}, \ \text{Re} \ \lambda > 0, \ \text{Re} \ \mu > 0 \).

5.10. For any \( \lambda \in \mathbb{C} \) define the fractional integral \( I^\lambda \) by
\[
I^\lambda f = \left( \frac{d}{dx} \right)^k I^{\lambda+k} f \quad \text{for} \ f \in \mathcal{D}'(\mathbb{R}_+),
\]
where \( k \in \mathbb{Z}_+ \) is such that Re \( \lambda + k > 0 \). Prove that \( I^\lambda \) is well defined, i.e., the result does not depend on the choice of \( k \in \mathbb{Z}_+ \). Prove that \( I^\lambda I^\mu = I^{\lambda+\mu} \) for any \( \lambda, \mu \in \mathbb{C} \). Prove that \( I^0 f = f \) and \( I^{-k} f = f^{(k)} \) for \( k \in \mathbb{Z}_+ \) and any \( f \in \mathcal{D}'(\mathbb{R}_+) \). So \( I^{-\lambda} f \) for \( \lambda > 0 \) can be considered a fractional derivative (of order \( \lambda \)) of the distribution \( f \).
Harmonic functions

6.1. Mean-value theorems for harmonic functions

Harmonic function in an open set \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) is a (real-valued or complex-valued) function \( u \in C^2(\Omega) \) satisfying the Laplace equation

\[
\Delta u = 0,
\]

where \( \Delta \) is the standard Laplacian on \( \mathbb{R}^n \). As we already know from Ch. 5, any harmonic function is in fact in \( C^\infty(\Omega) \), and even real-analytic in \( \Omega \). This is true even if we a priori assume only that \( u \in \mathcal{D}'(\Omega) \), i.e. if \( u \) is a distribution in \( \Omega \).

For \( n = 1 \) the harmonicity of a function on an interval \((a, b)\) means that this function is linear. Though in this case all statements, discussed here, are also true, they are usually trivial, so we can concentrate on the case \( n \geq 2 \).

In this Chapter we will provide a sample of classical properties of harmonic functions and related topics. All proofs are relatively elementary: we do not use advanced functional analysis, such as Sobolev spaces (they will be discussed later in this book). Among other things, we illustrate, how a local equation (6.1) may imply a range of global properties of solutions.

We will start with a property of harmonic functions which is called mean value theorem, or Gauss’s law of the arithmetic mean.

**Theorem 6.1.** (Mean Value Theorem for spheres) Let \( B = B_r(x) \) be an open ball in \( \mathbb{R}^n \) with the center at \( x \) and the radius \( r > 0 \), \( S = S_r(x) \) is its boundary (the sphere with the same center and radius), \( \bar{B} \) is the closure
of $B$ (so $\bar{B} = B \cup S$). Let $u$ be a continuous function on $\bar{B}$ such that its restriction to $B$ is harmonic. Then the average of $u$ over $S$ equals $u(x)$ (the value of $u$ at the center). In other words,

$$u(x) = \frac{1}{\text{vol}(S)} \int_S u(y)dS = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{S_r(x)} u(y)dS_y,$$

where $dS_y$ means the standard area element on the sphere $S$, $\sigma_{n-1}$ is the $(n-1)$-volume of the unit sphere.

**Proof.** Let us rewrite the average of $u$ in (6.2) as an average over the unit sphere. To this end, denote $z = r^{-1}(y - x)$ and make the change of variables $y \mapsto z$ in the last integral. We get then

$$\frac{1}{\sigma_{n-1} r^{n-1}} \int_{S_r(x)} u(y)dS_y = \frac{1}{\sigma_{n-1}} \int_{|z|=1} u(x + rz)dS_z.$$ 

Now, if we replace $r$ by a variable $\rho$, $0 \leq \rho \leq r$, then the integral becomes a function $I = I(\rho)$ on $[0, r]$, which is continuous due to the uniform continuity of $u$ on $B$. Clearly, $I(0) = u(x)$. Our theorem will be proved if we show that $I(\rho) = \text{const}$, or, equivalently, that $I'(\rho) \equiv 0$ on $[0, r)$. To this end, let us calculate the derivative $I'(\rho)$ by differentiating under the integral sign (which is clearly possible):

$$I'(\rho) = \frac{1}{\sigma_{n-1}} \int_{|z|=1} \frac{d}{d\rho} u(x + \rho z)dS_z = \frac{1}{\sigma_{n-1}} \int_{|z|=1} \frac{\partial u}{\partial n}(x + \rho z)dS_z,$$

where $\hat{n}_\rho$ is the outgoing unit normal to the sphere $S_\rho(x)$, at the point $x + \rho z$. We will see that the last integral vanishes, if we apply the following corollary of Green’s formula (4.32), which is valid for any bounded domain with a $C^2$ boundary

$$\int_\Omega \Delta u \ dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} dS$$

(take $v \equiv 1$ in (4.32)). For our purposes we should take $\Omega = B_\rho(x)$. Since $u$ is harmonic, we come to the desired conclusion that $I'(\rho) = 0$.

The sphere can be replaced by ball in Theorem 6.1:

**Theorem 6.2.** (Mean Value Theorem for balls) In the same notation as in Theorem 6.1, let $u$ be a continuous function on $\bar{B} = B_\rho(x)$ such that its restriction to $B$ is harmonic. Then the average of $u$ over $B$ equals $u(x)$. In other words,

$$u(x) = \frac{1}{\text{vol}(B)} \int_B u(y)dy = \frac{n}{\sigma_{n-1} r^n} \int_{B_\rho(x)} u(y)dy,$$
where \( dy \) means the standard volume element (Lebesgue measure) on \( \mathbb{R}^n \).

**Proof.** Let \( \rho \) denote the distance from \( x \), considered as a function on \( B_r(x) \), as in the proof of Theorem 6.2. Since \( |\text{grad}\, \rho| = 1 \), we can replace integration over \( B_r(x) \) by two subsequent integrations: first integrate over the spheres \( S_\rho(x) \) for every \( \rho \in [0, r] \), then integrate with respect to the radii \( \rho \) (no additional Jacobian type factor is needed). Using Theorem 6.1, we obtain:

\[
\int_{B_r(x)} u(y)dy = \int_0^r \int_{S_\rho(x)} u(y) dS_\rho(y) d\rho = \int_0^r u(x) \text{vol}_{n-1}(S_\rho(x)) d\rho = u(x) \text{vol}(B_r(x)),
\]

which is what we need. \( \square \)

**Remark 6.3.** Proof of Theorem 6.1 shows a way how a local equation can imply a global property (in this case, the mean value property (6.2)). It suggests that we can try to connect the desired global property with a trivial one by a path (in our case, the path is parametrized by the radius \( \rho \in [0, r] \)) and expand the validity of the desired property along this path, starting from a trivial case (\( \rho = 0 \) in our case), relying on the local information when moving by an infinitesimal step with respect to the parameter.

The proof of Theorem 6.2 does not belong to this type, because it simply deduces a global result from another global one.

### 6.2. The maximum principle

In this section we will often assume for simplicity of formulations, that \( \Omega \) is an open connected subset in \( \mathbb{R}^n \). (In case if \( \Omega \) is not connected, we can then apply the results on each connected component separately.) Recall that connectedness of an open set \( \Omega \subset \mathbb{R}^n \) can be defined by one of the following equivalent conditions:

(i) \( \Omega \) can not be presented as a union of two disjoint non-empty open subsets.

(ii) If \( F \subset \Omega \) is non-empty open and at the same time closed (in \( \Omega \)), then \( F = \Omega \).

(iii) Every two points \( x, y \in \Omega \) can be connected by a continuous path in \( \Omega \) (that is, there exists a continuous map \( \gamma : [0, 1] \to \Omega \), such that \( \gamma(0) = x \) and \( \gamma(1) = y \)).

(For the proofs see e.g. Armstrong [1], Sect. 3.5.)
The maximum principle for harmonic functions can be formulated as follows:

**Theorem 6.4. (The Maximum Principle for Harmonic Functions)** Let \( u \) be a real-valued harmonic function in a connected open set \( \Omega \subset \mathbb{R}^n \). Then \( u \) cannot have a global maximum in \( \Omega \) unless it is constant. In other words, if \( x_0 \in \Omega \) and \( u(x_0) \geq u(x) \) for all \( x \in \Omega \), then \( u(x) = u(x_0) \) for all \( x \in \Omega \).

**Proof.** Let \( x_0 \in \Omega \) be a point of global maximum for \( u \), \( B = B_r(x_0) \) is a ball in \( \mathbb{R}^n \) (with the center at \( x_0 \)), such that \( \bar{B} \subset \Omega \) (this is true if the radius \( r \) is sufficiently small). Now, applying (6.4) with \( x = x_0 \), we obtain

\[
0 = u(x_0) - \frac{1}{\text{vol}(B)} \int_B u(y) dy = \frac{1}{\text{vol}(B)} \int_B (u(x_0) - u(y)) dy.
\]

Since \( u(x_0) - u(y) \geq 0 \) for all \( y \in \Omega \), we conclude that \( u(y) = u(x_0) \) for all \( y \in B \).

Let us consider the set

\[
M = \{ y \in \Omega | u(y) = u(x_0) \}.
\]

As we just observed, for every point \( y \in M \) there exist a ball \( B \) centered at \( y \), such that \( B \subset M \). This means that \( M \) is open. But it is obviously closed too. It is non-empty because it contains \( x_0 \). Since \( \Omega \) is connected, we conclude that \( M = \Omega \) which means that \( u(y) = u(x_0) = \text{const} \) for all \( y \in \Omega \).

**Corollary 6.5.** Let \( u \) be a real-valued harmonic function in a connected open set \( \Omega \subset \mathbb{R}^n \). Then \( u \) cannot have a global minimum in \( \Omega \) unless it is constant.

**Proof.** Applying Theorem 6.4 to \( -u \), we obtain the desired result. \( \square \)

**Corollary 6.6.** Let \( u \) be a complex-valued harmonic function in a connected open set \( \Omega \subset \mathbb{R}^n \). Then \( |u| \) cannot have a global maximum in \( \Omega \) unless \( u = \text{const} \).

**Proof.** Assume that \( x_0 \in \Omega \) and \( |u(x_0)| \geq |u(x)| \) for all \( x \in \Omega \). If \( |u(x_0)| = 0 \), then \( u \equiv 0 \) and the result is obvious. Otherwise, present \( u(x_0) = |u(x_0)| e^{i\varphi_0} \), \( 0 \leq \varphi_0 < 2\pi \), and consider a new function \( \tilde{u} = e^{-i\varphi_0}u \), which is also harmonic. Denoting the real and imaginary parts of \( \tilde{u} \) by \( v, w \), so that \( \tilde{u} = v + iw \), we see that \( v \) and \( w \) are real-valued harmonic functions on \( \Omega \), and for all \( x \in \Omega \)

\[
v(x_0) = |u(x_0)| \geq |u(x)| = |\tilde{u}(x)| = \sqrt{v^2(x) + w^2(x)} \geq v(x).
\]
Applying Theorem 6.4 to \( v \), we see that \( v(x) = v(x_0) = \text{const} \) for all \( x \in \Omega \), and all the inequalities in (6.5) become equalities. In particular, we have \( v(x_0) = \sqrt{v^2(x_0) + w^2(x)} \), hence \( w \equiv 0 \), which implies the desired result. \( \square \)

For bounded domains \( \Omega \) we can take into account what happens near the boundary to arrive to other important facts.

**Theorem 6.7.** 1) Let \( u \) be a real-valued harmonic function in an open bounded \( \Omega \subset \mathbb{R}^n \), and let \( u \) can be extended to a continuous function (still denoted by \( u \)) on the closure \( \overline{\Omega} \). Then for any \( x \in \overline{\Omega} \)

\[
(6.6) \quad \min_{\partial \Omega} u \leq u(x) \leq \max_{\partial \Omega} u,
\]

where \( \partial \Omega = \overline{\Omega} \setminus \Omega \) is the boundary of \( \Omega \).

2) Let \( u \) be a complex-valued harmonic function on \( \Omega \), which can be extended to a continuous function on \( \overline{\Omega} \). Then for any \( x \in \overline{\Omega} \)

\[
(6.7) \quad |u(x)| \leq \max_{\partial \Omega} |u|.
\]

3) The inequalities in (6.6) and (6.7) become strict for \( x \) in those connected components of \( \Omega \) where \( u \) is not constant.

**Proof.** Since \( \overline{\Omega} \) is compact, \( u \) should attain its maximum and minimum somewhere on \( \overline{\Omega} \). But due to Theorem 6.4 and Corollary 6.5 this may happen only on the boundary or in the connected components of \( \Omega \) where \( u \) is constant. The desired results for a real-valued \( u \) immediately follow. For complex-valued \( u \) the same argument works if we use Corollary 6.6 instead of Theorem 6.4 and Corollary 6.5. \( \square \)

**Corollary 6.8.** Let \( \Omega \) be an open bounded set in \( \mathbb{R}^n \), \( u, v \) be harmonic functions, which can be extended to continuous functions on \( \overline{\Omega} \), and such that \( u = v \) on \( \partial \Omega \). Then \( u \equiv v \) everywhere in \( \Omega \).

**Proof.** Taking \( w = u - v \), we see that \( w \) is harmonic and \( w = 0 \) on \( \partial \Omega \). Now it follows from (6.7) that \( w \equiv 0 \) in \( \Omega \). \( \square \)

This corollary means that any harmonic function \( u \) in \( \Omega \), which can be extended to a continuous function on \( \overline{\Omega} \), is uniquely defined by its restriction to the boundary \( \partial \Omega \). This fact can be reformulated as uniqueness of solution for the following boundary-value problem for the Laplace equation, which is
called Dirichlet’s problem:

\[
\begin{align*}
\Delta u(x) &= 0, \quad x \in \Omega; \\
u|_{\partial \Omega} &= \varphi(x), \quad x \in \partial \Omega.
\end{align*}
\]

Here, for the start, let us assume that \( \varphi \in C(\partial \Omega) \), and we are interested in a classical solution \( u \), that is \( u \in C(\overline{\Omega}) \), \( u \) is harmonic in \( \Omega \). We just proved that for any bounded \( \Omega \) and any continuous \( \varphi \), there may be at most one solution of this problem. It is natural to ask whether such solution exists for a fixed bounded \( \Omega \) but with any given \( \varphi \). This proves to be true if the boundary \( \partial \Omega \) is smooth or piecewise smooth. But in general case the complete answer, characterizing open sets \( \Omega \) for which the Dirichlet problem is solvable for any continuous \( \varphi \), is highly non-trivial. This answer was given by N. Wiener, and it is formulated in terms of capacity, the notion which appeared in electrostatics and was brought by Wiener to mathematics in the 1920’s. The whole story is beyond the scope of this course and can be found in the books on Potential theory (see e.g. Wermer [33]).

However, it occurs that in some weaker sense the problem is always solvable. The appropriate language and necessary technique (functional analysis, Sobolev spaces) will be discussed in Chapter 8.

### 6.3. Dirichlet’s boundary-value problem

To require the existence and uniqueness of solution of a mathematical problem which appears as a model of a real-life situation, is quite natural. (For example, if it is not unique, then we should additionally investigate which one corresponds to reality.) But this is not enough. Another reasonable requirement is well-posedness of the problem which means that the solution continuously depends on the data of the problem, where the continuity is understood with respect to some natural topologies in the spaces of data and solutions.

The well-posedness of the Dirichlet problem in the sup-norm immediately follows from the maximum principle. More precisely, we have

**Corollary 6.9.** Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \). Let \( u_1, u_2 \) be harmonic functions in \( \Omega \), which can be extended to continuous functions on \( \overline{\Omega} \), \( \varphi_j = u_j|_{\partial \Omega}, j = 1, 2 \). Then

\[
\max_{\Omega} |u_1 - u_2| \leq \max_{\partial \Omega} |\varphi_1 - \varphi_2|.
\]

**Proof.** The result immediately follows from (6.7) in Theorem 6.7. □
Now assume that \( \{u_k\} \) is a sequence of harmonic functions in \( \Omega \), continuously extendable to \( \overline{\Omega} \), whose restrictions on \( \partial \Omega \) (call them \( \varphi_k \)) converge uniformly to \( \varphi \) on \( \partial \Omega \) as \( k \to \infty \). Then \( \{\varphi_k\} \) is a Cauchy sequence in the Banach space \( C(\partial \Omega) \) with the sup-norm. It follows from (6.9) that \( \{u_k\} \) is a Cauchy sequence in \( C(\overline{\Omega}) \). Hence, \( u_k \to u \in C(\overline{\Omega}) \) with respect to this norm. It is easy to see that the limit function \( u \) is also harmonic. Indeed, the uniform convergence in \( \Omega \) obviously implies convergence \( u_k \to u \) in distributions (that is, in \( \mathcal{D}'(\Omega) \)), hence \( \Delta u_k \to \Delta u \) and \( \Delta u = 0 \) in \( \Omega \), so \( u \) is harmonic. ▲

\[ \nabla \]

The Laplace equation and the Dirichlet problem appear in electrostatics as the equation and boundary-value problem for a potential in a domain \( \Omega \) which is free of electric charges. It is also natural to consider a modification which allows charges, including point charges and also the ones distributed in some volume, or on a surface. To this end we should replace the Laplace equation by the Poisson equation \( \Delta u = f \) (see (4.41)), where \( f \) is an appropriate distribution. Allowing charges inside \( \Omega \), we can consider Dirichlet’s problem for the Poisson equation:

\[
\begin{cases}
\Delta u(x) = f(x), & x \in \Omega; \\
u|_{\partial \Omega} = \varphi(x), & x \in \partial \Omega.
\end{cases}
\]

(6.10)

To make it precise, we can assume that \( u \) is a distribution in \( \Omega \), such that it is in fact a continuous function in a neighborhood \( U \) of the boundary \( \partial \Omega \) in \( \overline{\Omega} \). We can also take \( f \in \mathcal{D}'(\Omega) \), and \( \varphi \in C(\partial \Omega) \). Then we can extend the result of Corollary 6.8 to the problem (6.10):

**Corollary 6.10.** Assume that \( u, v \in \mathcal{D}'(\Omega) \) are solutions of the boundary-value problem (6.10) with the same \( f \) and \( \varphi \). Then \( u = v \) in \( \Omega \).

**Proof.** Clearly, \( w = u - v \) is a distribution in \( \Omega \), satisfying \( \Delta w = 0 \). Therefore, by Theorem 5.14, \( w \) is a harmonic function in \( \Omega \). By our assumptions, \( w \) is continuous up to the boundary and \( w = 0 \) on \( \partial \Omega \). Hence, \( w \equiv 0 \) by the maximum principle. □

### 6.4. Hadamard’s example

Now I will explain an example, due to Jacques Hadamard, which shows that there exist uniquely solvable problems which are not well-posed with respect to natural norms, i.e. small perturbations of the data may lead to uncontrollably large perturbations of the solution.
Let us consider the initial-value problem for the Laplacian in a two-dimensional strip $\Pi_T = \{(x, y) \in \mathbb{R}^2 | x \in \mathbb{R}, 0 \leq y < T\}$:

\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Pi_T, \\
u(x, 0) &= \varphi(x), \quad \frac{\partial u}{\partial y}(x, 0) = \psi(x) \quad \text{for all} \quad x \in \mathbb{R},
\end{align*}
\]

where we assume that $u \in C^2(\Pi_T)$, $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$.

**PICTURE BELOW HADAMARD (6.12)**

A solution of this problem may not exist (see Problem 6.5), but if it exists, then it is unique. To prove the uniqueness we will need the following

**Lemma 6.11.** Let $\Omega = \mathbb{R}^n$ with the coordinates $(x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Assume that a function $u \in C^1(\Omega)$ is harmonic in the open sets

$\Omega^\pm = \{x \in \Omega | \pm x_n > 0\}$,

which are obtained as two subsets of $\Omega$ lying above and below the hyperplane $x_n = 0$ respectively. Then $u$ can be extended from $u \in L^1_{loc}(\Omega^+ \cup \Omega^-)$ to a harmonic function in $\Omega$.

**Proof.** It suffices to prove that $\Delta u = 0$ in the sense of distributions, i.e. that for any test function $\Phi \in C_0^\infty(\Omega)$ we have

\[
\int_{\Omega} u \Delta \Phi \, dx = 0.
\]

Note that it is sufficient to prove this equality locally, in a small neighborhood of any chosen point $x_0 \in \Omega$, that is for the test functions $\Phi$ supported in such a neighborhood. Since this is obvious for the points $x_0$ which do not belong to the separating hyperplane, we can assume that $x_0 = (x'_0, 0)$, and that $\Omega$ is a small ball centered at such a point. Then, due to Green’s formula (4.32) we can write:

\[
\int_{\Omega_-} u \Delta \Phi \, dx = \int_{\Omega \cap \{x_n = 0\}} \left( u \frac{\partial \Phi}{\partial x_n} - \Phi \frac{\partial u}{\partial x_n} \right) \, dx',
\]

where we used that $\Delta u = 0$ in $\Omega_-$. Similarly,

\[
\int_{\Omega_+} u \Delta \Phi \, dx = -\int_{\Omega \cap \{x_n = 0\}} \left( u \frac{\partial \Phi}{\partial x_n} - \Phi \frac{\partial u}{\partial x_n} \right) \, dx',
\]

where the minus sign in the right hand side appears due to the fact that the direction of the $x_n$-axis is for $\Omega_-$ and incoming for $\Omega_+$. We also used the continuity of $u$ and $\partial u/\partial x_n$ which implies that these quantities are the same from both sides of $\{x_n = 0\}$. Adding these equalities leads to (6.12), which ends the proof. \(\square\)
6.4. Hadamard’s example

Now let us turn to the proof of the uniqueness in (6.11). For two solutions
\( u_1, u_2 \), their difference \( u = u_1 - u_2 \) is a solution of (6.11) with \( \varphi \equiv \psi \equiv 0 \). Extend \( u \) by reflection with respect to the line \( y = 0 \), as an odd function
with respect to \( y \), to a function \( \tilde{u} \) on a doubled strip

\[
\tilde{\Pi}_T = \{(x, y) \in \mathbb{R}^2 | x \in \mathbb{R}, y \in (-T, T)\}.
\]

Namely, define \( \tilde{u}(x, y) = -u(x, -y) \) for \( -T < y < 0 \). Due to the initial
conditions on \( u \) we clearly have \( \tilde{u} \in C^1(\tilde{\Pi}_T) \). Also, \( \tilde{u} \) is harmonic in the
divided strip \( \tilde{\Pi}_T \setminus \{(x, 0) | x \in \mathbb{R} \} \), i.e. everywhere in \( \tilde{\Pi}_T \) except at the \( x \)-axis. It follows from Lemma 6.11 that \( \tilde{u} \) is harmonic in \( \tilde{\Pi}_T \). Therefore,
it is real-analytic in \( \tilde{\Pi}_T \) by Theorem 5.17 (or Corollary 5.18). But it is
also easy to check that the equation \( \Delta \tilde{u} = 0 \) and the initial conditions
\( \tilde{u}|_{y=0} = \partial \tilde{u}/\partial y|_{y=0} = 0 \) imply by induction in \( |\alpha| \), that \( \partial^\alpha \tilde{u}|_{y=0} = 0 \) for every
2-multiindex \( \alpha \). Hence, by the Taylor expansion of \( \tilde{u} \) at the point \( x = y = 0 \),
we conclude that \( \tilde{u} \equiv 0 \) in \( \tilde{\Pi}_T \), which proves the desired uniquenss.

Remark 6.12. Uniqueness of \( C^m \)-solution of the initial-value problem for
any order \( m \) equation with real-analytic coefficients and the initial condi-
tions on any non-characteristic hypersurface, is the content of a Holmgren
theorem (see e.g. John [15], Sect.3.5). This theorem is deduced, by a du-
ality argument, from the existence of solution for an adjoint problem in
real-analytics functions, which is the content of the Cauchy-Kovalevsky the-
orem (see Theorem 5.17). In the next chapter we will explain Holmgren’s
method in detail but for the heat equation.

Now we can proceed to the Hadamard example . Let us try to find the
exponential solutions of the Laplace equation \( \Delta u(x, y) = 0 \), that is, solutions
of the form

\[
u(x, y) = e^{kx+ly},\]

where \( k, l \in \mathbb{C} \). Substituting this function into the Laplace equation, we
immediately see that it is a solution if and only if \( k^2 + l^2 = 0 \). For example
we can take \( k = i\omega, l = \omega \) for a real \( \omega \), to get

\[
u(x, y) = e^{i\omega x} e^{\omega y}.
\]

We can also take the real part of this solution which is

\[
u(x, y) = u_\omega(x, y) = e^{\omega y} \cos \omega x.
\]
Let us try to understand whether we can use the $C^k$-norms on functions $\varphi \in C^k(\mathbb{R})$:

\begin{equation}
\|\varphi\|_{(k)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}} |\partial^\alpha \varphi(x)|,
\end{equation}

for the initial data of the problem (6.11), and investigate whether a value of the solution $u$ at a point can be done small if the $C^k$-norms of the initial data are small. Equivalently, we can ask whether a small deviation of $\varphi, \psi$ from the equilibrium values $\varphi \equiv \psi \equiv 0$ necessarily leads to small perturbation of the solution $u$. More precisely, fixing an arbitrarily large $k$ and choosing, for example, a point $(0, \sigma) \in \Pi_T$, where $\sigma > 0$, we would like to know the answer to the following question: is it true, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\varphi\|_{(k)} + \|\psi\|_{(k)} < \delta$ implies that $|u(0, \sigma)| < \varepsilon$?

The answer to this question is emphatically negative. Indeed, for $u = u_\omega$ with $\omega > 1$, we obtain $\varphi(x) = \cos \omega x$, $\psi(x) = -\omega \sin \omega x$, and

\begin{equation}
\|\varphi\|_{(k)} + \|\psi\|_{(k)} \leq 2(k + 1)\omega^{k+1}.
\end{equation}

At the same time $u(0, \sigma) = e^{\omega \sigma}$, which is much bigger if $\omega$ is large. In particular, taking instead the solution $\tilde{u}_\omega = e^{-\omega \sigma/2}u_\omega$ with the corresponding initial data

\begin{equation}
\varphi_\omega(x) = e^{-\omega \sigma/2} \cos \omega x, \quad \psi_\omega(x) = -e^{-\omega \sigma/2} \omega \sin \omega x,
\end{equation}

we see that $u(0, \sigma) = e^{\omega \sigma/2}$ exponentially grows as $\omega \to +\infty$, whereas

\begin{equation}
\|\varphi_\omega\|_{(k)} + \|\psi_\omega\|_{(k)} \leq 2(k + 1)\omega^{k+1} e^{-\omega \sigma/2},
\end{equation}

which exponentially decays as $\omega \to +\infty$. In particular, now the negative answer to the question becomes obvious: the value of $u$ at a point can not be controlled by the $C^k$-norms of the initial data.

As we have seen in Chapter 2, the situation is quite opposite for the wave equation (also in dimension 2): there the values of $u$ are estimated by the sup-norms of the initial data. In fact, this holds for hyperbolic equations in higher dimensions too. We will see this for the wave equation in Chapter 10.

### 6.5. Green’s function: first steps

In Sect. 3.5 we introduced Green’s function of the Sturm-Liouville operator on $[0, l]$ with vanishing boundary conditions at the ends 0 and $l$. Here we will define a similar object for the Dirichlet problem (6.8) in a bounded open set $\Omega \subset \mathbb{R}^n$. Let us start with the fundamental solution $E_n = E_n(x)$ of the
Laplacian $\Delta$ in $\mathbb{R}^n$. Then $\mathcal{E}_n = \mathcal{E}_n(x)$ is a distribution in $\mathbb{R}^n$, which has a point singularity at $0 \in \mathbb{R}^n$ and satisfies the equation $\Delta \mathcal{E}_n(x) = \delta(x)$. Such a solution is defined up to adding a constant. To choose a definite copy in case $n \geq 3$, we can first require that $\mathcal{E}_n(x) \to 0$ as $|x| \to \infty$. Then it is easy to show that $\mathcal{E}_n(x)$ is given by (4.38).

In case $n = 2$ the fundamental solution can not decay at infinity, but we can select a radially symmetric solution by (4.39), which looks like an arbitrary choice of the constant, but note that $\nabla \mathcal{E}_2(x)$ does not depend upon this choice.

Note that for any $n \geq 2$ the fundamental solution $\mathcal{E}_n(x)$ is harmonic for all $x \neq 0$.

In this section we will define, study and apply Green’s function $G = G(x, y)$, where $x, y \in \Omega$, $x \neq y$. More precisely, we will assume $G$ to be a distribution on $\Omega \times \Omega$, which is defined by a locally integrable function $G = G(x, y)$. For every fixed $y \in \Omega$, it should be harmonic with respect to $x \in \Omega$ if $x \neq y$, and we also impose the boundary condition $G(x, y) = 0$ if $x$ or $y$ is in the boundary of $\Omega$.

As usual, two such functions are identified if they coincide almost everywhere in $\Omega \times \Omega$, or, equivalently, if the corresponding distributions coincide in $\Omega \times \Omega$. (For simplicity we will simply say that $G$ is locally integrable.)

Summarizing, we get

**Definition 6.13.** The Green function $G = G(x, y)$ of the Laplacian $\Delta$ in $\Omega$ with the Dirichlet boundary condition, is a locally integrable solution of the equation

$$\Delta_x G(x, y) = \delta(x - y),$$

for any fixed $y \in \Omega$ and any $x \in \Omega$, $x \neq y$, satisfying the boundary condition

$$G(x, y) = 0, \quad y \in \Omega, \quad x \in \Omega, \quad x \neq y, \text{ and } y \in \partial \Omega \text{ or } x \in \partial \Omega.$$

Define also $\Omega_y = \{(z, y) \in \Omega \times \Omega | z \in \Omega \}$ for every fixed $y \in \Omega$, $\Delta_x$ is the Laplacian with respect to $x$ in $\Omega_y$.

**Comments to Definition 6.13.**

1) The equation (6.14) with the Dirichlet boundary condition corresponds to a physical requirement that the resulting solution represents the electrostatic Coulomb interaction between electric charges, so that the charge
located at \( y \in \Omega \), has the potential \( G(\cdot, y) \) with the fixed \( y \). Due to the symmetry principle, which we will prove later, \( G(x, y) = G(y, x) \) for all \( x, y \in \Omega \), hence the same will hold for any charge located at \( x \) and tested at \( y \).

2) The Dirichlet boundary condition means that \( G(x, y) = 0 \) when \( x \) or \( y \) is at the boundary of \( \Omega \).

3) In more detail, we will assume that \( \Omega \) and \( \partial \Omega \) are in \( C^2(\bar{\Omega}) \) and \( \partial C^2(\bar{\Omega}) \) respectively. The Green function \( G \) will have a singularity on the diagonal \( \text{diag}(\Omega \times \Omega) = \{(x, x) | x \in \Omega\} \subset \Omega \times \Omega \), and it is in \( C^2 \) outside the diagonal. (Sometimes we will need \( C^3 \) instead of \( C^2 \).) The singularities of \( G(x, y) \) near the diagonal \( x = y \) are similar to the simplest singularity in case \( \Omega = \mathbb{R}^n \).

The simplest singularity with \( \Omega = \mathbb{R}^n \) has an unbounded \( \Omega \), but it is useful to keep its structure in mind. In this case, normalizing it to have charge 1 and singularity at \( x = y \), we have, by definition,

\[
G(x, y) = G_0(x, y) = \mathcal{E}_n(x - y), \quad x, y \in \mathbb{R}^n,
\]

where \( \mathcal{E}_n = \mathcal{E}_n(x) \) is the standard fundamental solution of the Laplacian \( \Delta \) in \( \mathbb{R}^n \), \( x, y \in \mathbb{R}^n \), \( n \geq 2 \) (see (4.38), (4.39)).

In an analogy with the case \( n = 1 \), we will define Green’s function of the Laplacian in \( \Omega \subset \mathbb{R}^n \) for \( n \geq 2 \) and

\[
G = G(x, y), \quad x, y \in \Omega, \quad \text{by the formula}
\]

\[
G(x, y) := \mathcal{E}_n(x - y) + g(x, y),
\]

where \( g = g(x, y) \) is called the remainder, which is expected to satisfy the following properties:

\[
Gu(x) = \int_{\Omega} G(x, y) u(y) dy, \quad u \in C_0^\infty(\Omega), \quad x \in \Omega.
\]
Green’s function: first steps

(6.5) \( f(x) = \delta_y(x) = \delta(x-y) \) and \( \varphi \equiv 0 \). Here \( y \in \Omega \) is considered as a parameter. So we will have then

\[
\begin{cases}
\Delta_{x,y} G(x, y) = \delta(x-y), & x, y \in \Omega; \\
\Delta_{x,y} G(x, y)|_{x \in \partial \Omega} = 0.
\end{cases}
\]

As follows from Corollary 6.10, if for some \( y \in \Omega \) such a distribution \( G(\cdot, y) \) exists, then it is unique. Concerning the existence, we have

**Lemma 6.14.** Assume that the Dirichlet problem (6.8) has a solution for any \( \varphi \in C(\partial \Omega) \). Then Green’s function \( G = G(\cdot, y) \) exists for all \( y \in \Omega \).

**Proof.** Let us start with a particular solution \( u(x) \) of the equation \( \Delta u(x) = \delta(x-y) \), which is \( u(x) = \mathcal{E}_n(x-y) \), where \( \mathcal{E}_n(x) \) is the fundamental solution of the Laplacian \( \Delta \), given by (4.38) and (4.39). Then the remainder

\[(6.20) \quad g(x, y) := G(x, y) - \mathcal{E}_n(x-y)\]

should be harmonic with respect to \( x \) and satisfy the boundary condition \( g(x, y) = -\mathcal{E}_n(x-y) \) for \( x \in \partial \Omega \). This means that \( g(\cdot, y) \) is a solution of the Dirichlet problem (6.8) with the boundary condition \( \varphi(x) = -\mathcal{E}_n(x-y) \), for any fixed \( y \in \Omega \). Due to our assumption about the solvability of Dirichlet’s problem in \( \Omega \), such a function \( g \) exists, which ends the proof. \( \square \)

**Remark 6.15.** In fact, for the existence of Green’s function \( G(x, y) \) we need the existence of a solution of Dirichlet’s problem (6.8) not for all \( \varphi \in C(\partial \Omega) \), but only for specific functions of the form \( \varphi_y = -\mathcal{E}_n(\cdot-y)|_{\partial \Omega} \), \( y \in \Omega \). In the future, when we mention Green’s function, then we assume that it exists in the sense of this remark.

Let us consider the set \( L \) of all \( \varphi \in C(\partial \Omega) \), such that Dirichlet’s problem (6.8) has a solution \( u = u_{\varphi} \in C(\bar{\Omega}) \). Clearly, \( L \) is a linear subspace in \( C(\partial \Omega) \). Moreover, \( L \) is closed in the Banach space \( C(\partial \Omega) \). The linear operator \( S : L \to C(\bar{\Omega}) \), \( \varphi \mapsto u_{\varphi} \), is bounded. (It has norm 1 due to the maximum principle.) The existence of Green’s function is equivalent to inclusion \( \{ \varphi_{y} | y \in \Omega \} \subset L \), which is equivalent to saying that the closed linear span of the set of all \( \varphi_y \)’s is a subset of \( L \).

We will later refer to \( S \) as the solving operator for Dirichlet’s problem (6.8).

Let us investigate the behavior of \( G(x, y) \) as a function of \((x, y)\) on \( \bar{\Omega} \times \Omega \). Since we know \( \mathcal{E}_n \) explicitly, it is sufficient to investigate the behavior of the remainder \( g(x, y) \) in (6.20).
Lemma 6.16. Let $G$ and $g$ be the corresponding Green function and the remainder (as in (6.20)). Then

(a) $G(x, y)$ can be extended to a continuous function of $(x, y)$ on the set $ar{\Omega} \times \Omega \setminus \text{diag}(\Omega \times \Omega) \subset \mathbb{R}^n$, where $\text{diag}(\Omega \times \Omega) = \{(x, x) | x \in \Omega\}$.

(b) The function $x \mapsto G(x, y)$ is harmonic in $\Omega \setminus \{y\}$ for every fixed $y \in \Omega$.

(c) $g(x, y)$ can be extended to a continuous function of $(x, y)$ on the set $ar{\Omega} \times \Omega \setminus \text{diag}(\Omega \times \Omega) \subset \mathbb{R}^n$.

(d) The function $x \mapsto g(x, y)$ is harmonic in $\Omega \setminus \{y\}$ for every fixed $y \in \Omega$.

Proof. The result follows if we note that the function $x, y \mapsto \mathcal{E}_n(x - y)$ on $\bar{\Omega} \times \Omega \setminus \text{diag}(\Omega \times \Omega)$, together with the functions $G, g$, posses the continuity and harmonicity properties indicated in the formulation of the Lemma for $G(x, y)$ and $g(x, y)$. It remains to note that all the properties are linear. □

Corollary 6.17. Under the same conditions

$$G = G(x, y) \in C(\bar{\Omega} \times \Omega \setminus \text{diag}(\Omega \times \Omega)).$$

6.6. Symmetry and regularity of Green’s functions

Up to this moment we did not impose any smoothness restrictions upon $\Omega$, except that it is an open bounded subset in $\mathbb{R}^n$. But such restrictions are needed if we want to know more about the behavior of the solutions of Dirichlet’s problem near the boundary, aside of just being continuous. We will introduce such restrictions in the future. Now we will start from a conditional result.

Proposition 6.18. Let $\Omega$ be a bounded domain, $\partial \Omega \in C^2$. Assume that Green’s function $G = G(x, y)$ for Dirichlet’s problem (for the Laplacian $\Delta$ in $\Omega$) exists and, besides,

(6.21) $G(\cdot, y) \in C^2(\bar{\Omega} \setminus \{y\})$ for every $y \in \Omega$,

or, equivalently,

(6.22) $g(\cdot, y) \in C^2(\bar{\Omega} \setminus \{y\})$ for every $y \in \Omega$,

where $g$ is the remainder, $g(x, y) = G(x, y) - \mathcal{E}_n(x - y)$. Then both $G$ and $g$ are symmetric, that is, $G(x, y) = G(y, x)$ for all $x, y \in \Omega, x \neq y$, and $g(x, y) = g(y, x)$ for all $x, y \in \Omega, x \neq y$. 
6.6. Symmetry and regularity of Green’s function

Proof. Let us recall Green’s formula

\begin{equation}
\int_{\Omega} (u \Delta v - v \Delta u) dx = \int_{\partial \Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,
\end{equation}

where \( u, v \in C^2(\Omega) \) (see (4.32) and the discussion which follows it). Let us try to apply (6.23) to the functions \( u = G(\cdot, y) \) and \( v = G(\cdot, z) \), where \( y, z \in \Omega \), \( y \neq z \). Due to the boundary conditions on \( G \), we have \( u|_{\partial \Omega} = v|_{\partial \Omega} = 0 \), hence the right hand side vanishes (for any fixed \( y, z \in \Omega \)). We also have \( \Delta u = \delta(\cdot - y) \), \( \Delta v = \delta(\cdot - z) \), so the left hand side equals \( u(z) - v(y) = G(z, y) - G(y, z) \), if the integrals there are understood in the sense of distributions. Therefore, if Green’s formula can be extended to such \( u, v \), then we will get the desired symmetry of Green’s function.

Note that the only reason that (6.23) can not be directly applied to our \( u \) and \( v \) is that they have singularities at the points \( y \) and \( z \) respectively. So a natural way is to approximate \( u, v \) by functions \( u_\varepsilon, v_\varepsilon \in C^2(\Omega) \), where \( \varepsilon \in (0, \varepsilon_0) \), so that all the terms in (6.23) are limits, as \( \varepsilon \downarrow 0 \), of the corresponding terms in this formula with \( u, v \) replaced by \( u_\varepsilon, v_\varepsilon \). To this end, let us take an arbitrary function \( \chi \in C^\infty(\mathbb{R}^n) \), such that \( \chi = 0 \) in \( B_{1/2}(0) \), \( \chi = 1 \) in \( \mathbb{R}^n \setminus B_1(0) \), \( 0 \leq \chi(x) \leq 1 \) for all \( x \in \mathbb{R}^n \). Then take \( \chi_\varepsilon(x) = \chi(\varepsilon^{-1}x) \). Finally, define \( u_\varepsilon(x) = \chi_\varepsilon(x - y)u(x) \) and \( v_\varepsilon(x) = \chi_\varepsilon(x - z)v(x) \).

We will choose \( \varepsilon_0 > 0 \) so that the closed balls \( B_{\varepsilon_0}(y) \) and \( B_{\varepsilon_0}(z) \) are disjoint subsets in \( \Omega \) (that is \( \varepsilon_0 < |y - z|/2 \), and \( \varepsilon_0 \) is strictly less than the distances from \( y \) and \( z \) to \( \partial \Omega \)).

Clearly, \( |u_\varepsilon(x)| \leq |u(x)| \) and \( |v_\varepsilon(x)| \leq |v(x)| \) for all \( x \in \overline{\Omega} \). Also, \( u_\varepsilon = u \) in \( \Omega \setminus B_{\varepsilon}(y) \), and \( v_\varepsilon = v \) in \( \Omega \setminus B_{\varepsilon}(z) \). In particular, these equalities hold in a neighborhood of \( \partial \Omega \).

Note that \( u \) and \( v \) are locally integrable near their singularities, hence integrable in \( \Omega \). By the Lebesgue dominated convergence theorem \( u_\varepsilon \to u \), \( v_\varepsilon \to v \), as \( \varepsilon \downarrow 0 \), in \( L^1(\Omega) \), hence, in \( \mathcal{D}'(\Omega) \). It follows that \( \Delta u_\varepsilon \to \Delta u \) and \( \Delta v_\varepsilon \to \Delta v \) in \( \mathcal{D}'(\Omega) \).

Since \( u_\varepsilon = u \in C^\infty \) in a neighborhood of \( \text{supp} \Delta v \) (which is a subset of \( B_{\varepsilon_0}(z) \)), we clearly have

\[
\int_{\Omega} u_\varepsilon \Delta v_\varepsilon dx = \langle \Delta v_\varepsilon, u_\varepsilon \rangle = \langle \Delta v_\varepsilon, u \rangle \to \langle \Delta v, u \rangle = \langle \delta(\cdot - z), u \rangle = u(z) = G(z, y).
\]

Here the angle brackets \( \langle \cdot, \cdot \rangle \) stand for the duality between \( \mathcal{E}'(\Omega_1) \) and \( C^\infty(\Omega_1) \), where \( \Omega_1 \) is an appropriate open subset in \( \Omega \).
Similarly, we obtain that
\[
\int_{\Omega} v_\varepsilon \Delta u_\varepsilon \, dx = \langle \Delta u_\varepsilon, v_\varepsilon \rangle = \langle \Delta u, v \rangle = \langle \delta(\cdot - y), v \rangle = v(y) = G(y, z).
\]

Now Green’s formula (6.23) with \( u = G(\cdot, y), \ v = G(\cdot, z) \) (which is equivalent to the symmetry of Green’s function) can be obtained if we replace \( u, v \) by \( u_\varepsilon, v_\varepsilon \) respectively, and then take limit as \( \varepsilon \downarrow 0 \). \( \square \)

**Remark 6.19.** The symmetry of Green’s function can be interpreted as the self-adjointness of an operator in \( L^2(\Omega) \) which is inverse to the Laplacian with Dirichlet’s boundary condition. The boundary condition defines an appropriate domain of the Laplacian, and Green’s function becomes the Schwartz kernel of this operator. We will discuss this approach in later chapters. It requires a powerful technique of the Sobolev spaces, which will be discussed in Chapter 8. It is noticeable that this technique does not require any restrictions on \( \Omega \) except its boundedness.

\[\blacktriangledown\] Now we will discuss how to relieve Proposition 6.18 of the redundant regularity requirements. This becomes possible if we introduce more delicate measure of smoothness than simply \( C^k \): *Hölder spaces*, which allow a precise descriptions of the smoothness of solutions of the Dirichlet problem for the Poisson equation (6.10).

First, for functions \( u \in C^k(\Omega), \ k = 0, 1, 2, \ldots, \) introduce the norm
\[
\|u\|_{(k)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha u(x)|,
\]
defined whenever \( u \in C^k(\Omega) \) and the right hand side makes sense (i.e. if it is finite). (We already used this norm on \( \mathbb{R} \) – see (6.13).) With this norm, \( C^k(\Omega) \) becomes a Banach space.

Now for any \( \gamma, \ 0 < \gamma < 1, \) and any function \( u \in C(\bar{\Omega}) \) introduce a seminorm
\[
|u|_\gamma = \sup_{x, y \in \Omega, \ x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.
\]
If this seminorm is finite, then we will say that this function is a *Hölder function* with the exponent \( \gamma \). (If the same is true with \( \gamma = 1 \), then one says that \( u \) is a *Lipschitz function*, but we will rarely use Lipschitz functions in this book.) We will denote the space of all Hölder functions with the exponent \( \gamma \) by \( C^\gamma(\Omega) \).
Note that replacing $\Omega$ by $\bar{\Omega}$ may change the seminorm $|u|_{(\gamma)}$, though this does not happen in case of a good boundary (say, of class $C^1$ and non-selfintersecting).

Similarly, for any integer $k \geq 0$ and $\gamma \in (0, 1)$ define the space $C^{k,\gamma}(\Omega)$, consisting of functions $u$ on $\Omega$, such that $u \in C^k(\bar{\Omega})$ and $\partial^\alpha u \in C^\gamma(\Omega)$ for all multiindices $\alpha$ with $|\alpha| = k$. It is a Banach space with the norm
\[
\|u\|_{k,\gamma} = \|u\|_{(k)} + \sum_{\{\alpha: |\alpha|=k\}} |\partial^\alpha u|_{(\gamma)}.
\]

We can also use two versions of the Hölder spaces: $C^{k,\gamma}(\Omega)$ and $C^{k,\gamma}(\bar{\Omega})$, where $\bar{\Omega}$ means the closure of an open set $\Omega \subset \mathbb{R}^n$, $k$ is a non-negative integer, $\gamma \in (0, 1)$. But for practical purposes only the first version is needed. (For example, these two versions coincide for bounded domains $\Omega$ with a sufficiently smooth boundary.)

The spaces $C^{k,\gamma}$ can be used to measure the smoothness of the boundary as well. Namely, we will say that $\Omega$ has a $C^{k,\gamma}$ boundary and write $\partial \Omega \in C^{k,\gamma}$ if $\partial \Omega \in C^k$ and the functions $h(x')$, which enter into local representations of the boundary as the graph $x_n = h(x')$ (see Definition 4.8), are in fact in $C^{k,\gamma}$ on every compact subset of the domain where they are defined.

If $k \geq 1$, then it is easy to see that the space $C^{k,\gamma}(\bar{\Omega})$ and $C^{k,\gamma}$ smoothness of $\partial \Omega$ are invariant under $C^{k,\gamma}$ diffeomorphisms. (This follows from the statements formulated in Problem 6.6.)

In fact, on the $C^{k,\gamma}$ boundary $\partial \Omega$ itself, for $k \geq 1$, there exists a structure of a $C^{k,\gamma}$ manifold, which is given by the sets of local coordinates related to each other in the usual way, by $C^{k,\gamma}$ diffeomorphisms between two domains in $\mathbb{R}^{n-1}$, which are images of the intersection of the two coordinate neighborhoods in $\bar{\Omega}$. Such local coordinates can be taken to be $x'$ for the point $(x', h(x'))$ in the local graph presentation of $\partial \Omega$.

Let us return to (6.10) which is the Dirichlet problem for the Poisson equation.

Here is a remarkable theorem by J. Schauder, which gives a precise solvability and regularity, and also shows that the Hölder spaces $C^{k,\gamma}$ work good for the problem (6.10).

**Theorem 6.20.** (a) (Existence and uniqueness) Assume that $k$ is an integer, $k \geq 0$, $\gamma \in (0, 1)$, $\partial \Omega \in C^{k+2,\gamma}$, $f \in C^{k,\gamma}(\Omega)$ and $\varphi \in C^{k+2,\gamma}(\partial \bar{\Omega})$. Then a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the boundary-value problem (6.10) exists and is unique.
(b) (Regularity) In fact, the unique solution \( u \) belongs to \( C^{k+2,\gamma}(\Omega) \).

(c) (Isomorphism Operator) The operator

\[
A : C^{k+2,\gamma}(\Omega) \to C^{k,\gamma}(\Omega) \times C^{k+2,\gamma}(\partial\Omega), \quad Au = \{\Delta u, u_{|\partial\Omega}\},
\]

is a linear topological isomorphism (that is, a bounded linear operator, which has a bounded inverse).

**Corollary 6.21.** Assume that \( \partial\Omega \in C^\infty \), \( f \in C^\infty(\Omega) \) and \( \varphi \in C^\infty(\partial\Omega) \). Then the problem (6.10) has a solution \( u \in C^\infty(\Omega) \). A solution \( u \) of this problem, such that \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \), is unique and therefore in fact we have \( u \in C^\infty(\bar{\Omega}) \). Moreover, the operator

\[
A : C^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega), \quad Au = \{\Delta u, u_{|\partial\Omega}\},
\]

is a linear topological isomorphism of Fréchet spaces.

**Proof.** The statement immediately follows from Theorem 6.20 if we note that for any fixed \( \gamma \in (0,1) \), the space \( C^\infty(\bar{\Omega}) \) is the intersection of all spaces \( C^{k,\gamma}(\Omega) \), \( k = 0,1,2,\ldots \), and the Fréchet topology in \( C^\infty(\bar{\Omega}) \) can be defined by norms \( \| \cdot \|_{k,\gamma} \). \( \square \)

**Remarks.** 1. I will not provide a proof of Theorem 6.20. The interested reader can find complete proofs in the monographs by Ladyzhenskaya and Ural’tseva [17], Ch. III; Gilbarg and Trudinger [8], Ch. 6. In both books a substantial extension of this result to general second-order elliptic operators is proved. (It is also due to Schauder.)

2. All three statements of Theorem 6.20 fail in the usual classes \( C^k(\Omega) \) even locally. Namely, the equation \( \Delta u = f \) with \( f \in C(\Omega) \) may fail to have a solution \( u \in C^2(\Omega_1) \) where \( \Omega_1 \) is a fixed subdomain of \( \Omega \). Also \( f \in C(\bar{\Omega}) \) does not even imply \( u \in C^2(\Omega) \). However, it is possible to deduce a weaker existence and (separately) regularity statements in terms of the usual \( C^k \) spaces.

Here is the regularity result:

**Corollary 6.22.** Assume that \( k \geq 1 \) is an integer, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), such that \( \partial\Omega \in C^{k+2} \). Let \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \) be a solution of (6.10) with \( f = \Delta u \in C^k(\bar{\Omega}) \) and \( \varphi = u_{|\partial\Omega} \in C^{k+2}(\partial\Omega) \). Then \( u \in C^{k+1}(\Omega) \).

In particular, if \( \partial\Omega \in C^{k+2} \), \( u \in C(\bar{\Omega}) \) is harmonic in \( \Omega \), and \( \varphi = u_{|\partial\Omega} \in C^{k+2}(\partial\Omega) \), then \( u \in C^{k+1}(\bar{\Omega}) \).
Proof. Choose \( \gamma \in (0, 1) \), and note that the given conditions imply the inclusions \( \partial \Omega \in C^{k+1,\gamma} \), \( f \in C^{k-1,\gamma}(\Omega) \), and \( \varphi \in C^{k+2,\gamma}(\partial \Omega) \). Then Theorem 6.20 implies that \( u \in C^{k+1,\gamma}((\Omega) \), hence \( u \in C^{k+1}(\Omega) \).

About the existence result see Problem 6.8. ▲

Now let us return to the existence, smoothness and symmetry of Green’s function.

**Theorem 6.23.** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), such that
\( \partial \Omega \in C^{2,\gamma} \), where \( \gamma \in (0, 1) \). Then the following statements hold:

(a) There exists Green’s function \( G = G(x,y) \) of the operator \( \Delta \) with Dirichlet’s boundary conditions in \( \Omega \) (see (6.19));

(b) For the remainder \( g(x,y) = G(x,y) - \mathcal{E}_n(x-y) \) we have \( g \in C^{2,\gamma}(\Omega \times \Omega \setminus \text{diag}(\Omega \times \Omega)) \), and \( g(\cdot,y) \in C^{2,\gamma}(\Omega \setminus \{y\}) \) for any \( y \in \Omega \);  

(c) Green’s function is symmetric, i.e. \( G(x,y) = G(y,x) \) for all \( x,y \in \Omega \), \( x \neq y \).

Proof. Due to Theorem 6.20, for any fixed \( y \in \Omega \), we can find \( g = g(x,y) \) such that \( g(\cdot,y) \in C^{2,\gamma}(\Omega \setminus \{y\}) \), by solving Dirichlet’s problem (6.8) with \( \varphi(x) = -\mathcal{E}_n(x-y) \). Therefore, Green’s function exists, and (a), (b) are satisfied. Now (c) follows by a straightforward application of Proposition 6.18. □

**Corollary 6.24.** Assume \( \partial \Omega \in C^{2,\gamma} \). Then \( g(x,y) \) can be extended to a continuous function in \( \bar{\Omega} \times \Omega \setminus \text{diag}(\Omega \times \Omega) \).

Proof. The statement immediately follows from Lemma 6.16, and the symmetry of Green’s function and the remainder \( g(x,y) \).

\[ 6.7. \text{Green’s function and Dirichlet’s problem} \]

We will start by deducing an integral formula for the solution \( u \) of the non-homogeneous Dirichlet problem (6.10) through the right hand side \( f \) and the boundary data \( \varphi \), assuming that we know Green’s function \( G = G(x,y) \) of \( \Omega \). (In particular, we assume that Green’s function exists.)

Let us also assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary. (More precisely, it suffices to assume that \( \partial \Omega \) is piecewise \( C^2 \).) This will guarantee that the Green-Stokes type formulas work for \( \Omega \). Specifically, we need the formula (6.23) to hold for all functions \( u, v \in C^2(\Omega) \).

We will also require that \( G \in C^2 \) on \( \{(x,y) \in (\bar{\Omega} \times \Omega) \cup (\Omega \times \bar{\Omega}) \mid x \neq y\} \).
All conditions above (both on \( \Omega \) and on Green’s function) will be satisfied if we impose a slightly stronger Hölder regularity conditions on the boundary \( \partial \Omega \) and the given data:

\[
\partial \Omega \in C^{2,\gamma}, \quad f \in C^\gamma(\Omega), \quad \varphi \in C^{2,\gamma}(\partial \Omega).
\]

(Here \( 0 < \gamma < 1 \); see a discussion about necessary definitions and facts above, before Theorem 6.20.) By Schauder’s Theorem 6.20 the conditions above imply that \( u \in C^{2,\gamma}(\Omega) \).

Starting with the general Green’s formula (6.23), take an arbitrary \( u = u(x), \ u \in C^2(\bar{\Omega}) \), and \( v = v(\cdot) = v(x) = G(\cdot, y) = v(x, y) \), where the notation \( G(\cdot, y) \) means that \( G(x, y) \) is considered as a function (or distribution) of \( x \in \Omega \) for a fixed \( y \in \bar{\Omega} \) (that is, \( y \) is considered to be a parameter).

Note, however, that the function \( v = G(\cdot, y) \) has a singularity at \( x = y \), so that \( v \) is not in \( C^2 \) (with respect to \( x \)) in any neighborhood of \( y \). But (6.23) still holds if \( v \) has a point singularity at \( y \in \Omega \), as is clear from the arguments given above in the proof of Proposition 6.18, which are nothing else but just an appropriate approximation of \( v \) by smooth functions.

To substitute the functions \( u \) and \( v \) into (6.23), note that

\[
\Delta v = \Delta_x G(x, y) = \delta(x-y), \quad \int_{\Omega} u(x) \Delta_x G(x, y) \, dx = \int_{\Omega} u(x) \delta(x-y) \, dx = u(y).
\]

So the left hand side of (6.23) has the form

\[
u(y) - \int_{\Omega} G(x, y) \Delta u(x) \, dx,
\]

whereas the right hand side is

\[-\int_{\partial \Omega} \frac{\partial G(x, y)}{\partial n_x} u(x) \, dS_x,\]

because \( G(x, y) = 0 \) if \( x \in \partial \Omega \). Equality of these two expressions, after interchanging of \( x \) and \( y \), together with the symmetry of Green’s function imply the following Proposition:

**Proposition 6.25.** Let us assume that \( \Omega \) is a bounded open set with a piecewise smooth \((C^2)\) boundary \( \partial \Omega \) in \( \mathbb{R}^n \), and the Dirichlet Problem for the Laplacian in \( \Omega \) has Green’s function \( G = G(x, y) \), which is defined for all \((x, y) \in (\Omega \times \Omega) \cup (\Omega \times \bar{\Omega})\). Then for every \( u \in C^2(\bar{\Omega}) \) the following integral representation holds:

\[
(6.27) \quad u(x) = \int_{\Omega} G(x, y) \Delta u(y) \, dy - \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial n_y} u(y) \, dS_y, \quad x \in \bar{\Omega}.
\]
In other words, if \( u \in C^2(\overline{\Omega}) \) is the (unique) solution of the Dirichlet problem (6.10), that is \( \Delta u = f \) in \( \Omega \) and \( u|_{\partial\Omega} = \varphi \) on \( \partial\Omega \), then

\[
(6.28) \quad u(x) = \int_{\Omega} G(x,y)f(y)dy - \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n_y} \varphi(y) dS_y, \quad x \in \Omega.
\]

This Proposition gives an explicit presentation of the solution for the Dirichlet problem in terms of Green’s function. Two important particular cases are obtained when \( \varphi \equiv 0 \) or \( f \equiv 0 \): if \( \varphi \equiv 0 \), then

\[
(6.29) \quad u(x) = \int_{\Omega} G(x,y)f(y)dy, \quad x \in \Omega;
\]

if \( f \equiv 0 \), then

\[
(6.30) \quad u(x) = -\int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n_y} \varphi(y) dS_y, \quad x \in \Omega.
\]

**Corollary 6.26.** Let \( u \in C^2(\overline{\Omega}) \) satisfy the same conditions as in (6.30) above. In particular, it suffices for \( \Omega \) to be bounded, for \( \partial\Omega \) to be in \( C^2 \) and for \( u \) to be harmonic in \( \Omega \). Then \( u \) satisfies (6.30).

**Remark 6.27.** Here are more details about the physical meaning of the formulas (6.28)–(6.30) in electrostatics. (About the simplest notions of electrostatics see physics textbooks, e.g. [7], Vol. 2, Chapters 4–8, and also Chapters 4 and 5 here.)

The situation to describe is a unit charge sitting in a body \( \overline{\Omega} \), which is bounded by a thin closed conductive shell \( \partial\Omega \). Let us explain what these words actually mean.

Consider \( \overline{\Omega} \) to be a closed vessel bounded by a thin shell \( \partial\Omega \), made of a conductive material (e.g. a metal), which means a material where the electrons can move freely. Equivalently, we may say that there are no forces acting on the electrons, hence the potential is locally constant on the boundary \( \partial\Omega \), i.e. it is constant on each of the connected components of \( \partial\Omega \). By adding an appropriate solution of the Dirichlet problem in \( \Omega \), we can force these constants to vanish. Then all connected components of the boundary \( \partial\Omega \) are grounded, i.e. connected with the Earth by wires. It is assumed that the Earth has a potential which vanishes on its surface, hence the boundary condition \( u|_{\partial\Omega} = 0 \) is satisfied. For simplicity, assume that the interior of the vessel is made of a dielectric (that is non-conducting) material.

Now let us consider Green’s function \( G(x,y) \), \( x, y \in \Omega \), \( x \neq y \). It is interpreted as a static (i.e. time-independent) potential of a unit charge
located at \( y \in \tilde{\Omega} \), whereas the potential itself is \( u = u(x) = G(., y) \). We are interested in the electric field in the vessel \( \Omega \), which has no charges inside \( \Omega \) (except at \( y \)). The boundary condition \( G(x, y) = 0 \) for \( x \in \partial \Omega \), requires \( \Omega \) to be a closed vessel bounded by a thin conductive shell \( \partial \Omega \).

Green’s function \( G(x, y) \) replaces the fundamental solution \( \mathcal{E}_n(x - y) \), when the whole space \( \mathbb{R}^n \) is replaced by the domain \( \Omega \).

For a system of charges the potentials add up by the superposition principle, and for a distributed charge the potential of this charge becomes an integral, as in (6.28)–(6.30).

Note that for \( n \geq 3 \) all potentials with compactly supported charges tend to 0 as \( |x| \to \infty \). This reflects a physically natural requirement that the potential \( u \) should be 0 at infinity (up to an additive constant, which can be chosen so that, indeed, \( u(\infty) = 0 \)).

By the superposition principle, the integral in (6.29) is the potential \( u \) of the system of charges which are distributed in \( \Omega \) with the density \( f(y) \) (or \( f(y)dy \)). Assuming that \( u|_{\partial \Omega} = 0 \) we obtain the (unique) solution of the boundary-value problem

\[
\begin{aligned}
\Delta u(x) &= f(x), \quad x \in \Omega; \\
u|_{\partial \Omega} &= 0.
\end{aligned}
\]

If the boundary is not connected, then the condition, that the boundary is conductive, means that the potential \( u \) is constant on each of the connected components of the boundary. This is a weaker condition compared with the vanishing condition for Green’s function \( G(x, y) = 0 \), \( x \in \partial \Omega \).

Physically, in the same electrostatic model, the property of the boundary to be conductive is equivalent to one that no energy is needed to transport a small charge along the boundary. (The latter statement follows from the fact that the energy needed to transport a small charge along a curve is equal to the difference between the potentials of the ends of the curve, where the value of the potential at the end of the curve should be subtracted from the value at the beginning of the curve.)

**Remark 6.28.** Turning back to the integral formulas (6.28)–(6.30), we see that the solution of the Dirichlet boundary-value problem for the Poisson equation (6.31), under the smoothness requirements described above, is given by (6.29), if such a solution exists. Moreover, it suffices to require \( f \in C(\Omega) \), \( \partial \Omega \in C^2 \) and \( u \in C^2(\Omega) \) to assure the possibility of substitution of \( u \) to (6.31), and directly check both conditions. (Even this is often not
needed; the conditions are satisfied automatically, if all data are sufficiently smooth, e.g. \( f \in C^1(\partial \Omega), \partial \Omega \in C^2 \) and \( u \in C^3(\bar{\Omega}) \). But it may happen that such solution does not always exist.

Similar statements are true for the boundary-value problem (6.8) and the integral representation (6.30).

As to the more general problem (6.10), its solution can be presented as a sum \( u = u_f + u_\varphi \), where \( u_f, u_\varphi \) are solutions of (6.10) with \( \varphi = 0 \) and \( f = 0 \) respectively. So to solve the boundary-value problem (6.10) we need to add corresponding solutions of (6.29) and (6.30).

**Remark 6.29.** If Green’s function is known, the solution \( u \) of the boundary-value problem (6.31) is found by a simple integration from the right hand side \( f \), according to (6.29). But to find solution of (6.8), given Green’s function and the boundary data \( \varphi \), we still need to calculate the normal derivative \( \partial G/\partial \bar{n}_y \) and then integrate over the boundary. This significant difference appears because the solution operator needs to produce a function of \( n \) variables from a function of \( n - 1 \) variables. Here one degree of freedom gets consumed by the requirement for the solution \( u \) to be a harmonic function in \( \Omega \).

Note also that the right hand side of the formula (6.30) is nothing else but the double layer potential (see Ch. 5).

### 6.8. Explicit formulas for Green’s functions

The cases, where Green’s function can be found explicitly and as a finite expression, are rare. Nevertheless, they are worth learning because they show what can you expect in the general case. We will discuss method of images (sometimes called method of reflections).

#### 6.8.1. Method of images (generalities).

This method allows to find explicitly Green’s function and formulas for solving the Dirichlet problem for the Laplace operator in some important examples.

Let us first explain the idea of the method for domains in \( \mathbb{R}^n \) where \( n \geq 3 \). It is convenient to consider some unbounded domains too. In this case, the condition \( G(x, y)|_{x \in \partial \Omega} = 0 \) may be not sufficient to select a unique Green function, and we will impose additionally the following natural condition:

**Decay Assumption.** For every fixed \( y \in \Omega \),

\[
G(x, y) \to 0 \quad \text{as} \quad |x| \to \infty, \quad x \in \bar{\Omega}.
\]
Lemma 6.30. If $n \geq 3$, then there is at most one Green’s function $G = G(x, y)$, existing for a given domain $\Omega \subset \mathbb{R}^n$ (possibly unbounded) and satisfying (6.32).

Proof. Let $B_R(0)$ be the open ball with the radius $R > 0$ and center at 0 in $\mathbb{R}^n$. Denote $\Omega_R = \Omega \cap B_R(0)$ which is a bounded open set in $\mathbb{R}^n$. (It may happen that $\Omega_R$ is empty.) Clearly, union of all $\Omega_R$ with $R > 0$ is $\Omega$.

Now, let $u = u(x) = G_1(x, y) - G_2(x, y)$, where $G_1, G_2$ are two Green functions (of $\Delta$ in $\Omega$), both satisfying (6.32), $y \in \Omega$ is fixed. Then $u$ is harmonic in $\Omega$, and $u|_{\partial \Omega} = 0$. If $\Omega$ was bounded, then we would be able to conclude that $u \equiv 0$ due to the maximum principle (6.7). Unfortunately, this argument does not work for unbounded $\Omega$. To modify it, note first, that $\partial \Omega_R \subset (\partial \Omega \cap \partial B_R(0)) \cup (\bar{\Omega} \cap \partial B_R(0))$, hence,

$$\max_{\partial \Omega_R} |u| \leq \max_{\partial \Omega \cap \partial B_R(0)} |u| + \max_{\bar{\Omega} \cap \partial B_R(0)} |u| = \max_{\partial B_R(0)} |u|,$$

because $u|_{\partial \Omega} = 0$. It follows, again by the maximum principle, that

$$\max_{\bar{\Omega}_R} |u| \leq \max_{\bar{\Omega} \cap \partial B_R(0)} |u|.$$

Due to (6.32), the right hand side tends to 0 as $R \to \infty$, whereas the left hand side increases as $R$ increases. It follows that $u \equiv 0$ in $\bar{\Omega}_R$ for sufficiently large $R$, which implies that $u \equiv 0$ in $\Omega$. □

Main idea

Consider the fundamental solution $\mathcal{E}_n = \mathcal{E}_n(z)$ of the Laplacian $\Delta$ in $\mathbb{R}^n$ (see (4.38),(4.39)), $n \geq 3$. Due to the equation $\Delta_z \mathcal{E}(z) = \delta(z)$ and Liouville’s Theorem 5.28, this is the only fundamental solution with $\mathcal{E}_n(z) \to 0$ as $|z| \to \infty$. It is clear from the definition of Green’s function, that for $\Omega = \mathbb{R}^n$ we have

$$G(x, y) = G_{\mathbb{R}^n}(x, y) = \mathcal{E}_n(x - y).$$

The idea of the method of images is to construct Green’s function of a domain $\Omega \subset \mathbb{R}^n$ by taking the fundamental solution of the Laplacian in $\mathbb{R}^n$ and adding a correction term (6.20) which has the form of the potential of a point charged particle or a system of such particles. These additional charged particles must be located outside $\bar{\Omega}$ (to make the additional sum of the potentials harmonic in $\Omega$ and continuous in $\bar{\Omega}$), and also assure that the total potential of all charges vanishes on the boundary of $\Omega$. Thus, the total potential will be Green’s function of $\Omega$. ▲
We will start from the case when the correcting term is the potential of just one charged particle. Let the original charged particle be located at \( y \in \Omega \) and carry a unit charge \( q_0 = 1 \), so it creates an electric field with the potential

\[
u_0 = u_0(\cdot, y) = \mathcal{E}_n(\cdot - y) = \mathcal{E}_n(x - y), \quad x, y \in \mathbb{R}^n, \ x \neq y.\]

Let us denote by \( y^* \) the location of the correcting charge, which means \( y^* = y^*(y) \in \mathbb{R}^n \setminus \bar{\Omega} \), and let us denote the corresponding correcting charge by \( q^* = q^*(y) \in \mathbb{R} \). The total potential of the system of these two charged particles is

\[
u = u(x) = u(x, y) = \mathcal{E}_n(x - y) + q^*(y)\mathcal{E}_n(x - y^*),
\]

where \( y \) is considered a parameter (a fixed vector which can be chosen arbitrary). For Green’s function of \( \Omega \) to coincide with the right hand side of (6.33), we need to require that \( u|_{x \in \partial \Omega} = 0 \), for all \( x \in \partial \Omega, y \in \bar{\Omega}, x \neq y \). In other words,

\[
\mathcal{E}_n(x - y) + q^*(y)\mathcal{E}_n(x - y^*) = 0, \quad x \in \partial \Omega, \ y \in \bar{\Omega}.
\]

In this case, Green’s function of \( \Omega \) is

\[
G(x, y) = \mathcal{E}_n(x - y) + q^*(y)\mathcal{E}_n(x - y^*),
\]

where \( x, y \in \Omega, x \neq y, y^* \in \mathbb{R}^n \setminus \bar{\Omega} \).

**Remark 6.31.** In case of unbounded domain the above discussed conditions on \( \Omega \) should be supplemented by the condition (6.32) which requires vanishing of \( u \) at infinity. We will abbreviate it as \( u(\infty, y) = 0 \) (for all \( y \in \Omega \)).

Note that \( q^*(y) < 0 \) for all \( y \in \Omega \), due to (6.34).

**Remark 6.32.** Since \( \mathcal{E}_n(x - y) \) is actually well-defined and real-analytic for all \( x \neq y \), the sum in (6.35) is naturally defined for all \( x, y, y^* \in \mathbb{R}^n \), such that \( x \neq y, x \neq y^* \), whatever is \( q^* \in \mathbb{R} \) (which can be also variable). In particular, we can take charges, which are arbitrarily located with respect to \( \Omega \). This means physically that we actually consider the charges at \( y, y^* \) to be much larger by absolute value than the charge at \( x \), which is therefore considered as a “small, weakly charged test particle”. The influence of the latter upon the charges at \( y, y^* \) is considered negligible and therefore ignored. Also, the diameters of all particles are considered negligible if compared with the distances between them, so they are thought to be point particles.
6. Harmonic functions

In (classical) gravitation the equations are the same as in electrostatics, except all charges are positive (they are masses) and they only attract each other (there is no repulsion), this would mean that we have 3 “point particles”: two heavy stars which are considered “attached” to their places in the sky, or, more precisely, moving very slowly, compared with the motion of the third particle which is a light planet, located at $x$ and moving under the influence of the stars gravitation attraction. The problem of describing the motion of such a system of three gravitating point masses, is referred to, in the language of Celestial Mechanics, as one of the simplified (restricted) versions of so-called classical (as opposed to quantum) three-body problem.

**Description of admissible domains**

Let us attempt to calculate Green’s function by finding $y^* = y^*(y)$ and $q^* = q^*(y)$, which are both functions on $\Omega$ with values in $\mathbb{R}^n \setminus \overline{\Omega}$ and $\mathbb{R} \setminus \{0\}$, respectively, and satisfy (6.34). We will see that such functions exist only for very special domains $\Omega$, and we will completely describe such domains.

The explicit form of $E_n$ allows to rewrite (6.34) in the form

$$(6.36) \quad |x - y|^2 - n + q^*|x - y^*|^2 = 0, \quad x \in \partial \Omega,$$

or, equivalently (for $x \neq y$),

$$(6.37) \quad \frac{|x - y^*|^2}{|x - y|^2} = (-q^*)^{2/(n-2)}, \quad x \in \partial \Omega, y \in \Omega,$$

which implies, in particular, that the left-hand side is independent of $x$. Let us denote

$$(6.38) \quad Q^* = Q^*(q^*) = (-q^*)^{-2/(n-2)}, \quad Q^* > 0,$$

which is equivalent to

$$(6.39) \quad q^* = -(Q^*)^{-(n-2)/2}.$$ 

Now we can rewrite (6.37) as

$$(6.40) \quad |x - y|^2 - Q^*|x - y^*|^2 = 0, \quad x \in \partial \Omega, y \in \Omega.$$ 

**Proposition 6.33.** Under the conditions above, let $u = u(x) = u(x, y) = u(x, y, x^*, y^*, q^*)$ be a two-charges potential (as in (6.33)) such that $Q^* > 0$, $Q^* \neq 1$. Then the zero nodal domain $u^{-1}(0)$ is a geometric sphere.

It is a hyperplane in case $Q^* = 1$. 


Proof. It follows that for every fixed \( y \in \Omega \), the set of all \( x \), satisfying (6.40), is a quadric, which depends upon \( y \in \Omega \). It is easy to see by looking at the principal part of the quadric equation (second degree terms with respect to \( x \)), that this quadric is a sphere or a hyperplane. Let us recall that under our assumptions \( Q^* > 0 \). We will prove below that it is a sphere if \( Q^* \neq 1 \) and a hyperplane if \( Q^* = 1 \). We will determine the precise parameters of the quadric later. □

Remark 6.34. We can simplify the equation of the quadric (6.40) (or (6.37)). Let us choose \( y \in \Omega \), and find \( y^* = y^*(y) \) and \( q^* = q^*(y) \). (We still have to specify how to do this explicitly.) Then \( y^* \in A \). Take \( x_0 \) to be the intersection point of \( \partial \Omega \) with the straight line segment \([y, y^*]\), connecting \( y \) and \( y^* \). This segment intersects \( \partial \Omega \) by exactly one intersection point, because the points \( y, y^* \) are located on the different sides of \( \partial \Omega \) which is a sphere or a hyperplane. Then the right hand side of (6.37) is equal to its value for \( x = x_0 \), which means that

\[
(6.41) \quad \frac{|x - y^*|}{|x - y|} = \frac{|x_0 - y^*|}{|x_0 - y|}, \quad x \in \partial \Omega, \ y \in \Omega.
\]

Here we will usually take \( y \in A \setminus \{0\} \), but sometimes it is better to use one-point compactification \( S^n = S^n \cup \{\infty\} \), extending \( y^* = y^*(y) \) to a continuous map \( S^n \to S^n \).

It is interesting that (6.41) does not contain dimension \( n \).

\( \mathbf{\Box} \) Denoting the standard scalar product of vectors \( x, y \in A \) by \( x \cdot y \), we obtain

\[
0 = Q^*(|x|^2 - 2x \cdot y^* + |y^*|^2) - (|x|^2 - 2x \cdot y + |y|^2)
\]

\[
= (Q^* - 1)|x|^2 - 2x \cdot (Q^*y^* - y) + (Q^*|y^*|^2 - |y|^2).
\]

If \( Q^* \neq 1 \), then completing the square in the right hand side gives

\[
0 = (Q^* - 1) \left[ |x|^2 - \frac{2x \cdot (Q^*y^* - y)}{Q^* - 1} + \frac{|Q^*y^* - y|^2}{(Q^* - 1)^2} \right]
\]

\[
- \frac{|Q^*y^* - y|^2 - (Q^*|y^*|^2 - |y|^2)(Q^* - 1)}{Q^* - 1}
\]

\[
= (Q^* - 1) \left| x - \frac{Q^*y^* - y}{Q^* - 1} \right|^2 - \frac{Q^*}{Q^* - 1} |y^* - y|^2,
\]

or, equivalently,

\[
(6.42) \quad \left| x - \frac{Q^*y^* - y}{Q^* - 1} \right|^2 = \frac{Q^*}{(Q^* - 1)^2} |y^* - y|^2.
\]
This means that our quadric is a sphere \( S_R(c) \) with the center \( c \in \mathbb{R}^n \) and the radius \( R > 0 \), given by the formulas

(6.43) \[
    c = \frac{Q^*y^* - y}{Q^* - 1}, \quad R = \frac{(Q^*)^{1/2}}{|Q^* - 1|} |y^* - y|.
\]

\( \blacktriangleleft \) Similarly to the relations deduced above, we can also obtain relations

(6.44) \[
    |(Q^* - 1)x - (Q^*y^* - y)|^2 = Q^*|y^* - y|^2,
\]

(6.45) \[
    |Q^*(x - y^*) - (x - y)|^2 = Q^*|y^* - y|^2.
\]

Their advantage is that they also hold for \( Q^* = 1 \), in which case the sphere becomes hyperplane. \( \blacktriangleright \)

Solving the first equation in (6.43) with respect to \( y^* \) and \( y \), we get

(6.46) \[
    y^* = \frac{1}{Q^*} y + \left(1 - \frac{1}{Q^*}\right) c, \quad y = Q^*y^* + (1 - Q^*)c.
\]

It follows that \( c, y \) and \( y^* \) are on the same line in \( \mathbb{R}^n \). If \( Q^* - 1 > 0 \) (resp. \( Q^* - 1 < 0 \)), then they follow in the order \( c, y^*, y \) (resp. \( c, y, y^* \)).

Substituting the expression for \( y^* \) from (6.46) into (6.43), and then solving the resulting equation with respect to \( Q^* \), we easily find that

(6.47) \[
    Q^* = \frac{|y - c|^2}{R^2}.
\]

Now using the first equation of (6.43), we obtain

(6.48) \[
    y^* - c = \frac{R^2}{|y - c|} \cdot \frac{y - c}{|y - c|},
\]

which implies that \( y^* - c \) and \( y - c \) are proportional and

\[
    |y - c| \cdot |y^* - c| = R^2.
\]

This means that the transformation from \( y \) to \( y^* \) is the standard inversion with respect to the sphere \( S_R(c) \).

**Corollary 6.35.** The variables \( y^*, Q^* \) can be real-analytically expressed in terms of \( 2n + 1 \) geometric quantities \( c, R, y \), except on a finite number of algebraic submanifolds.

The formulas above also imply that the geometric parameters \( c, R, y \) uniquely define all other parameters \( y^* \) (inverse to \( y \) with respect to the sphere \( S_R(c) \)), \( Q^* \) (see (6.47)), \( q^* \) (see (6.39)). By translation we can achieve \( c = 0 \) (translating all other parameters respectively). \( \blacktriangleright \)
Remark 6.36. If (6.48) is considered as a map

\[ * : \mathbb{R}^n \setminus \{c\} \rightarrow \mathbb{R}^n \setminus \{c\}, \quad y \mapsto y^*, \]

then it is involutive or an involution, which means that \( *^2 = \text{Id} \) (the identity map), with \( S_R(c) \) being its fixed point set.

The particular involution \( * : y \mapsto y^* \) of \( \mathbb{R}^n \setminus \{c\} \) given by (6.48), is called reflection with respect to the sphere \( S_R(c) \).

Remark 6.37. In the limit \( R \rightarrow +\infty \), with the appropriate choice of the center \( c = c(R) \), we can obtain a limit relation \( S_R(c(R)) \rightarrow S_\infty \), where \( S_\infty \) is a hyperplane in \( \mathbb{R}^n \) and the convergence is in \( C^2 \)-topology of hypersurfaces (defined by \( C^2 \)-topologies on functions representing \( S_\infty \) as graphs of \( C^2 \)-functions in local coordinates). Then, for any fixed \( y \) we get the reflection \( y^* = y^*(R) \), such that \( y^*(R) \rightarrow y^*_\infty \), where \( y^*_\infty \) is the mirror image of \( y \) with respect to the limit hyperplane \( S_\infty \). (See more about this below in this Section.)

\[ \nabla \]

To understand the general cause of involutivity, we can consider general charged particles \( \{y, q\} \), where \( y \in \mathbb{R}^n, q \in \mathbb{R} \setminus \{0\} \), and \( q \) is the charge located at the point \( y \). Let \( \{y_1, q_1\} = \{y^*, q^*\} \) be the “conjugate” pair (to \( \{y, q\} \)) with respect to a fixed sphere \( S_R(c) \). This means that \( y_1 \neq y \), and

\[ \frac{q}{|x - y|^{n-2}} = -\frac{q_1}{|x - y_1|^{n-2}}, \quad x \in S_R(c). \]

Clearly, the nodal hypersurface for this system depends only upon the ratio \( q/q_1 \) (i.e. does not change under the transformations \( (q, q_1) \rightarrow (tq, tq_1) \), where \( t \in \mathbb{R} \setminus \{0\} \). Previously we only considered the “normalized” pairs \( \{y, q\} \), satisfying the condition \( q = 1 \). Assuming that we also have a charged particle \( \{y_2, q_2\} \), which is conjugate to \( \{y_1, q_1\} \), then we also have \( y_2 \neq y_1 \), and

\[ \frac{q_1}{|x - y_1|^{n-2}} = -\frac{q_2}{|x - y_2|^{n-2}}, \quad x \in S_R(c). \]

It follows that

\[ \frac{q}{|x - y|^{n-2}} = \frac{q_2}{|x - y_2|^{n-2}}, \quad x \in S_R(c). \]

Using (6.48) (implying the uniqueness of \( y^* \) for fixed \( c, R, y \)), we see that \( y_2 = y \), which means \( (y^*)^* = y \) for all \( y \), i.e. involutivity of the map \( * \). \( \nabla \)

We will summarize the arguments above in the following theorem.
Theorem 6.38. Let $\Omega$ be a bounded domain with a $C^\infty$ boundary $\partial \Omega$ in $\mathbb{R}^n$, $n \geq 3$. Assume that for every $y \in \Omega$ and every $q > 0$ there exist a point $y^* \in \mathbb{R}^n \setminus \Omega$ and a number $q^* < 0$, such that the potential $u$ of the system of two charges $1, q^*$ at the points $y, y^*$ respectively, has the nodal hypersurface

$$u(0)^{-1} = \{ x \in \mathbb{R}^n : u(x) = 0 \},$$

which coincides with $\partial \Omega$. Then the following statements hold:

(a) $\Omega$ is a ball $B_R(c)$ with the center $c \in \Omega$ and radius $R > 0$.

(b) With $Q^* = (-q^*)^{-2/(n-2)}$ the quantities $Q^*, R, c, q^*, y^*$ are unique and given by the formulas (6.43), (6.33).

(c) The identities (6.37), (6.40), (6.41), (6.44), (6.45), (6.46), (6.47), (6.48) hold.

Making the calculations in the opposite direction, we can conclude that the following proposition holds:

Proposition 6.39. For every sphere $S_R(c)$ and every point $y \in \mathbb{R}^n \setminus S_R(c)$ ($n \geq 3$), there exist a unique point $y^* \in \mathbb{R}^n$ and a unique number $q^* \in \mathbb{R} \setminus \{0\}$, such that the potential $u$ of the system of two charged point particles at $y, y^*$ with the charges $1, q^*$ respectively, has the nodal hypersurface which coincides with the sphere $S_R(c)$.

Proof. There are two excluding possibilities:

(a) $\partial \Omega \subset S_R(c)$;

(b) $\exists y \in (\mathbb{R}^n \setminus \partial \Omega) \cap S_R(c)$.

It is easy to see that (a) implies the equality $\partial \Omega = S_R(c)$, and (b) contradicts the conditions of Proposition 6.39. □

• Green's function for a half-space, $n \geq 2$.

Now we lift the restriction $n \geq 3$ and assume $n \geq 2$. The Laplacian $\Delta_2$ does not have a fundamental solution $E_2(\cdot)$ satisfying $E_2(x) \to 0$ as $|x| \to \infty$. Also, there is no natural fundamental solution for $\Delta_2$. The general one has the form $E_2 + C$, where $C$ is an arbitrary constant. Fixing $C$, we choose $E_2$, and this choice does not make any difference. In this section we will chose the constant so that $E_2(x) = (2\pi)^{-1} \ln |x|$, as in (4.39).

Let $L$ be a hyperplane in $M = \mathbb{R}^n$, $n \geq 3$, $y \in M \setminus L$. Then $M \setminus L = M^+ \cup M^-$, where $M^\pm$ are the disjoint connected components of $M \setminus L$, i.e. $M^+$ and $M^-$ are open half-spaces of $M$, defined by $L$. Assume that $y \in M^-$. 
Let us denote $\Omega = M^*$. Then $\mathbb{R}^n \setminus \tilde{\Omega} = M^+$. So we can define $y^*$, the conjugate point to $y$. It is easy to see that $y^*$ is the mirror image of $y$ in the mirror $L$.

We can choose orthogonal coordinates $\{x_1, \ldots, x_n\}$ in $M$, such that in these coordinates $L = \{x \in \mathbb{R}^n \mid x_n = 0\}$, $M^+ = \{x \in M \mid x_n > 0\}$. Then for $y = (y_1, \ldots, y_n)$ we have

\begin{equation}
(6.49) \quad y^* = (y_1, \ldots, y_{n-1}, -y_n).
\end{equation}

Now it is easy to check that the function

\begin{equation}
(6.50) \quad G(x, y) = \mathcal{E}_n(x - y) - \mathcal{E}_n(x - y^*)
\end{equation}

satisfies all conditions required from Green’s function (including the Decay Assumption (6.32), which is required from all unbounded domains).

In the notations, introduced earlier in this Chapter, the reflected charge is $q^* = -1$, $y^*$ is given by (6.49), that is, $y^*$ is the reflected in the mirror (which is $L$) unit charge at $y$, which also changed sign at reflection.

\[\nabla\] The desired result also follows from symmetry considerations, more precisely, from the uniqueness of Green’s function combined with the invariance of the problem with respect to all translations parallel to the mirror $L$, rotations with a fixed point $c \in L$ and mirror reflections with respect to the mirror $L$. \[\nabla\]

- **Green’s function in 2 dimensions**

Even though the fundamental solution $\mathcal{E}_2(z)$ of the Laplacian in $\mathbb{R}^2$ (that is, for $n = 2$) behaves differently at infinity (for example, it does not go to 0 when $|z| \to \infty$), almost nothing changes for other objects, e.g. for derivatives of Green’s functions. In particular, a Green function of the half-plane

\[\Omega = \{x = (x_1, x_2) : x_2 < 0\} \subset \mathbb{R}^2,\]

can be taken in the form

\[G(x, y) = \mathcal{E}_2(x - y) + g(x, y),\]

where $\mathcal{E}_2(x) = \frac{1}{2\pi} \log |x|$, $g = g(x, y)$ is the compensating reflected term, $g(x, y) = -\mathcal{E}_2(x - y^*)$, $x \in \Omega$,

with

\[y^* = (y_1, y_2)^* = (y_1, -y_2)\]
In particular, \( G(x, y) \) generally stays unbounded if \( x \to \infty \) and \( |y| \) is bounded. We also see from (6.40) that

\[
G(x, y) = \mathcal{E}_2(x - y) - \mathcal{E}_2(x - y^*). \tag{6.51}
\]

### 6.8.2. Green’s function of a ball.

Consider the open ball \( \Omega = B_R(0) \) with the radius \( R > 0 \), centered at the origin \( 0 \in \mathbb{R}^n \), \( n \geq 2 \). Let us fix \( y \in \Omega \), and take the fundamental solution \( \mathcal{E}_n = \mathcal{E}_n(x) \) of the Laplace operator. Now choose a smooth function \( g \) in \((\bar{\Omega} \times \Omega) \cup (\Omega \times \bar{\Omega})\),

\[
g = g(\cdot, y) = g(x, y) = q^*(y)\mathcal{E}_n(x - y^*(y)), \tag{6.52}
\]

the other notations the same as in (6.33). We would like the sum

\[
G(x, y) = \mathcal{E}_n(x - y) + g(x, y) = \mathcal{E}_n(x - y) + q^*(y)\mathcal{E}_n(x - y^*(y)), \tag{6.53}
\]

to be Green’s function of Dirichlet’s problem for the ball \( \Omega \). Simplifying the sufficient conditions on \( \Omega \), we see that it is sufficient to determine the coefficients \( q^*, y^* \) in (6.52) by substitution the expression (6.53) into the conditions for Green’s functions:

(a) \( g \in C^2(\bar{\Omega} \times \bar{\Omega}) \);

(b) \( \Delta_x G(\cdot, y) = \delta(\cdot, y) \) for all \( (x, y) \in (\bar{\Omega} \times \Omega) \cup (\Omega \times \Omega) \),

\[
\Delta_y G(x, \cdot) = \delta(x, \cdot) \quad \text{for all} \quad (x, y) \in (\bar{\Omega} \times \Omega) \cup (\Omega \times \bar{\Omega});
\]

(c) \( G(x, y) = 0 \) if \( (x, y) \in (\partial \Omega \times \Omega) \cup (\Omega \times \partial \Omega) \).

Note that due to the location and local structure of singularities, we have, for \( G \) defined on \( \Omega \times \Omega \),

\[
\text{sing supp } G = \text{diag}(\Omega \times \Omega).
\]

Moreover, \( y \in \Omega \) implies \( y^* \in \mathbb{R}^n \setminus \bar{\Omega} \), so all interior singularities have the same form as \( \mathcal{E}_n(x - y) \). More precisely, \( G(x, y) - \mathcal{E}_n(x - y) \in C^\infty(\Omega \times \Omega) \). We also established above that one corrective term is sufficient only if \( \Omega \) is a ball or a half space, in which case the formulae (6.39), (6.42), (6.43), (6.46), (6.47), (6.48) provide effective relations between \( q^*, Q^*, y, y^*, c \) and \( R \).

Using the arguments on Green’s function in a smooth bounded domain above, we see that a solution \( g = g(x, y) \) of the Laplace equations \( \Delta_x g = \Delta_y g = 0 \) with any prescribed Dirichlet boundary condition \( g(x, y) \) for all \( x \in \partial \Omega, y \in \Omega \) (see also (6.30)) will necessarily coincide with the original solution \( u \). This also could be seen from the Maximum Principle. Hence, it is enough to investigate the hypothetic solution from the formula (6.30).

Note first that for any \( y \in \Omega \) we have \( q^* \in (-\infty, -1) \) (or, equivalently, \( Q^* \in (0, 1) \), see (6.38) and (6.39)). Due to the above mentioned uniqueness of
Green’s function, this leads to the same Green function and other parameters as above.

The argument above makes it natural to try test functions of the form (6.35) and adjust them to become Green’s functions. Fixing \( y^* = y^*(y) \), we see, as above, that \( y^* \) must be a conjugate point to \( y \) for every \( y \in \Omega \). It also follows that if we choose \( y^* \) to be a conjugate point to \( y \) (for every fixed \( y \in \Omega \)), then, after some natural analytic continuations, the set of all \( y^* \)’s becomes a sphere, and so does the set of all \( y^* \)’s.

Now we can choose the set of common zeros of all functions (6.35) as a sphere \( B_R(0) \), and arrive to a sphere which depend upon the choices of \( c \in \mathbb{R}^n \), \( R > 0 \) and \( q^* < -1 \).

Straightforward construction, starting from the sphere, described above, may be done as follows.

A. Given a sphere \( S_R(0) \), where \( R > 0 \) is its radius, \( 0 \) is the center of the sphere (the origin of \( \mathbb{R}^n \)), define \( \Omega = B_R(0) \) which is the open ball with the same center and radius. Introduce the natural inclusion map \( B_R(0) \subset \mathbb{R}^n \) (naturally with respect to the canonical analytic inclusion).

B. Adding a point \( \infty \) to \( \mathbb{R}^n \), introduce topology on the union \( S^n = \mathbb{R}^n \cup \infty \), so that it naturally extends the inclusion \( \mathbb{R}^n \subset S^n = \mathbb{R}^n \cup \infty \) to the homeomorphism of the union to the standard sphere \( S^n \).

C. Define the involutions \(*: S^n \to S^n, y \mapsto y^*\), by
\[
y^* = \frac{R^2 y}{|y|^2},
\]
on \( \mathbb{R}^n \setminus \{0\} \) and extended to the points \( 0 \) and \( \infty \) by continuity.

D. Using the charge 1 at a point \( y \in \Omega = B_R(0) \) and the involution \(*\), introduced above (the reflection with respect to the sphere \( S_R(0) \)), define the reflected charge \( q^* \) at \( y^* \) for every \( y \in S^n \) and adjust its size so that the total potential of two charges at \( y, y^* \) vanishes on the sphere.

E. For Green’s function \( G = G(x, y) \) of \( \Omega \), we have \( G = \mathcal{E}(x-y) + g(x, y) \), where \( g \in C^\infty((\overline{\Omega} \times \Omega) \cup (\Omega \times \overline{\Omega})) \), \( \Delta_x g = \Delta_y g = 0 \) on \( \Omega \times \Omega \). Besides, \( G(x, y) = 0 \) if \( x \in \partial \Omega \) or \( y \in \partial \Omega \).

**Remark 6.40.** In fact, the condition E is sufficient for the uniqueness of Green’s function of a given domain \( \Omega \). But the introductory geometric terminology considerably simplifies exposition. Indeed, assuming \( n \geq 3 \), \( y \in \Omega = B_R(0) \), we can take the reflection \( y^* = R^2 |y|^{-2} y \), and then write \( G = G(x, y) \) by the formula above, finding \( g(x, y) \) as a one-charge potential.
\(q^*(y)E_n(x - y^*(y))\) with the coefficient \(q^*(y)\) to be found from the Dirichlet boundary condition \(G(x, y) = 0\) for all \(x \in \partial \Omega\) (or all \(y \in \partial \Omega\)).

In these notations we obtain

**Proposition 6.41.** Let us consider the set of all conjugate pairs \(\{y, y^*\}\) where \(y \in \Omega = B_R(0)\), and \(y^* = R^2|y|^{-2}y\), that is, \(y^*\) is conjugate to \(y\), \(y^* \in \mathbb{R}^n \setminus \bar{\Omega}\). Now take \(G = G(x, y) = E_n(x - y) + g(x, y)\), where \(g \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \bar{\Omega}))\) satisfies the equations and boundary conditions required from Green's functions, that is \(\Delta_x g = 0, \Delta_y g = 0\) on \(\Omega \times \Omega\), and \(G(x, y) = 0\) if \(x = 0\) or \(y = 0\).

Then \(G\) is Green's function of the Laplacian in \(\Omega\).

**Proof.** Immediately follows from the equations and boundary conditions imposed on \(G, g\).

\(\square\)

\(\blacktriangleleft\) Now we can establish the geometric description of the electric fields in the ball \(\Omega = B_R(0)\), when the equilibrium of the main point charge (located at \(x \in \Omega\)) is given by exactly one additional charge which is located outside the ball. According to our present notations, we can take \(c = 0\) and \(y \in \Omega\) arbitrary. This means that the following relations hold:

(i) 0 is the center of the ball \(\Omega\),
(ii) \(R > 0\) is the radius of \(\Omega\),
(iii) \(q^*\) and \(Q^*\) are related by \((6.38)\) and \((6.39)\),
(iv) \(Q^*\) is expressed through \(R\) by \((6.47)\),
(v) \(y^*\) is related with \(y\) by \((6.48)\): \(y^* = R^2|y|^{-2}y\).

It is easy to see that all other quantities like \(q^*, Q^*\), etc. can be easily expressed through the geometric parameters \(c, R\). This allows to restrict consideration of the problems relating any quantities like this, to elementary problems relating \(c, R, y\). \(\blacktriangleright\)

\(\blacktriangleleft\) Let us write down explicit formulas for Green’s function of the ball \(\Omega = B_R(0)\). According to the arguments above, it is reasonable to try

\[(6.54)\] \(G(x, y) = E_n(x - y) + q^*(y)E_n(x - R^2|y|^{-2}y), \ x \in \bar{\Omega}, y \in \Omega,\)

where \(q^*(\cdot)\) is found by properties of \(G\) which we have not used in relation to \(q^*\). In particular, we can use the symmetry of Green’s function, i.e. the relation \(G(x, y) = G(y, x)\) for all \(x, y \in \Omega\). This means

\(q^*(y)E_n(x - y) = q^*(x)E_n(y - x), \ x, y \in \Omega, \ x \neq y.\)
Since $\mathcal{E}_n(-z) = \mathcal{E}_n(z)$ for all $z \in \mathbb{R}^n \setminus \{0\}$, we can cancel the factor $\mathcal{E}_n$ and conclude that $q^*(x) = q^*(y)$ for all $x, y \in \Omega$. Therefore, we conclude that $q^*(\cdot) = \text{const}$. Denote this constant simply $q^*$. Then, we can rewrite the equality (6.34) as

$$\mathcal{E}_n(x - y) + q^* \mathcal{E}_n(x - R^2|y|^{-2}y) = 0,$$

for every $x \in S_R(0) = \partial \Omega$, with a constant $q^*$. (Compare (6.34).)

### 6.9. Problems

**6.1.** Prove the statements inverse to the ones of Theorems 6.1, 6.2: if a continuous function $u$ in an open set $\Omega \subset \mathbb{R}^n$ has mean value property for all balls $B$ with $\bar{B} \subset \Omega$ or for all spheres which are boundaries of such balls (that is, either (6.2) holds for such spheres or (6.4) holds for all such balls), then $u$ is harmonic in $\Omega$. Moreover, it suffices to take balls or spheres, centered at every point $x$ with sufficiently small radii $r \in (0, \varepsilon(x))$, where $\varepsilon(x) > 0$ continuously depends upon $x \in \Omega$.

**6.2.** Let $u$ be a real-valued $C^2$ function on an open ball $B = B_r(x) \subset \mathbb{R}^n$, which extends to a continuous function on the closure $\bar{B}$, and $\Delta u \geq 0$ everywhere in $B$. Prove that

\begin{equation}
(6.55) \quad u(x) \leq \frac{1}{\text{vol}_{n-1}(S)} \int_S u(y) dS_y = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{S_r(x)} u(y) dS_y,
\end{equation}

and

\begin{equation}
(6.56) \quad u(x) \leq \frac{1}{\text{vol}(B)} \int_B u(y) dy = \frac{n}{\sigma_{n-1} r^n} \int_{B_r(x)} u(y) dy,
\end{equation}

where the notations are the same as in Section 6.1.

**6.3.** Prove the inverse statement to the one in the previous problem: if $u \in C^2(\Omega)$ is real-valued and for all balls $B$ with $\bar{B} \subset \Omega$ (or for all spheres which are boundaries of such balls), (13.2) (respectively, (13.1)) holds, then $\Delta u \geq 0$ in $\Omega$. Moreover, it suffices to take balls or spheres, centered at every point $x$ with sufficiently small radii $r \in (0, \varepsilon(x))$, where $\varepsilon(x) > 0$ continuously depends upon $x \in \Omega$.

**Remark 6.42.** Let a real-valued function $u \in C^2(\Omega)$ be given. It is called subharmonic if $\Delta u \geq 0$ in $\Omega$. Note, however, that the general notion of subharmonicity does not require that $u \in C^2$ and allows more general functions $u$ (or even distributions, in which case $\Delta u \geq 0$ means that $\Delta u$ is a positive Radon measure).
6.4. Let \( u \) be a real-valued subharmonic function in a connected open set \( \Omega \subset \mathbb{R}^n \). Prove that \( u \) cannot have local maxima in \( \Omega \), unless it is constant. In other words, if \( x_0 \in \Omega \) and \( u(x_0) \geq u(x) \) for all \( x \) in a neighborhood of \( x_0 \), then \( u(x) = u(x_0) \) for all \( x \in \Omega \).

6.5. Consider the initial value problem (6.11) with \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( \psi \equiv 0 \). Assume that it has a solution \( u \). Prove that in this case \( \varphi \equiv 0 \) on \( \mathbb{R}^n \) and \( u \equiv 0 \) on \( \Pi_T \).

6.6. Let \( U, V, W \) be open sets in \( \mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p \) respectively, and \( g : U \to V \), \( f : V \to W \) be maps which are locally \( C^{k,\gamma} \) (that is, they belong to \( C^{k,\gamma} \) on any relatively compact subdomain of the corresponding domain of definition).

(a) Assume that \( k \geq 1 \). Then prove that the composition \( f \circ g : U \to W \) is also locally \( C^{k,\gamma} \).

(b) Prove that the statement (a) above does not hold for \( k = 0 \).

(c) Prove that if \( k \geq 1 \), \( \Omega \) is a domain in \( \mathbb{R}^n \) with \( \partial \Omega \in C^{k,\gamma} \), \( U \) is a neighborhood of \( \Omega \) in \( \mathbb{R}^n \), and \( \psi : U \to \mathbb{R}^n \) is a \( C^{k,\gamma} \) diffeomorphism of \( U \) onto a domain \( V \subset \mathbb{R}^n \), then the image \( \Omega' = \psi(\Omega) \) is a domain with a \( C^{k,\gamma} \) boundary, \( \overline{\Omega'} \subset V \).

6.7. (a) Prove that if \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \partial \Omega \in C^{k,\gamma} \) with \( k \geq 1 \), then \( \partial \Omega \) is a \( C^{k,\gamma} \) manifold, \( \dim_{\mathbb{R}} \partial \Omega = n - 1 \).

(b) Prove that for any non-negative integers \( k, k' \) and numbers \( \gamma, \gamma' \in (0, 1) \), such that \( k + \gamma \geq k'+\gamma' \), \( k \geq 1 \), the class \( C^{k',\gamma'} \) is locally invariant under \( C^{k,\gamma} \) diffeomorphisms. In particular, function spaces \( C^{k',\gamma'}(\partial \Omega) \) are well defined.

(c) Under the same assumptions as in (a) and (b), for a real-valued function \( \varphi : \partial \Omega \to \mathbb{R} \), the inclusion \( \varphi \in C^{k',\gamma'}(\partial \Omega) \) is equivalent to the existence of an extension \( \hat{\varphi} \in C^{k',\gamma'}(\Omega) \), such that \( \hat{\varphi}|_{\partial \Omega} = \varphi \).

6.8. Assume that \( k \geq 1 \) is an integer, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), such that \( \partial \Omega \in C^{k+2} \), \( f \in C^k(\Omega) \) and \( \varphi \in C^{k+2}(\partial \Omega) \). Using Schauder's Theorem 6.20, prove that the problem (6.10) has a solution \( u \in C^{k+1}(\Omega) \).

6.9. Assume that \( \partial \Omega \in C^2 \), and there exists Green's function \( G = G(x,y) \) of \( \Omega \), such that \( G \in C^2 \) for \( x \neq y \), \( x, y \in \Omega \). Consider an integral representation of a function \( u \in C^2(\Omega) \) given by the formula (6.28), with \( f \in C(\Omega) \) and
\( \varphi \in C(\partial \Omega) \). Prove that the pair \( \{f, \varphi\} \), representing a fixed function \( u \in C^2(\bar{\Omega}) \) by (6.28), is unique.

**Hint.** The function \( f \) can be recovered from \( u \) by the formula \( f = \Delta u \), whereas at any regular point of the boundary \( x \in \partial \Omega \), \( \varphi \) is the jump of the outgoing normal derivative of \( u \) on \( \partial \Omega \):

\[
\varphi(x) = \lim_{\epsilon \to 0^+} \left( \frac{\partial u}{\partial \bar{n}}(x + \epsilon \bar{n}) - \frac{\partial u}{\partial \bar{n}}(x - \epsilon \bar{n}) \right).
\]

**6.10.** Provide detailed proofs of identities (6.44) and (6.45).

**6.11.** Provide detailed proofs of identities (6.46), (6.47) and (6.48).

**6.12.** Deduce the expression for Green’s function in a half-space of \( \mathbb{R}^n \), \( n \geq 3 \), from symmetry considerations.

**Hint.** Use the uniqueness of Green’s function combined with the invariance of the problem with respect to all translations parallel to the mirror \( L \), rotations with a fixed point \( c \in L \) and mirror reflections with the mirror \( L \).
Chapter 7

The heat equation

7.1. Physical meaning of the heat equation

The heat equation is

\[ \frac{\partial u}{\partial t} = a^2 \Delta u, \]

where \( u = u(t, x) \), \( t \in \mathbb{R}, x \in \mathbb{R}^n \), \( a > 0 \), and \( \Delta \) is the Laplace operator with respect to \( x \). This equation is an example of a parabolic type equation. It describes, for instance, the temperature distribution of a homogeneous and isotropic medium (in which case \( u(t, x) \) is the temperature at the point \( x \) at the time \( t \)). This equation is also satisfied by the density of a diffusing matter, e.g. the density of Brownian particles when there are sufficiently many of them so that we can talk about their density and consider the evolution of this density as a continuous process. This is a reason why the equation (7.1) is often called the diffusion equation.

Deriving of the heat equation is based on a consideration of the energy balance. More precisely, we will deal with the transmission of the heat energy, which is the kinetic energy of the atoms and molecules chaotic motion in the gases and liquids or the energy of the atoms vibrations in metals, alloys and other solid materials. Instead of “heat energy” we will use the word “heat”, which is shorter and historically customary in the books on thermodynamics.

▼ The heat equation for temperature \( u(t, x) \) is derived from the following natural empirically verified physical assumptions:
1) The heat $Q$ needed to warm up a part of matter with the mass $m$ from the temperature $u_1$ to $u_2$ is proportional to $m$ and $u_2 - u_1$: 

$$Q = cm(u_2 - u_1).$$

The coefficient $c$ is called the specific heat (per unit of mass and unit of temperature).

2) Fourier’s law. The quantity of heat $\Delta Q$ passing through a small surface element with the area $S$ during the time $\Delta t$, is proportional to $S$, $\Delta t$, and $\frac{\partial u}{\partial \hat{n}}$ (the rate of growth of $u$ in the direction $\hat{n}$ of the unit normal vector to the surface). More precisely,

$$\Delta Q = -kS\frac{\partial u}{\partial \hat{n}}\Delta t,$$

where $k > 0$ is the heat conductivity coefficient characterizing heat properties of the media; the minus sign indicates that heat is moving in the direction opposite to that of the growth of the temperature. ▲

One should keep in mind that both hypotheses are only approximations, and so is the equation (7.1) which they imply. As we will see below, a consequence of equation (7.1) is an infinite velocity for propagation of heat, which is physically absurd. In the majority of applied problems, however, these hypotheses and equation (7.1) are sufficiently justified. A more precise model of heat distribution should take into account molecular structure of the matter leading to problems of statistical physics far more difficult than the solution of the model equation (7.1).

Note, however, that, regardless of its physical meaning, equation (7.1) and its analogues play an important role in mathematics. They are used, e.g., in the study of elliptic equations.

▼ Let us derive the heat equation (for $n = 3$). We use the energy conservation law (heat conservation in our case) in a volume $\Omega \subset \mathbb{R}^3$. Let us assume that $\Omega$ has a smooth boundary $\partial \Omega$. The rate of change of the heat energy of the matter in $\Omega$ is obviously equal to

$$\frac{dQ}{dt} = \frac{d}{dt} \int_\Omega c\rho u \, dV,$$

where $\rho$ is the volume density of the matter (mass per unit volume), $dV$ is the volume element in $\mathbb{R}^3$. Assuming that $c = \text{const}$ and $\rho = \text{const}$, we get

$$\frac{dQ}{dt} = c\rho \int_\Omega \frac{\partial u}{\partial t} \, dV.$$
Now, let $P$ be the rate of the heat escape through $\partial \Omega$. Clearly,

$$P = - \int_{\partial \Omega} k \frac{\partial u}{\partial \bar{n}} dS,$$

where $\bar{n}$ is the outward unit normal to $\partial \Omega$, $dS$ the area element of $\partial \Omega$. Here in the right hand side we have the flow of the vector $-k \text{grad } u$ through $\partial \Omega$. By the divergence theorem (4.34) we get

$$P = - \int_{\Omega} \text{div}(k \text{grad } u) dV$$

or, for $k = \text{const}$,

$$P = -k \int_{\Omega} \Delta u dV.$$

The heat energy conservation law in the absence of heat sources means that

$$\frac{dQ}{dt} = -P.$$

Substituting the above expressions for $\frac{dQ}{dt}$ and $P$, we get

$$\int_{\Omega} c_p \frac{\partial u}{\partial t} dV = \int_{\Omega} k \Delta u dV,$$

which, since $\Omega$ is arbitrary, results in (7.1) with $a^2 = \frac{k}{c_p}$. ▲

### 7.2. Boundary value problems for the heat and Laplace equations

A physical interpretation of the heat equation suggests natural setting for boundary value problems for this equation. The Cauchy problem (sometimes also called the initial value problem) is the simplest one:

\[
\begin{aligned}
  \frac{\partial u}{\partial t} &= a^2 \Delta u, \quad t > 0, x \in \mathbb{R}^n, \\
  u|_{t=0} &= \varphi(x) \quad x \in \mathbb{R}^n
\end{aligned}
\]

which, since $\Omega$ is arbitrary, results in (7.1) with $a^2 = \frac{k}{c_p}$. ▲

(7.2)
The first initial-boundary value problem (or Cauchy-Dirichlet’s initial-boundary value problem):

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= a^2 \Delta u, \quad t > 0, \ x \in \Omega, \\
|u|_{t=0} &= \varphi(x), \quad x \in \Omega, \\
|u|_{\partial t} &= \alpha(t, x), \quad x \in \partial \Omega, \ t > 0.
\end{aligned}
\]

The second boundary value problem (or Cauchy-Neumann’s boundary value problem) is obtained by replacing the latter of conditions (7.3) by

\[
\left. \frac{\partial u}{\partial \bar{n}} \right|_{\partial \Omega} = \beta(t, x), \quad x \in \partial \Omega, \ t > 0,
\]

where \( \bar{n} \) is the outward unit normal to \( \partial \Omega \).

For time-independent boundary conditions it is often interesting to determine temperature behavior as \( t \to +\infty \). Suppose that a limit \( v(x) \) of the solution \( u(t, x) \) of the heat equation exists as \( t \to +\infty \) (we say that the solution “stabilizes” or “tends to an equilibrium”). Then it is natural to expect that \( v(x) \) satisfies the Laplace equation \( \Delta v(x) = 0 \), i.e., \( v(x) \) is a harmonic in \( x \), at least because the only possible finite limit of \( \partial u(t, x) / \partial t \) can be 0. It is very easy to make the meaning of this statement precise but we will not do so at the moment in order not to divert our attention.

The first and second initial-boundary value problems for the heat equation turn in the limit into the following boundary value problems for the Laplace equation:

**Dirichlet’s problem**

\[
\begin{aligned}
\Delta v(x) &= 0, \quad x \in \Omega; \\
v|_{\partial \Omega} &= \alpha(x), \quad x \in \partial \Omega.
\end{aligned}
\]

**Neumann’s problem:**

\[
\begin{aligned}
\Delta v(x) &= 0, \quad x \in \Omega; \\
\left. \frac{\partial v}{\partial \bar{n}} \right|_{\partial \Omega} &= \beta(x), \quad x \in \partial \Omega.
\end{aligned}
\]

In this model Dirichlet’s problem means the following: find the steady state temperature inside \( \Omega \) when the temperature \( \alpha(x) \) is maintained on the boundary. The meaning of Neumann’s problem is to find the steady state temperature inside \( \Omega \) if the heat flow \( \beta(x) \) on the boundary is given.

We already mentioned the Dirichlet problem in the previous chapter – see Corollary 6.8, where the uniqueness of the classical solution was established. Clearly, the solution of Neumann’s problem is not unique: any constant can
always be added to the solution. Further, it is clear that the steady state
distribution of heat can exist in $\Omega$ only if the total heat flux through the
boundary vanishes, i.e.,

\begin{equation}
\int_{\partial \Omega} \beta(x) \, dS = 0.
\end{equation}

(This also follows from the formula (6.3), which implies that if $v$ is a solution
of (7.5) then (7.6) holds.)

We will see later that the above problems for the heat and Laplace
equations are well-posed in the sense that they have a unique solution (up
to adding a constant in case of the Neumann problem) in natural classes
of functions, and, besides, the solutions depend continuously on initial and
boundary values for an appropriate choice of spaces.

Note that the Dirichlet and Neumann problems for the Laplace equation
have many physical interpretations apart from the above ones. For instance,
a small vertical deflection of a membrane $u(t, x_1, x_2)$ and a small displace-
ment $u(t, x_1, x_2, x_3)$ of an elastic solid body satisfy the wave equation

\[ \frac{\partial^2 u}{\partial t^2} = a^2 \Delta u. \]

In the time-independent case we come again to the Laplace equation and
then the Dirichlet problem. For instance, for a membrane this is the problem
of recovering the membrane form from that of its boundary. The boundary
condition of Neumann’s problem is interpreted as prescribing the boundary
value of the vertical component of the force acting on the membrane along its
boundary. This topic is related with the famous question by Mark Kac “Can
One Hear the Shape of a Drum?” (1968), which led to a wide development
of Spectral Geometry.

### 7.3. A proof that the limit function is harmonic.

There are several ways to justify the harmonicity of the limit function. We
will give one of them.

**Proposition 7.1.** Let a bounded solution $u(t, x)$ of the heat equation be
defined and bounded for $t > 0$, $x \in \Omega$ (for an open subset $\Omega \subset \mathbb{R}^n$). Suppose
that for almost all $x \in \Omega$ there exists a limit

\begin{equation}
v(x) = \lim_{t \to +\infty} u(t, x).
\end{equation}

Then $v(x)$ is a harmonic function in $\Omega$. 

Remark 7.2. The solution $u(t,x)$ may be a priori considered as a bounded measurable function (here the equation (7.1) should be understood in the distributional sense). However we will see below that the heat operator has a fundamental solution which is infinitely differentiable for $|t| + |x| \neq 0$ and, therefore, actually, $u(t,x) \in C^\infty(\mathbb{R}_+ \times \Omega)$, where $\mathbb{R}_+ = \{ t : t > 0 \}$. Moreover, under the assumptions of Proposition 7.1, the limit in (7.7) is actually uniform together with any finite number of derivatives on any compact set $K \subset \Omega$, i.e., it is the limit in the topology of $C^\infty(\Omega)$.

Proof of Proposition 7.1. Clearly,

$$\lim_{\tau \to +\infty} u(t+\tau,x) = v(x)$$

for all $t > 0$ and almost all $x \in \Omega$. If $\varphi(t,x) \in \mathcal{D}(\mathbb{R}_+ \times \Omega)$, then the dominated convergence theorem implies

$$(7.8) \quad \lim_{\tau \to +\infty} \int u(t+\tau,x)\varphi(t,x)dtdx = \int v(x)\varphi(t,x)dtdx,$$

or, in other words,

$$\lim_{\tau \to +\infty} u(t+\tau,x) = v(x) \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \Omega)$$

(here $v(x)$ should be understood as $1 \otimes v$, i.e., as the right-hand side of (7.8)). Applying the heat operator $\partial_t - a^2 \Delta_x$ to the equality above, we can interchange this operator with the passage to the limit (this can be done in the space of distributions; for example, in our case everything is reduced to replacing $\varphi(t,x)$ in (7.8) by another function $(-\partial_t - a^2 \Delta_x)\varphi(t,x) \in \mathcal{D}(\mathbb{R}_+ \times \Omega)$). Then $(\partial_t - a^2 \Delta_x)v(x) = 0$, i.e. $v(x)$ is a distributional solution of $\Delta v(x) = 0$. But, as we know, $v(x)$ is then a regular harmonic function, as required. \qed

7.4. A solution of the Cauchy problem for the heat equation and Poisson’s integral

Let us investigate the Cauchy problem (7.2). First, assume that $\varphi(x) \in S(\mathbb{R}^n)$ and $u(t,x) \in S(\mathbb{R}^n_+)$ for any $t \geq 0$, so that $u(t,x)$ is a continuous function of $t$ for $t \geq 0$ with values in $S(\mathbb{R}^n_+)$ and, moreover, $C^\infty$-function of $t$ for $t > 0$ with values in $S(\mathbb{R}^n_+)$. As we will see below, a solution $u(t,x)$ with the described properties exists (and is unique in a very broad class of solutions). For the time being, note that we may perform the Fourier transform in $x$ and since the Fourier
transform is a topological isomorphism $F : S(\mathbb{R}^n_x) \rightarrow S(\mathbb{R}^n_\xi)$, it commutes with $\frac{\partial}{\partial t}$. Thus, let
\[
\hat{u}(t, \xi) = \int e^{-ix\cdot\xi} u(t, x) dx
\]
(hereafter, integration is performed over the whole $\mathbb{R}^n$ unless otherwise indicated).

The standard integration by parts gives
\[
\int e^{-ix\cdot\xi} \left( \frac{\partial u}{\partial t} - a^2 \Delta_x u \right) dx = \left\{ \frac{\partial}{\partial t} + a^2 |\xi|^2 \right\} \int e^{-ix\cdot\xi} u(t, x) dx = \frac{\partial \hat{u}}{\partial t} + a^2 |\xi|^2 \hat{u}.
\]
Therefore (7.1) is equivalent to
\[
(7.9) \quad \frac{\partial \hat{u}(t, \xi)}{\partial t} + a^2 |\xi|^2 \hat{u}(t, \xi) = 0.
\]
The initial condition takes the form
\[
(7.10) \quad \hat{u} \big|_{t=0} = \hat{\varphi}(\xi), \text{ where } \hat{\varphi} = F\varphi.
\]
Equation (7.9) is an ordinary differential equation with the independent variable $t$ and with a parameter $\xi$. It is easy to solve it, and we get with (7.10):
\[
\hat{u}(t, \xi) = e^{-ta^2|\xi|^2} \hat{\varphi}(\xi).
\]
From this formula we see that $\hat{u}(t, \xi)$ is actually a (continuous for $t \geq 0$ and $C^\infty$ for $t > 0$) function of $t$ with values in $S(\mathbb{R}^n_\xi)$. Therefore, applying the inverse Fourier transform, we get a solution $u(t, x)$ of the Cauchy problem satisfying the above conditions.

Let us derive an explicit formula. We have
\[
u(t, x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{u}(t, \xi) d\xi = (2\pi)^{-n} \int e^{ix\cdot\xi - ta^2|\xi|^2} \hat{\varphi}(\xi) d\xi
\]
\[
= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi - ta^2|\xi|^2} \varphi(y) dy d\xi.
\]
Interchanging the order of integration (by Fubini's theorem) we get
\[
u(t, x) = \int \Gamma(t, x-y) \varphi(y) dy,
\]
where
\[
(7.11) \quad \Gamma(t, x) = (2\pi)^{-n} \int e^{ix\cdot\xi - ta^2|\xi|^2} d\xi
\]
(we have essentially repeated the argument proving that the Fourier transform turns multiplication into convolution). Let us express $\Gamma(t, x)$ explicitly,
computing the integral in (7.11). For this, complete the square in the exponent:

\[ix \cdot \xi - ta^2|\xi|^2 = -ta^2 \cdot \xi + ix \cdot \xi = -ta^2 \left( \frac{ix}{2ta^2} \right) \cdot \left( \frac{ix}{2ta^2} \right) - \frac{|x|^2}{4ta^2}.
\]

The change of coordinates \(\xi - \frac{ix}{2ta^2} = \eta\), i.e., the shift of the contour of integration over each \(\xi_j\), yields:

\[\Gamma(t, x) = (2\pi)^{-n} e^{-\frac{|x|^2}{4ta^2}} \int e^{-ta^2|\eta|^2} d\eta = (2\pi)^{-n} e^{-\frac{|x|^2}{4ta^2}} (a\sqrt{t})^{-n} \int e^{-|\xi|^2} d\xi
\]

(we have also made the change of variables \(\xi = a\sqrt{t}\eta\)).

Using the formula

\[\int e^{-|\xi|^2} d\xi = \pi^{n/2},
\]

we finally get

\[\Gamma(t, x) = (2a\sqrt{\pi t})^{-n} \exp\left(-\frac{|x|^2}{4ta^2}\right).
\]

The solution of the Cauchy problem is expressed in the form

\[u(t, x) = \frac{1}{(2a\sqrt{\pi t})^n} \int e^{-\frac{|x-y|^2}{4a^2t}} \varphi(y) dy,
\]

called Poisson’s integral.

Let us analyze this formula. First, observe that the formula makes sense for a larger class of initial functions \(\varphi(x)\) than the class \(S(\mathbb{R}^n)\) from which we have started. For instance, it is clear that if \(\varphi(x)\) is continuous and

\[|\varphi(x)| \leq Ce^{b|x|^2}, \quad b > 0,
\]

then the integral (7.12) converges if \(\frac{1}{4a^2 t} > b\), i.e., for \(0 < t < \frac{1}{4a^2 b}\). Here the integral itself and integrals obtained from it after differentiation with respect to \(t\) or \(x\) any number of times uniformly converge for \(t \in [A, B], |x| \leq R\), where \(0 < A < B < \frac{1}{4a^2 b}\). Therefore, we may differentiate any number of times under the integral sign. The function \(u(t, x)\) is a solution of the heat equation (7.1) for the indicated values of \(t\). Indeed, it suffices to show that \(\Gamma(t, x)\) is a solution of (7.1) for \(t > 0\). This can be directly verified, but to avoid this calculation we may also note that, as we already know,

\[0 = \left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) u(t, x) = \int \left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) \Gamma(t, x-y) \varphi(y) dy
\]

for any \(\varphi \in S(\mathbb{R}^n)\), implying

\[\left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) \Gamma(t, x-y) = 0, \quad t > 0.
\]
It is easy to verify that if $\varphi$ is continuous and satisfies (7.13) and $u$ is given by (7.12), then $u_{t=0} = \varphi(x)$ in the sense that

$$\lim_{t \to 0^+} u(t, x) = \varphi(x), \quad \text{for any } x \in \mathbb{R}^n. \tag{7.14}$$

We will even prove this in two ways.

**1st method.** Note that $\Gamma(t, x) = \varepsilon^{-n} f(\frac{x}{\varepsilon})$, where $\varepsilon = 2a\sqrt{t}$ and $f(x) = \pi^{-n/2} \exp(-|x|^2)$, so that $f(x) \geq 0$, $\int f(x) dx = 1$ and $\varepsilon \to 0^+$ as $t \to 0^+$. So we deal with a classical $\delta$-like family of positive functions (in particular, $\Gamma(t, x) \to \delta(x)$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0^+$). Now, (7.14) is verified via the standard scheme of the proof of $\delta$-likeness, see Example 4.5. We skip the corresponding details; the reader should consider this a recommended exercise.

**2nd method.** Suppose we wish to verify (7.14) for $x = x_0$. Let us decompose $\varphi(x)$ into the sum $\varphi(x) = \varphi_0(x) + \varphi_1(x)$, where $\varphi_0(x)$ is continuous, coincides with $\varphi(x)$ for $|x - x_0| \leq 1$ and vanishes for $|x - x_0| \geq 2$. Accordingly, $\varphi_1(x)$ is continuous, vanishes for $|x - x_0| \leq 1$ and satisfies (7.13). Poisson’s integral (7.12) for $\varphi$ splits into the sum of similar integrals for $\varphi_0$ and $\varphi_1$ (we denote them by $u_0(t, x)$ and $u_1(t, x)$). It suffices to verify (7.14) for $u_0(t, x)$ and $u_1(t, x)$ separately.

First, let us deal with $u_1(t, x)$. We have

$$u_1(t, x_0) = \int_{|y-x_0| \geq 1} \Gamma(t, x_0 - y)\varphi_1(y) dy.$$ 

Clearly, the integrand tends to zero as $t \to 0^+$ (since $\lim_{t \to 0^+} \Gamma(t, x) = 0$ for $x \neq 0$) and is majorized for $0 < t \leq B < \frac{1}{4a^2b}$, by an independent of $t$ function of the form $C_1 \exp(-\varepsilon|y|^2)$, where $\varepsilon > 0, C_1 > 0$. By the dominated convergence theorem $\lim_{t \to 0^+} u_1(t, x_0) = 0$.

It remains to consider the case of an initial function with a compact support. Here the argument of Example 4.5 will do without any modifications. There is, however, another method based on approximation by smooth functions with a subsequent passage to the limit. It is based on the following important

**Lemma 7.3.** (The maximum principle) *Let a function $u(t, x)$ be defined by Poisson’s integral (7.12). Then*

$$\inf_{x \in \mathbb{R}^n} \varphi(x) \leq u(t, x) \leq \sup_{x \in \mathbb{R}^n} \varphi(x). \tag{7.15}$$
If one of these inequalities turns into the equality for some \( t > 0 \) and \( x \in \mathbb{R}^n \), then \( u = \text{const} \).

**Proof.** We have

\[
    u(t, x) = \int \Gamma(t, x - y) \varphi(y) dy \leq \sup_{x \in \mathbb{R}^n} \varphi(x) \cdot \int \Gamma(t, x - y) dy = \sup_{x \in \mathbb{R}^n} \varphi(x).
\]

The second inequality in (7.15) is proved similarly. Since \( \Gamma(t, x) > 0 \) for all \( t > 0, x \in \mathbb{R}^n \), then the equality is only possible if \( \varphi(x) = \text{const} \) almost everywhere, and then \( u = \text{const} \) as required. \( \square \)

**Remark 7.4.** As we will see later, a solution \( u(t, x) \) of the Cauchy problem (7.2), satisfying the growth restriction of type (7.13) with respect to \( x \) uniformly in \( t \), is unique. It will follow that the solution \( u(t, x) \) in fact can be represented by the Poisson integral, hence satisfies the maximum principle (7.15).

Let us finish the proof of (7.14). Let \( \varphi \) be a continuous function with a compact support; \( \varphi_k, k = 1, 2, \ldots, \) a sequence of functions from \( C_0^\infty(\mathbb{R}^n) \) such that

\[
    \sup_{x \in \mathbb{R}^n} |\varphi(x) - \varphi_k(x)| \to 0 \quad \text{for} \quad k \to +\infty.
\]

(It is possible to obtain a sequence \( \varphi_k \), for example, via mollifying, cf. Section 4.2.) Let

\[
    u_k(t, x) = \int \Gamma(t, x - y) \varphi_k(y) dy.
\]

Since \( \varphi_k \in S(\mathbb{R}^n) \), it follows that \( \lim_{t \to +0} u_k(t, x) = \varphi_k(x) \) in the topology of \( S(\mathbb{R}^n) \) (in particular, uniformly on \( \mathbb{R}^n \)). But by Lemma 7.3

\[
    \sup_{t>0, x \in \mathbb{R}^n} |u_k(t, x) - u(t, x)| \leq \sup_{x \in \mathbb{R}^n} |\varphi_k(x) - \varphi(x)|;
\]

this, clearly, implies \( \lim_{t \to +0} u(t, x) = \varphi(x) \). Indeed, given \( \varepsilon > 0 \), we may choose \( k \) such that \( \sup_{x \in \mathbb{R}^n} |\varphi_k(x) - \varphi(x)| < \frac{\varepsilon}{3} \), and then \( t_0 > 0 \) such that

\[
    \sup_{x \in \mathbb{R}^n} |u_k(t, x) - \varphi_k(x)| < \frac{\varepsilon}{3} \quad \text{for} \quad 0 < t < t_0.
\]

This clearly implies

\[
    |u(t, x) - \varphi(x)| \leq |u(t, x) - u_k(t, x)| + |u_k(t, x) - \varphi_k(x)| + |\varphi_k(x) - \varphi(x)| \leq 2 \sup_{x \in \mathbb{R}^n} |\varphi_k(x) - \varphi(x)| + |u_k(t, x) - \varphi_k(x)| < \varepsilon \quad \text{for} \quad 0 < t < t_0,
\]

as required. \( \square \)
Observe also that Poisson’s integral often makes sense for distributions \( \varphi \) as well. For instance, if \( \varphi \in S'(\mathbb{R}^n) \), then Poisson’s integral makes the following natural sense:

\[
u(t, x) = \langle \varphi(y), \Gamma(t, x - y) \rangle,
\]

since \( \Gamma(t, x - y) \in S(\mathbb{R}_y^n) \) for any \( t > 0, x \in \mathbb{R}^n \). It is also easily verified that \( u(t, x) \) is a solution of (7.1). The relation \( \lim_{t \to +0} u(t, \cdot) = \varphi(\cdot) \) now holds in the sense of weak convergence in \( S'(\mathbb{R}^n) \). Indeed, it is easy to verify that

\[
\int u(t, x) \psi(x) dx = \int \langle \varphi(\cdot), \Gamma(t, x - \cdot) \rangle \psi(x) dx = \langle \varphi(\cdot), \int \Gamma(t, x - \cdot) \psi(x) dx \rangle = \langle \varphi(\cdot), v(t, \cdot) \rangle,
\]

where \( v(t, x) \) is the Poisson integral corresponding to the initial function \( \psi(x) \) (we justify the possibility to interchange the integration with respect to \( x \) and application of the functional \( \varphi \) by the convergence of the integral in the topology of \( S(\mathbb{R}^n_y) \), cf. an identical argument in the proof of Proposition 5.6).

Since \( v(t, \cdot) \to \psi(\cdot) \) as \( t \to +0 \) in the topology of \( S(\mathbb{R}^n) \), it follows that

\[
\lim_{t \to +0} \int u(t, x) \psi(x) dx = \langle \varphi, \psi \rangle,
\]

implying that

\[
\lim_{t \to +0} u(t, \cdot) = \varphi(\cdot) \text{ in } S'(\mathbb{R}^n).
\]

We have already noted above that \( \lim_{t \to +0} \Gamma(t, x) = \delta(x) \) in \( S'(\mathbb{R}^n) \). This relation is often written in a shorter form

\[
\Gamma|_{t=0} = \delta(x)
\]

and is a particular case of the statement just proved.

Finally, let (7.13) be true for any \( b > 0 \) (with a constant \( C > 0 \) depending on \( b \)). Then the Poisson integral and, therefore, solution of the Cauchy problem are defined for any \( t > 0 \). This is the case e.g. if \( |\varphi(x)| \leq C \exp(b_0|x|^{2-\varepsilon}) \) for some \( \varepsilon > 0, b_0 > 0 \), and, in particular, if \( \varphi(x) \) is bounded.

If \( \varphi \in L^p(\mathbb{R}^n) \) for \( p \in [1, \infty) \), then Poisson’s integral converges by Hölder’s inequality. In this case \( \lim_{t \to +0} u(t, \cdot) = \varphi(\cdot) \) with respect to the norm \( L^p(\mathbb{R}^n) \) (the proof is similar to the argument proving the convergence of mollifiers with respect to the \( L^p \)-norm, see Section 4.2).
It is easy to verify that if \( \varphi(x) \) satisfies (7.13), then the function \( u(t, x) \) determined by Poisson’s integral satisfies a similar estimate:

\[
|u(t, x)| \leq C_1 e^{b_1|x|^2}, \quad 0 \leq t \leq B < \frac{1}{4a^2b},
\]

where the constants \( C_1, b_1 \) depend on \( \varphi \) and \( B \). It turns out that a solution satisfying such an estimate is unique; we will prove this later. At the same time it turns out that the estimate

\[
|u(t, x)| \leq C e^{b|x|^2+\varepsilon}
\]

does not guarantee the uniqueness for any \( \varepsilon > 0 \). (For the proof see e.g. John [15], Ch.7.)

The uniqueness of solution makes it natural to apply Poisson’s integral in order to find a meaningful solution (as a rule, solutions of fast growth have no physical meaning). For instance, the uniqueness of a solution takes place in the class of bounded functions. By the maximum principle (Lemma 7.3), the Cauchy problem is well-posed in the class of bounded functions (if \( \varphi \) is perturbed by adding a uniformly small function, then the same happens with \( u(t, x) \)).

In conclusion notice that the fact that \( \Gamma(t, x) > 0 \) for all \( t > 0, x \in \mathbb{R}^n \) implies that “the speed of the heat propagation is infinite” (since the initial perturbation is supported at the origin!). But \( \Gamma(t, x) \) decreases very fast as \( |x| \to +\infty \), which is practically equivalent to a finite speed of the heat propagation.

### 7.5. The fundamental solution for the heat operator.

**Duhamel’s formula**

**Theorem 7.5.** The following locally integrable function

\[
\mathcal{E}(t, x) = \theta(t)\Gamma(t, x) = (2a\sqrt{\pi t})^{-n}\theta(t)\exp\left(-\frac{|x|^2}{4a^2t}\right)
\]

is a fundamental solution for \( \frac{\partial}{\partial t} - a^2 \Delta_x \) in \( \mathbb{R}^{n+1} \). Here \( \theta(t) \) is the Heaviside function.

**Proof.** First, let us give a heuristic explanation why \( \mathcal{E}(t, x) \) is a fundamental solution. We saw that \( \Gamma(t, x) \) is a continuous function in \( t \) with values in \( S' (\mathbb{R}^n_x) \) for \( t \geq 0 \). The function \( \mathcal{E}(t, x) = \theta(t)\Gamma(t, x) \) is not continuous as a function of \( t \) with values in \( S' (\mathbb{R}^n_x) \), but has a jump at \( t = 0 \) equal to
δ(x). Therefore, applying $\frac{\partial}{\partial t}$, we get $δ(t) \otimes δ(x) + f(t, x)$, where $f(t, x)$ is continuous in $t$ in the similar sense. But then

$$\left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) E(t, x) = δ(t) \otimes δ(x) = δ(t, x),$$

since the summand which is continuous with respect to $t$, vanishes because $(\frac{\partial}{\partial t} - a^2 \Delta_x) E(t, x) = 0$ for $|t| + |x| \neq 0$. This important argument makes it possible to write down the fundamental solution in all cases when we can write the solution of the Cauchy problem in a Poisson integral-like form. We will not bother with its justification, however, since it is simpler to verify directly that $E(t, x)$ is a fundamental solution.

We begin with the verification of local integrability of $E(t, x)$. Clearly, $E(t, x) \in C^\infty(\mathbb{R}^{n+1})$. So we only have to verify local integrability in a neighborhood of the origin. Since

$$\exp \left( -\frac{1}{\tau} \right) \leq C_\alpha \tau^\alpha \quad \text{for } \tau > 0, \text{ for any } \alpha \geq 0,$$

we have

(7.16) $$|E(t, x)| \leq C_\alpha t^{-n/2} \left( \frac{t}{|x|^2} \right)^\alpha = C_\alpha t^{-n/2+\alpha| |x|}^{-2\alpha}.$$

Let $\alpha \in (\frac{n}{2} - 1, \frac{n}{2})$ so that $-\frac{n}{2} + \alpha > -1$ and $-2\alpha > -n$. It is clear from (7.16) that $E(t, x)$ is majorized by an integrable function in a neighborhood of the origin; therefore, it is locally integrable itself.

Now, let $f(t, x) \in C^0(\mathbb{R}^{n+1})$. We have

$$\left\langle \left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) E(t, x), f(t, x) \right\rangle = \left\langle E(t, x), \left( -\frac{\partial}{\partial t} - a^2 \Delta_x \right) f(t, x) \right\rangle =$$

(7.17) $$= \lim_{\varepsilon \to 0} \int_{t \geq \varepsilon} E(t, x) \left( -\frac{\partial}{\partial t} - a^2 \Delta_x \right) f(t, x) dt dx.$$

Let us move $\frac{\partial}{\partial t}$ and $\Delta_x$ in the integral back onto $E(t, x)$. Since $(\frac{\partial}{\partial t} - a^2 \Delta_x) E(t, x) = 0$ for $t > 0$, the result only contains a boundary integral over the plane $t = \varepsilon$ obtained after integrating by parts with respect to $t$:

$$\int E(\varepsilon, x) f(\varepsilon, x) dx = \int \Gamma(\varepsilon, x) f(\varepsilon, x) dx.$$

The latter integral is $u_\varepsilon(\varepsilon, 0)$, where $u_\varepsilon(t, x)$ is Poisson’s integral with the initial function $\varphi(x) = f(\varepsilon, x)$. Consider also Poisson’s integral $u(t, x)$ with the initial function $\varphi(x) = f(0, x)$. By the maximum principle

$$|u(\varepsilon, 0) - u_\varepsilon(\varepsilon, 0)| \leq \sup_x |\varphi(x) - \varphi(x)| = \sup_x |f(\varepsilon, x) - f(0, x)|.$$
This, clearly, implies that
\[ u(\varepsilon, 0) - u_\varepsilon(\varepsilon, 0) \to 0 \quad \text{for } \varepsilon \to +0. \]

But, as we have already seen,
\[ u(\varepsilon, 0) \to \varphi(0) = f(0, 0) \quad \text{for } \varepsilon \to +0, \]
implying
\[ \lim_{\varepsilon \to +0} u_\varepsilon(\varepsilon, 0) = f(0, 0) \]
which, in turn, yields
\[ \lim_{\varepsilon \to +0} \int_{t \geq \varepsilon} E(t, x) \left( -\frac{\partial}{\partial t} - a^2 \Delta_x \right) f(t, x) dt dx = f(0, 0). \]

This, together with (7.17), proves the theorem. \(\Box\)

Corollary 7.6. If \(u(t, x) \in D'(\Omega)\), where \(\Omega\) is a domain in \(\mathbb{R}^{n+1}\), and
\[ (\frac{\partial}{\partial t} - a^2 \Delta_x) u = 0, \]
then \(u \in C^\infty(\Omega)\).

Therefore, the heat operator is an example of a hypoelliptic but not elliptic operator. Note that the heat operator has \(C^\infty\) but non-analytic solutions (e.g. \(E(t, x)\) in a neighborhood of \((0, x_0)\), where \(x_0 \in \mathbb{R}^n \setminus \{0\}\)).

The knowledge of the fundamental solution makes it possible to solve the non-homogeneous equation. For instance, a solution \(u(t, x)\) of the problem
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - a^2 \Delta_x u = f(t, x), & t > 0, \ x \in \mathbb{R}^n; \\
u|_{t=0} = \varphi(x), & x \in \mathbb{R}^n;
\end{array} \right.
\end{aligned}
\]
(7.18)
can be written in the form
\[
u(t, x) = \int_{\tau \geq 0} E(t - \tau, x - y) f(\tau, y) d\tau dy + \int \Gamma(t, x - y) \varphi(y) dy =
\]
(7.19)
\[
\int_0^t d\tau \int_{\mathbb{R}^n} \Gamma(t - \tau, x - y) f(\tau, y) dy + \int \Gamma(t, x - y) \varphi(y) dy
\]
under appropriate assumptions concerning \(f(t, x)\) and \(\varphi(x)\) which we will not formulate exactly. The first integral in (7.19) is of interest. Its inner integral
\[ v(\tau, t) = \int \Gamma(t - \tau, x - y) f(\tau, y) dy \]
is a solution of the Cauchy problem
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(\frac{\partial}{\partial \tau} - a^2 \Delta_x) v(\tau, t, x) = 0, & t > \tau, \ x \in \mathbb{R}^n; \\
v|_{t=\tau} = f(\tau, x).
\end{array} \right.
\end{aligned}
\]
(7.20)
The Duhamel formula is a representation of the solution of the Cauchy problem (7.18) with zero initial function \( \varphi \) in the form

\begin{equation}
\tag{7.21}
\begin{aligned}
  u(t, x) &= \int_{0}^{t} v(\tau, t, x) d\tau,
\end{aligned}
\end{equation}

where \( v(\tau, t, x) \) is the solution of the Cauchy problem (7.20) (for a homogeneous equation!).

A similar representation is possible for any evolution equation. In particular, precisely this representation will do for operators of the form \( \partial_t - A(x, D_x) \), where \( A(x, D_x) \) is a linear differential operator with respect to \( x \) with variable coefficients. Needless to say, in this case \( a^2 \Delta_x \) should be replaced by \( A(x, D_x) \) in (7.20). The formal verification is very simple: by applying \( \partial_t - A(x, D_x) \) to (7.21) we get in the right hand side

\begin{equation}
\begin{aligned}
  v(t, t, x) + \int_{0}^{t} \left( \frac{\partial}{\partial t} - A(x, D_x) \right) v(\tau, t, x) d\tau &= v(t, t, x) = f(t, x),
\end{aligned}
\end{equation}

as required.

### 7.6. Estimates of derivatives of a solution of the heat equation

Since the heat operator is hypoelliptic, we can use Proposition 5.23 to estimate derivatives of a solution of the heat equation through the solution itself on a slightly larger domain. We will only need it on an infinite horizontal strip.

**Lemma 7.7.** Let us assume that a solution \( u(t, x) \) of the heat equation (7.1) is defined in the strip \( \{ t, x : t \in (a, b), x \in \mathbb{R}^n \} \) and satisfies the estimate

\begin{equation}
\tag{7.22}
|u(t, x)| \leq Ce^{d|x|^p},
\end{equation}

where \( p \geq 0, \ C > 0, \ d \in \mathbb{R} \). Then in a somewhat smaller strip

\(\{ t, x : t \in (a', b'), x \in \mathbb{R}^n \} \), where \( a < a' < b' < b \),

similar estimates hold:

\begin{equation}
\tag{7.23}
|\partial^\alpha u(t, x)| \leq C_\alpha e^{d'|x|^p},
\end{equation}

where the constant \( p \) is the same as in (7.22); \( d' = 0 \) if \( d = 0 \) and \( d' = d + \varepsilon \) with an arbitrary \( \varepsilon > 0 \) if \( d \neq 0 \); \( C_\alpha \) depend upon the \((n + 1)\)-dimensional multi-index \( \alpha \) and also upon \( \varepsilon, \ a' \) and \( b' \).
Proof. Let us use Proposition 5.23 with

\[ \Omega_1 = \{ t, x : t \in (a, b), |x| < 1 \}, \quad \Omega_2 = \{ t, x : t \in (a', b'), |x| < 1/2 \}, \]

and estimate (5.23) applied to the solution \( u_y(t, x) = u(t, x + y) \) of the heat equation depending on \( y \in \mathbb{R}^n \) as on a parameter. We get

\[ |\partial^\alpha u(t, x)| \leq C_\alpha \sup_{|x' - x| < 1} |u(t, x')| \leq C'_\alpha \sup_{|x' - x| < 1} e^{d|x'|^p} \leq C'' e^{d|x|^p} \]

for \( t \in (a', b') \), as required. \( \square \)

7.7. Holmgren’s principle. The uniqueness of solution of the Cauchy problem for the heat equation

Holmgren’s principle is, roughly speaking, the fact that the uniqueness of the solution of the abstract Cauchy problem for the equation

\[ \frac{du}{dt} = A(t)u, \ t > 0, \]

with the initial condition \( u(0) = u_0 \), follows from the existence of a solution of the adjoint Cauchy problem directed opposite to the time axis:

\[ \begin{aligned}
\frac{dv}{dt} &= -A'(t)v, \quad t < t_0, \\
v|_{t=t_0} &= \psi.
\end{aligned} \]

Here \( u = u(t), v = v(t) \) are functions of \( t \) with values in vector spaces \( E \) and \( E' \) respectively, with a pairing \( \langle \cdot, \cdot \rangle \), i.e., a bilinear form \( \langle \cdot, \cdot \rangle : E \times E' \rightarrow \mathbb{C} \). This pairing should be non-degenerate, in the sense that if \( u \in E \) and \( \langle u, \psi \rangle = 0 \) for all \( \psi \in E' \), then \( u = 0 \). Further, \( A(t) \) and \( A'(t) \) are linear operators in \( E \) and \( E' \) respectively, which are assumed to be adjoint to each other with respect to the given pairing, i.e.

\[ \langle Au, v \rangle = \langle u, A'v \rangle, \quad u \in E, \ v \in E'. \]

To define the derivatives \( \frac{du}{dt} \) and \( \frac{dv}{dt} \), we need some topology in \( E \) and \( E' \) but we will not give a precise meaning to these derivatives. Instead, we will describe the scheme of the proof of the uniqueness of the solution of the Cauchy problem for the equation (7.24). This scheme will serve as a basis of such a proof in any concrete situation. Therefore, we postpone attaching a precise meaning to the subsequent arguments until a concrete situation arises.

So, let \( u(t) \) be a solution of (7.24) defined for \( 0 < t < T \) and having zero initial value \( u(0) = 0 \). We want to prove that \( u(t) \equiv 0 \). To this end consider
for some \( t_0 \in (0, T) \) a solution \( v(t) \) of the adjoint Cauchy problem \((7.25)\).

We have

\[
\frac{d}{dt}(u(t), v(t)) = \left< \frac{du(t)}{dt}, v(t) \right> + \left< u(t), \frac{dv(t)}{dt} \right>
= \langle A(t)u(t), v(t) \rangle - \langle u(t), A'(t)v(t) \rangle = 0, \quad 0 < t < t_0.
\]

This main calculation, which needs subsequent justifying, explains why the adjoint Cauchy problem appears. The formulas yield:

\[
\langle u(t), v(t) \rangle = \text{const} = \langle u(t_0), \psi \rangle = \langle u(0), v(0) \rangle = 0
\]
(passing to the limit as \( t \to t_0 \) and \( t \to 0 \) should be also justified). Now, if the problem \((7.25)\) is solvable (and the solution has properties which enable us to perform all the preceding calculations) for a class of initial values \( \psi \) such that \( \langle u, \psi \rangle = 0 \) for all \( \psi \) implies \( u = 0 \) then, clearly, \( u(t_0) = 0 \), but then \( u(t) \equiv 0 \), since \( t_0 \) is arbitrary.

Now, armed with the abstract scheme of Holmgren’s principle we will prove the uniqueness of the solution of the Cauchy problem for the heat equation.

**Theorem 7.8** (A.N. Tikhonov, 1935). Let a function \( u(t, x) \) be continuous in the strip \([0, T) \times \mathbb{R}^n\), satisfy the heat equation \((7.1)\), and satisfy the estimate

\[
|u(t, x)| \leq Ce^{b|x|^2}
\]

in the open strip \((0, T) \times \mathbb{R}^n\). If \( u(0, x) = 0 \) for all \( x \in \mathbb{R}^n \), then \( u(t, x) \equiv 0 \) in the whole strip.

**Proof.** Consider the adjoint Cauchy problem

\[
\left\{ \begin{array}{l}
\frac{dv(t, x)}{dt} = -a^2 \Delta_x v(t, x), \quad t < t_0, \\
\left. v \right|_{t=t_0} = \psi(x),
\end{array} \right.
\]

where \( \psi \in D(\mathbb{R}^n) \). Let us express the solution of this problem via Poisson’s integral:

\[ v(t, x) = \int \Gamma(t_0 - t, x - y)\psi(y)dy. \]

Since \( \psi \) has a compact support, this solution satisfies

\[
|v(t, x)| \leq C_\varepsilon \sup_{y \in \text{supp} \psi} \exp \left( -\frac{|x - y|^2}{4a^2(t_0 - t)} \right) \leq C'_\varepsilon \exp \left( -\frac{c|x|^2}{4a^2(t_0 - t)} \right),
\]

for \( t < t_0 - \varepsilon \), where \( \varepsilon > 0, c > 0, |x| \geq R, \) and \( R \) is sufficiently large.
The same estimate holds also for any derivative $\partial^\alpha v(t,x)$, where $\alpha$ is a $(n+1)$-dimensional multi-index. Let $t_0 > 0$ be so small that $\frac{c}{4a^2t_0} > b$, i.e., $t_0 < \frac{c}{4a^2b}$. Then the integral
\[
\langle u(t,x), v(t,x) \rangle = \int u(t,x)v(t,x)dx
\]
makes sense and converges uniformly for $0 \leq t \leq t_0 - \varepsilon$. As Lemma 7.7 shows, the integrals obtained by replacing $u$ or $v$ by their derivatives with respect to $t$ or $x$ also converge uniformly for $0 < \varepsilon \leq t \leq t_0 - \varepsilon$, where $\varepsilon > 0$ is arbitrary. Therefore, the function $\chi(t) = \langle u(t,x), v(t,x) \rangle$ is continuous at $t \in (0,t_0)$ is
\[
\frac{d\chi(t)}{dt} = \int \frac{\partial(u(t,x)v(t,x))}{\partial t}dx = \int \left( \frac{\partial u}{\partial t} \cdot v + u \cdot \frac{\partial v}{\partial t} \right) dx = a^2 \int [(\Delta_x u)v - u(\Delta_x v)]dx.
\]
The latter integral vanishes, since, by Green’s formula (4.32), it may be expressed in the form
\[
\lim_{R \to \infty} \int_{|x| \leq R} [(\Delta_x u)v - u(\Delta_x v)]dx = \lim_{R \to \infty} \int_{|x| = R} \left( \frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n} \right) dS = 0,
\]
because $\frac{\partial u}{\partial n} \cdot v$ and $u \cdot \frac{\partial v}{\partial n}$ tend to 0 exponentially fast as $|x| \to +\infty$. Therefore, $\chi(t) = \text{const} = 0$.

Now, let us verify that
\[
\int u(t_0,x)\psi(x)dx = \lim_{t \to t_0-0} \int u(t,x)v(t,x)dx.
\]
(Actually, the right hand side equals $\lim_{t \to t_0-0} \chi(t) = 0$.) We have
\[
\int u(t,x)v(t,x)dx = \int u(t,x)\Gamma(t_0 - t, x - y)\psi(y)dydx = \int \left( \int \Gamma(t_0 - t, x - y)u(t,x)dx \right) \psi(y)dy \to \int u(t_0,y)\psi(y)dy,
\]
since the argument we used in the consideration of Poisson’s integral shows that the convergence
\[
\lim_{t \to t_0-0} \int \Gamma(t_0 - t, x - y)u(t,x)dx = u(t_0,y)
\]
is uniform on any compact in $\mathbb{R}^n$. Thus, clearly,

$$\int u(t_0, x)\psi(x)dx = 0, \quad 0 \leq t_0 \leq \delta, \quad \psi \in \mathcal{D}(\mathbb{R}^n),$$

if $\delta < \frac{c}{4a^2b}$. Therefore, $u(t, x) \equiv 0$ for $0 \leq t \leq \delta$.

But now we may pass to defining initial conditions for $t = \delta$ and prove that $u(t, x) = 0$ already for $0 \leq t \leq 2\delta$, etc. Finally, we get $u(t, x) \equiv 0$ on the whole strip, as required.

Thus, in the class of functions satisfying (7.26), a solution of the Cauchy problem for the heat equation is unique.

**Corollary 7.9.** Let $u = u(t, x)$ be a solution of the Cauchy problem (7.2) in $[0, T) \times \mathbb{R}^n$ and $u$ satisfies the growth restriction (7.26). Then the maximum principle (7.15) holds for all $t \in [0, T)$ and $x \in \mathbb{R}^n$.

**Proof.** Let us prove the first inequality in (7.15). (The second one is easily reduced to the first if we replace $u$ by $-u$ and $\varphi$ by $-\varphi$.) Note that the result is obvious if $\inf_{x \in \mathbb{R}^n} \varphi(x) = -\infty$. So from now on assume that the infimum is finite, that is, $\varphi$ is bounded below. But then, adding a constant, we can reduce the general statement to the case when the infimum of $\varphi$ is 0, in particular, $\varphi \geq 0$, and we have to prove that $u \geq 0$ everywhere. Due to the uniqueness theorem, $u(t, x)$ can be presented by Poisson’s integral (7.12) for small $t$, $0 < t < \frac{1}{4a^2b}$. Then, clearly, $u(t, x) \geq 0$ for such $t$, due to positivity of the Poisson kernel $\Gamma$. But then, moving by small steps (e.g. $1/(8a^2b)$) in $t$, we obtain that $u \geq 0$ for all $t \in [0, T)$.

### 7.8. A scheme of solving the first and second
initial-boundary value problems by the Fourier method

Let us try to solve the first initial-boundary value problem with zero boundary conditions

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = a^2 u, & t > 0, \quad x \in \Omega, \\
u|_{x \in \partial \Omega} = 0, & t > 0, \quad x \in \partial \Omega, \\
u|_{t=0} = \varphi(x), & x \in \Omega
\end{cases}
\end{equation}

by the Fourier method. Here we assume $\Omega$ to be a bounded domain in $\mathbb{R}^n$.

If the function $u(t, x) = \psi(t)v(x)$ satisfies the first two conditions in (7.27), we formally get

\begin{equation}
\frac{\psi'(t)}{a^2 \psi(t)} = \frac{\Delta v(x)}{v(x)} = -\lambda, \quad v|_{\partial \Omega} = 0,
\end{equation}
leading to the eigenvalue problem for the operator $-\Delta$ with zero boundary conditions on $\partial \Omega$:

\[
\begin{cases}
-\Delta v = \lambda v, & x \in \Omega, \\
v|_{\partial \Omega} = 0.
\end{cases}
\]

(7.29)

For $n = 1$ this problem becomes the simplest Sturm-Liouville problem considered above (with the simplest boundary conditions: $v$ should vanish at the endpoints of the segment). We will prove below that problem (7.29) possesses properties very similar to those of the Sturm-Liouville problem. In particular, it has a complete orthogonal system of eigenfunctions $v_k(x)$, $k = 1, 2, \ldots$, with eigenvalues $\lambda_k$, such that $\lambda_k > 0$ and $\lambda_k \to +\infty$ as $k \to +\infty$.

It follows from (7.28) that $\psi(t) = C e^{-\lambda a^2 t}$ which makes it clear that it is natural to seek $u(t, x)$ as a sum of the series

\[u(t, x) = \sum_{k=1}^{\infty} C_k e^{-\lambda_k a^2 t} v_k(x).\]

The coefficients $C_k$ are chosen from the initial condition $u|_{t=0} = \varphi(x)$ and are coefficients of the expansion of $\varphi(x)$ in terms of the orthogonal system $\{v_k(x)\}_{k=1}^{\infty}$.

The second initial-boundary value problem

\[
\begin{cases}
\frac{\partial u}{\partial t} = a^2 \Delta u, & t > 0, x \in \Omega, \\
\frac{\partial u}{\partial n}|_{x \in \partial \Omega} = 0, & t > 0, x \in \partial \Omega, \\
u|_{t=0} = \varphi(x), & x \in \Omega
\end{cases}
\]

is solved similarly. It leads to the eigenvalue problem

\[
\begin{cases}
-\Delta v = \lambda v, & x \in \Omega \\
\frac{\partial v}{\partial n}|_{\partial \Omega} = 0
\end{cases}
\]

(7.30)

with the same properties as the problem (7.29), the only exception being that $\lambda = 0$ is an eigenvalue of (7.30).

The same approach, as for $n = 1$, yields solutions of more general problems, for example,

\[
\begin{cases}
\frac{\partial u}{\partial t} = a^2 \Delta u + f(t, x), & t > 0, x \in \Omega, \\
u|_{x \in \partial \Omega} = \alpha(t, x), & t > 0, x \in \partial \Omega, \\
u|_{t=0} = \varphi(x), & x \in \Omega.
\end{cases}
\]
For this, subtracting an auxiliary function, we make $\alpha(t, x) \equiv 0$ and then seek the solution $u(t, x)$ in the form of a series

$$u(t, x) = \sum_{k=1}^{\infty} \psi_k(t) v_k(x),$$

which leads to first order ordinary differential equations for $\psi_k(t)$.

The uniqueness of the solution of these problems may be proved by Holmgren’s principle or derived from the maximum principle: if $u(t, x)$ is a function continuous in the cylinder $[0, T] \times \Omega$ and satisfies the heat equation on $(0, T) \times \Omega$, then

$$u(t, x) \leq \max \left\{ \sup_{x \in \Omega} u(0, x), \sup_{x \in \partial \Omega, t \in [0, T]} u(t, x) \right\};$$

and if the equality takes place then $u = \text{const}$.

We will not prove this physically natural principle. Its proof is quite elementary and may be found, for instance, in Petrovskii [21].

In what follows, we give the proof of existence of a complete orthogonal system of eigenfunctions for the problem (7.29). The boundary condition in (7.29) will be understood in a generalized sense. The Dirichlet problem will also be solved in a similar sense. All this requires introducing Sobolev spaces whose theory we are about to present.

### 7.9. Problems

**7.1.** Given a rod of length $l$ with a heat-insulated surface (the lateral sides and ends). For $t = 0$, the temperature of the left half of the rod is $u_1$ and that of the right half is $u_2$. Find the rod temperature for $t > 0$ by the Fourier method. Find the behavior of the temperature as $t \to +\infty$. Estimate the relaxation time needed to decrease by half the difference between the maximal and minimal temperatures of the rod; in particular, find the dependence of this time on the parameters of the rod: length, thickness, density, specific heat and thermal conductivity.

**7.2.** The rod is heated by a constant electric current. Zero temperature is maintained at the ends and the side surface is insulated. Assuming that the temperature in any cross-section of the rod is a constant, find the distribution of the temperature in the rod from an initial distribution and describe the behavior of the temperature as $t \to +\infty$. 
7.3. Let a bounded solution $u(t,x)$ of the heat equation $u_t = a^2 u_{xx}$ ($x \in \mathbb{R}^1$) be determined for $t > 0$ and satisfy the initial condition $u|_{t=0} = \varphi(x)$, where \( \varphi \in C(\mathbb{R}^1), \lim_{x \to +\infty} \varphi(x) = b, \lim_{x \to -\infty} \varphi(x) = c \). Describe the behavior of $u(t,x)$ as $t \to +\infty$.

7.4. Prove the maximum principle for the solutions of the heat equation in the formulation given at the end of this section (see (7.31)).
Chapter 8

Sobolev spaces.
A generalized solution of Dirichlet’s problem

8.1. Spaces $H^s(\Omega)$

Let $s \in \mathbb{Z}_+$, $\Omega$ an open subset in $\mathbb{R}^n$.

**Definition 8.1.** The *Sobolev space* $H^s(\Omega)$ consists of $u \in L^2(\Omega)$ such that $\partial^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq s$, where $\alpha$ is a multiindex, and the derivative is understood in the distributional sense.

In $H^s(\Omega)$, introduce the inner product

$$(u, v)_s = \sum_{|\alpha| \leq s} (\partial^\alpha u, \partial^\alpha v),$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, and also the corresponding norm

$$\|u\|_s = (u, u)_s^\frac{1}{2} = \left[ \sum_{|\alpha| \leq s} \int_{\Omega} |\partial^\alpha u(x)|^2 dx \right]^\frac{1}{2}.$$  

Clearly, $H^s(\Omega) \subset H^{s'}(\Omega)$ for $s \geq s'$. Furthermore, $H^0(\Omega) = L^2(\Omega)$. In particular, each space $H^s(\Omega)$ is embedded into $L^2(\Omega)$.  

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Proposition 8.2. The inner product $(u, v)_s$ defines in $H^s(\Omega)$ the structure of a separable Hilbert space (i.e. a Hilbert space which admits at most countable orthonormal basis).

Proof. Only the completeness and separability are not obvious. Let us prove the completeness. Let a sequence $\{u_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $H^s(\Omega)$. The definition of $\|\cdot\|_s$ implies that all the sequences $\{\partial^\alpha u_k\}_{k=1}^{\infty}$ (for $|\alpha| \leq s$) are Cauchy sequences in $L^2(\Omega)$. But then, by the completeness of $L^2(\Omega)$, they converge, i.e., $\lim_{k \to \infty} \partial^\alpha u_k = v_\alpha$ with respect to the norm in $L^2(\Omega)$. In particular, $u_k \to v_0$ in $L^2(\Omega)$. But then $\partial^\alpha u_k \to \partial^\alpha v_0$ in $D'(\Omega)$ and since, on the other hand, $\partial^\alpha u_k \to v_\alpha$ in $D'(\Omega)$, it follows that $v_\alpha = \partial^\alpha v_0$. Therefore, $v_0 \in H^s(\Omega)$ and $\partial^\alpha u_k \to \partial^\alpha v_0$ in $L^2(\Omega)$ for $|\alpha| \leq s$. But this means that $u_k \to v_0$ in $H^s(\Omega)$.

Let us prove separability. The map $u \mapsto \{\partial^\alpha u\}_{|\alpha| \leq s}$ defines an isometric embedding of $H^s(\Omega)$ into the direct sum of several copies of $L^2(\Omega)$ as a closed subspace. Now, the separability of $H^s(\Omega)$ follows from that of $L^2(\Omega)$.

Let us consider the case $\Omega = \mathbb{R}^n$ in more detail. The space $H^s(\mathbb{R}^n)$ can be described in terms of the Fourier transform (understood in the sense of distributions, in $S'(\mathbb{R}^n)$).

Recall that, by Plancherel’s theorem, the Fourier transformation

$$F : u(x) \mapsto \hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx$$

determines an isometry of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n_\xi)$, where $L^2(\mathbb{R}^n_\xi)$ is defined by the measure $(2\pi)^{-n} d\xi$. Let us explain the meaning of this statement and its interpretation from distributional viewpoint. We know from Analysis that the Parseval identity

$$(8.2) \quad \int |u(x)|^2 dx = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 d\xi$$

holds for $u \in S(\mathbb{R}^n)$. The identity (8.2) for $u \in S(\mathbb{R}^n)$ follows easily, for instance, from the inversion formula (5.24). Indeed,

$$\int |\hat{u}(\xi)|^2 d\xi = \int \left( \int e^{-ix \cdot \xi} u(x) dx \right) \left( \int e^{iy \cdot \xi} \overline{u(y)} dy \right) d\xi$$

$$= \int e^{i(y-x) \cdot \xi} u(x) \overline{u(y)} dx dy d\xi = \int \overline{u(y)} \left[ \int e^{i(y-x) \cdot \xi} u(x) dx d\xi \right] dy$$

$$= (2\pi)^n \int |u(y)|^2 dy.$$
Since $F$ defines an isomorphism of $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, then $F$ can be extended to an isometry $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. The convergence in $L^2(\mathbb{R}^n)$ implies a (weak) convergence in $S'(\mathbb{R}^n)$, the space $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is continuous in restrictions of the weak topologies of $S'(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$. Therefore, this $F$ is, actually, a restriction of the generalized Fourier transform $F : S'(\mathbb{R}^n) \longrightarrow S'(\mathbb{R}^n)$.

Now, note that the Fourier transform maps $D^\alpha u(x)$ to $\xi^\alpha \tilde{u}(\xi)$. Therefore, $u \in H^s(\mathbb{R}^n)$ if and only if $\xi^\alpha \tilde{u}(\xi) \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq s$. But instead of this, we may write \( \sum_{|\alpha| \leq s} |\xi^\alpha|^2 |\tilde{u}(\xi)|^2 \in L^1(\mathbb{R}^n) \). Due to the obvious estimates

\[ C^{-1}(1 + |\xi|^2)^s \leq \sum_{|\alpha| \leq s} |\xi^\alpha|^2 \leq C(1 + |\xi|^2)^s, \]

where $C > 0$ does not depend on $\xi$, we have $u \in H^s(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^{s/2} \tilde{u}(\xi) \in L^2(\mathbb{R}^n)$ and the norm (8.1) for $\Omega = \mathbb{R}^n$ is equivalent to the norm

\[ (8.3) \quad \|u\|_s = \left[ \int (1 + |\xi|^2)^s |\tilde{u}(\xi)|^2 d\xi \right]^{1/2}. \]

We will denote the norm (8.3) by the same symbol as the norm (8.1); there should not be any misunderstanding. Thus, we have proved

**Proposition 8.3.** The space $H^s(\mathbb{R}^n)$ consists of $u \in S'(\mathbb{R}^n)$ such that $(1 + |\xi|^2)^{s/2} \tilde{u}(\xi) \in L^2(\mathbb{R}^n)$. It can be equipped with the norm (8.3) which is equivalent to the norm (8.1).

This proposition may serve as a basis for definition of $H^s(\mathbb{R}^n)$ for any real $s$. Namely, for any $s \in \mathbb{R}$ we may construct the space of $u \in S'(\mathbb{R}^n)$, such that $\tilde{u}(\xi) \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{s/2} \tilde{u}(\xi) \in L^2(\mathbb{R}^n)$, with the norm (8.3). We again get a separable Hilbert space denoted also, as for $s \in \mathbb{Z}_+$, by $H^s(\mathbb{R}^n)$.

In what follows we will need the Banach space $C^k_b(\mathbb{R}^n)$ (here $k \in \mathbb{Z}_+$) consisting of functions $u \in C^k(\mathbb{R}^n)$, whose derivatives $\partial^\alpha u(x)$, $|\alpha| \leq k$, are bounded for all $x \in \mathbb{R}^n$. The norm in $C^k_b(\mathbb{R}^n)$ is defined by the formula

\[ \|u\|_{(k)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)|. \]

**Theorem 8.4.** (Sobolev’s embedding theorem) For $s > \frac{n}{2} + k$, we have a continuous embedding

\[ (8.4) \quad H^s(\mathbb{R}^n) \subset C^k_b(\mathbb{R}^n). \]
Let us clarify the formulation. At the first glance, it seems that it is meaningless, since a function \( u \in H^s(\mathbb{R}^n) \) can be arbitrarily modified on any set of measure 0 without affecting the corresponding distribution (and the element of \( H^s(\mathbb{R}^n) \)). Changing \( u \) at all points with rational coordinates we may make it discontinuous everywhere. Therefore, we should more precisely describe the embedding (8.4), as follows: if \( u \in H^s(\mathbb{R}^n) \), then there exists a unique function \( u_1 \in C^k_0(\mathbb{R}^n) \) which coincides with the initial function \( u \) almost everywhere (and, for brevity we write \( u \) instead of \( u_1 \)). In other words, \( H^s(\mathbb{R}^n) \) is a subspace of \( C^k_0(\mathbb{R}^n) \) provided both are considered as subspaces of \( S'(\mathbb{R}^n) \) or \( D'(\mathbb{R}^n) \).

Observe that the uniqueness of a continuous representative is obvious, since in any neighborhood of \( x_0 \in \mathbb{R}^n \) there exist points of any set of total measure, so that any modification of a continuous function on a set of measure 0 leads to a function which is discontinuous at all points where the modification took place.

**Proof of Theorem 8.4.** First, let us prove the estimate

\[
\|u\|_{(k)} \leq C\|u\|_s, \ u \in S(\mathbb{R}^n),
\]

where \( C \) is a constant independent of \( u \). The most convenient way is to apply the Fourier transform. We have

\[
\partial^\alpha u(x) = \frac{1}{(2\pi)^n} \int (i\xi)^\alpha e^{ix\cdot\xi} \hat{u}(\xi) d\xi,
\]

implying

\[
|\partial^\alpha u(x)| \leq \frac{1}{(2\pi)^n} \int |\xi^\alpha \hat{u}(\xi)| d\xi;
\]

hence,

\[
\|u\|_{(k)} \leq C \int (1 + |\xi|^2)^{k/2} |\hat{u}(\xi)| d\xi.
\]

Dividing and multiplying the integrand by \( (1 + |\xi|^2)^{s/2} \), we have by the Cauchy-Schwarz inequality:

\[
\|u\|_{(k)} \leq C \int (1 + |\xi|^2)^{(k-s)/2} (1 + |\xi|^2)^{s/2} |\hat{u}(\xi)| d\xi \leq
\]

\[
\leq C \left( \int (1 + |\xi|^2)^{(k-s)} d\xi \right)^{\frac{1}{2}} \cdot \left( \int (1 + |\xi|^2)^{s/2} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

The integral in the first factor in the right hand side converges, since \( 2(k - s) < -n \) and, therefore, the integrand decays faster than \( |\xi|^{-n-\varepsilon} \) as \( |\xi| \to +\infty \) for a sufficiently small \( \varepsilon > 0 \). The second factor is equal to \( \|u\|_s \). Therefore, for \( s > k + \frac{n}{2} \) we get (8.5).
8.1. Spaces \( H^s(\Omega) \)

Now, note that \( S(\mathbb{R}^n) \) is dense in \( H^s(\mathbb{R}^n) \) for any \( s \in \mathbb{R} \). Indeed, introduce the operator \( \Lambda_s \) which acts on \( u = u(x) \) by multiplying the Fourier transform \( \tilde{u}(\xi) \) by \( (1 + |\xi|^2)^{s/2} \). This operator is a unitary map of \( H^s(\mathbb{R}^n) \) onto \( L^2(\mathbb{R}^n) \) mapping \( S(\mathbb{R}^n) \) isomorphically onto \( S(\mathbb{R}^n) \). Therefore, \( S(\mathbb{R}^n) \) is dense in \( H^s(\mathbb{R}^n) \), because \( S(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \).

Now, let \( v \in H^s(\mathbb{R}^n), u_m \in S(\mathbb{R}^n), u_m \rightarrow v \) in \( H^s(\mathbb{R}^n) \). But (8.5) implies that the sequence \( u_m \) is a Cauchy sequence with respect to the norm \( \| \cdot \|_{(k)} \). Therefore, \( u_m \rightarrow v_1 \) in \( C^k_0(\mathbb{R}^n) \). But then \( v \) and \( v_1 \) coincide as distributions and, therefore, almost everywhere. By continuity, (8.5) holds for all \( u \in H^s(\mathbb{R}^n) \) proving the continuity of the embedding (8.4). \( \square \)

In particular, Theorem 8.4 shows that it makes sense to talk about values of functions \( u \in H^s(\mathbb{R}^n) \) at a point for \( s > \frac{n}{2} \).

Also of interest is the question of a meaning of restriction (or trace) of a function from a Sobolev space on a submanifold of an arbitrary dimension. We will not discuss this problem in full generality, however, and only touch the most important case of codimension 1. For simplicity, consider only the case of a hyperplane \( x_n = 0 \) in \( \mathbb{R}^n \). A point \( x \in \mathbb{R}^n \) will be expressed in the form \( x = (x', x_n) \), where \( x' \in \mathbb{R}^{n-1} \).

**Theorem 8.5** (Sobolev’s theorem on restrictions or traces). If \( s > 1/2 \), then the restriction operator \( S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^{n-1}), u \mapsto u_{|x_n=0} \), can be extended to a linear continuous map

\[
H^s(\mathbb{R}^n) \rightarrow H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}).
\]

**Proof.** Let us express the restriction operator in terms of the Fourier transform (for \( u \in S(\mathbb{R}^n) \)):

\[
u(x',0) = \frac{1}{(2\pi)^n} \int e^{ix' \cdot \xi'} \tilde{u}(\xi', \xi_n) d\xi' d\xi_n.
\]

Denoting by \( F' \) the Fourier transform with respect to \( x' \), we get

\[
F'u(x',0) = \frac{1}{2\pi} \int \tilde{u}(\xi', \xi_n) d\xi_n.
\]

For brevity, set \( v(x') = u(x',0), \tilde{v}(\xi') = F'v(x') \), so that

\[
(8.6) \quad \tilde{v}(\xi') = \frac{1}{2\pi} \int \tilde{u}(\xi', \xi_n) d\xi_n.
\]

The needed statement takes on the form of the estimate

\[
(8.7) \quad \|v\|_{s-\frac{1}{2}} \leq C\|u\|_s,
\]
where \( \| \cdot \|_{s-\frac{1}{2}} \) is the norm in \( H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \), \( u \in S(\mathbb{R}^n) \) and \( C \) does not depend on \( u \). Let us prove this estimate. We have

\[
(8.8) \quad \left( \| v \|_{s-\frac{1}{2}} \right)^2 = \int (1 + |\xi'|^2)^{s-\frac{1}{2}} |\mathring{v}(\xi')|^2 d\xi'.
\]

From (8.6) we deduce for any \( s > 0 \):

\[
(8.9) \quad |\mathring{v}(\xi')|^2 = \frac{1}{4\pi^2} \left[ \int (1 + |\xi|^2)^{-s/2} (1 + |\xi|^2)^{s/2} \mathring{u}(\xi) d\xi_n \right]^2 \leq \frac{1}{4\pi^2} \int (1 + |\xi|^2)^{-s} d\xi_n \cdot \int (1 + |\xi|^2)^s |\mathring{u}(\xi)|^2 d\xi_n.
\]

Let us estimate the first factor. We have

\[
\int (1 + |\xi|^2)^{-s} d\xi_n = \int (1 + |\xi'|^2 + |\xi_n|^2)^{-s} d\xi_n
= (1 + |\xi'|^2)^{-s} \int \left( 1 + \frac{\xi_n}{\sqrt{1 + |\xi'|^2}} \right)^2 d\xi_n
= (1 + |\xi'|^2)^{-s} \frac{1}{2} \int_{-\infty}^{\infty} (1 + |t|^2)^{-s} dt = C(1 + |\xi'|^2)^{-s+\frac{1}{2}}.
\]

Here we used the fact that \( s > \frac{1}{2} \), which ensures the convergence of the last integral. Therefore, (8.9) is expressed in the form

\[
|\mathring{v}(\xi')|^2 \leq C(1 + |\xi'|^2)^{-s+\frac{1}{2}} \int (1 + |\xi|^2)^s |\mathring{u}(\xi)|^2 d\xi_n.
\]

By applying this inequality to estimate \( |\mathring{v}(\xi')|^2 \) in (8.8), we get the desired estimate (8.7). \( \square \)

**Remark.** Since \( s > 1/2 \), the restriction map \( H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \), \( u \mapsto u|_{x_n=0} \), is actually surjective (see e.g. Hörmander [12], Ch.II), so the statement of the theorem is the exact one (the index \( s - \frac{1}{2} \) cannot be increased).

Theorem 8.5 means that if \( u \in H^s(\mathbb{R}^n) \), then the trace \( u|_{x_n=0} \) of \( u \) on the hyperplane \( x_n = 0 \) is well-defined and is an element of \( H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \). It is obtained as follows: represent \( u \) in the form of the limit \( u = \lim_{m \to \infty} u_m \) of functions \( u_m \in S(\mathbb{R}^n) \) with respect to the norm \( \| \cdot \|_s \). Then the restrictions \( u_m|_{x_n=0} \) have a limit as \( m \to \infty \) with respect to the norm \( \| \cdot \|_{s-\frac{1}{2}} \) in the space \( H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \), and this limit does not depend on the choice of an approximating sequence.
If we wish to restrict ourselves to integer values of \( s \) we may use the fact that \( H^{s-rac{1}{2}}(\mathbb{R}^{n-1}) \subset H^{s-1}(\mathbb{R}^{n-1}) \). Therefore, the trace \( u|_{x_n=0} \) of \( u \in H^s(\mathbb{R}^n) \) belongs to \( H^{s-1}(\mathbb{R}^{n-1}) \).

The space \( H^s(M) \), where \( M \) is a smooth compact manifold (without boundary), can also be defined. Namely, we will write \( u \in H^s(M) \) if for any coordinate diffeomorphism \( \kappa : U \to \Omega \) (here \( U \) is an open subset of \( M \), \( \Omega \) an open subset of \( \mathbb{R}^n \)) and any \( \varphi \in C_0^\infty(\Omega) \) we have \( \varphi(u \circ \kappa^{-1}) \in H^s(\mathbb{R}^n) \). Here \( u \circ \kappa^{-1} \) is a distribution \( u \) pulled back to \( \Omega \) via \( \kappa^{-1} \). It is multiplied by \( \varphi \) to get a distribution with support in \( \Omega \). Therefore, \( \varphi(u \circ \kappa^{-1}) \in \mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n) \); in particular, \( \varphi(u \circ \kappa^{-1}) \in \mathcal{S}'(\mathbb{R}^n) \), and it makes sense to talk about the inclusion \( \varphi(u \circ \kappa^{-1}) \in H^s(\mathbb{R}^n) \). The localization helps us to prove the following analogue of Theorem 8.5:

if \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \), then for \( s > \frac{1}{2} \) the restriction map \( u \mapsto u|_{\partial \Omega} \) can be extended from \( C^\infty(\bar{\Omega}) \) to a linear continuous operator \( H^s(\Omega) \to H^{s-rac{1}{2}}(\partial \Omega) \).

8.2. Spaces \( \tilde{H}^s(\Omega) \)

Definition 8.6. The space \( \tilde{H}^s(\Omega) \) is the closure of \( \mathcal{D}(\Omega) \) in \( H^s(\Omega) \) (we assume that \( s \in \mathbb{Z}_+ \)).

Therefore, \( \tilde{H}^s(\Omega) \) is a closed subspace in \( H^s(\Omega) \). Hence, it is a separable Hilbert space itself.

Clearly, \( \tilde{H}^0(\Omega) = H^0(\Omega) = L^2(\Omega) \). But already \( \tilde{H}^1(\Omega) \) does not necessarily coincide with \( H^1(\Omega) \). Indeed, if \( \Omega \) is a bounded domain with a smooth boundary, then, as had been noted above, a function \( u \in H^1(\Omega) \) possesses a trace \( u|_{\partial \Omega} \in L^2(\partial \Omega) \). However, if \( u \in \tilde{H}^1(\Omega) \), then \( u|_{\partial \Omega} = 0 \). Therefore, if, e.g. \( u \in C^\infty(\bar{\Omega}) \) and \( u|_{\partial \Omega} \neq 0 \), then \( u \notin \tilde{H}^1(\Omega) \).

The space \( \tilde{H}^s(\Omega) \) is isometrically embedded into \( H^s(\mathbb{R}^n) \) as well. This embedding extends the embedding \( \mathcal{D}(\Omega) \subset \mathcal{D}(\mathbb{R}^n) \). Such an extension is defined by continuity, since the norm \( \| \cdot \|_s \), considered on \( \mathcal{D}(\Omega) \), is equivalent to the norm in \( H^s(\mathbb{R}^n) \).

The following theorem on compact embeddings is important.

Theorem 8.7. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( s, s' \in \mathbb{Z}_+, s > s' \). Then the embedding \( \tilde{H}^s(\Omega) \subset H^{s'}(\Omega) \) is a compact operator.

\( \blacktriangleright \) Recall, that a linear operator \( A : E_1 \to E_2 \), where \( E_1, E_2 \) are Banach spaces, is called compact if the image of the unit ball \( B_1 = \{ x : x \in \)
$E_1, \|x\|_{E_1} \leq 1$ is a precompact subset in $E_2$. In turn, a set $Q$ in a metric space $E$ is called precompact (in $E$) if its closure is compact or, equivalently, if from any sequence of points of $Q$ we can choose a subsequence converging to a point of $E$. If $E$ is a complete metric space, then a set $Q \subset E$ is precompact if and only if it is totally bounded, i.e. for any $\varepsilon > 0$ it has a finite $\varepsilon$-net: a finite set $x_1, \ldots, x_N \in E$, such that

\[
Q \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon)
\]

(here $B(x, r)$ means open ball in $E$ of radius $r$ centered at $x$), or, in other words, for any point $x \in Q$ there exists $k \in \{1, \ldots, N\}$ such that $\rho(x, x_k) < \varepsilon$, where $\rho$ is the metric in $E$.

Let $K$ be a compact subset in $\mathbb{R}^n$ (i.e., $K$ is closed and bounded). Consider the space $C(K)$ of continuous complex-valued functions on $K$. Let $F \subset C(K)$. The Arzela-Ascoli theorem gives a criterion of precompactness of $F$:

$F$ is precompact if and only if it is uniformly bounded and equicontinuous.

Here the uniform boundedness of the set $F$ means the existence of a constant $M > 0$, such that $\sup_{x \in K} |f(x)| \leq M$ for any $f \in F$; the equicontinuity means that for any $\varepsilon > 0$ there exists $\delta > 0$, such that $|x' - x''| < \delta$ with $x', x'' \in K$ implies $|f(x') - f(x'')| < \varepsilon$ for any $f \in F$.

Proofs of the above general facts on compactness and the Arzela-Ascoli theorem may be found in any textbook on functional analysis, see, e.g., Kolmogorov and Fomin [16], Chapter II, Sect. 6 and 7.

Note the obvious corollary of general facts on compactness: for a set $Q \subset E$, where $E$ is a complete metric space, to be precompact it suffices that for any $\varepsilon > 0$ there exists a precompact set $Q_\varepsilon \subset E$, such that $Q$ belongs to the $\varepsilon$-neighborhood of $Q_\varepsilon$. Indeed, a finite $\varepsilon/2$-net of $Q_{\varepsilon/2}$ is an $\varepsilon$-net for $Q$. ▲

**Proof of Theorem 8.7.** We will use the mollifying operation (see Proof of Lemma 4.5). We have introduced there a family $\varphi_\varepsilon \in D(\mathbb{R}^n)$, such that $\text{supp} \varphi_\varepsilon \subset \{x : |x| \leq \varepsilon\}$, $\varphi_\varepsilon \geq 0$ and $\int \varphi_\varepsilon(x)dx = 1$. Then from $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ we may form a mollified family:

\[
f_\varepsilon(x) = \int f(x - y)\varphi_\varepsilon(y)dy = \int f(y)\varphi_\varepsilon(x - y)dy.
\]
In Section 4.2 we have formulated and proved a number of important properties of mollifying. Now we will make use of these properties.

First, observe that a set \( F \) is precompact in \( H^{s'}(\Omega) \) if and only if all the sets \( F_\alpha = \{ \partial^\alpha f : f \in F \} \), where \( \alpha \) is a multi-index such that \( |\alpha| \leq s' \), are precompact in \( L^2(\Omega) \). Further, \( f \in \tilde{H}^s(\Omega) \) clearly implies \( \partial^\alpha f \in \tilde{H}^{s-|\alpha|}(\Omega) \), where \( |\alpha| \leq s \). Since \( s > s' \), it is clear that all sets \( F_\alpha \) belong to a bounded subset of \( \tilde{H}^1(\Omega) \). Therefore, it suffices to verify the compactness of the embedding \( \tilde{H}^1(\Omega) \subset L^2(\Omega) \).

Let \( F = \{ u : u \in \tilde{H}^1(\Omega), \| u \|_1 \leq 1 \} \). We should verify that \( F \) is precompact in \( L^2(\Omega) \) (or, which is the same, in \( L^2(\mathbb{R}^n) \)). The idea of this verification is as follows: consider the set \( F_\varepsilon = \{ f_\varepsilon(x), f \in F \} \) obtained by \( \varepsilon \)-mollifying of all functions from \( F \). Then prove that for a small \( \varepsilon > 0 \) the set \( F \) belongs to an arbitrarily small neighborhood of \( F_\varepsilon \) (with respect to the norm in \( L^2(\mathbb{R}^n) \)) and \( F_\varepsilon \) is precompact in \( C(\bar{\Omega}) \) for a fixed \( \varepsilon \) (hence, precompact in \( L^2(\Omega) \)). The remark before the proof shows then that \( F \) is precompact in \( L^2(\Omega) \).

Observe also that definition of the distributional derivative easily implies:

\[
\left( \frac{\partial f}{\partial x_j} \right)_\varepsilon (x) = \int \frac{\partial f(y)}{\partial y_j} \varphi_\varepsilon(x-y)dy = -\int f(y) \frac{\partial}{\partial y_j} \varphi_\varepsilon(x-y)dy = \int f(y) \frac{\partial \varphi_\varepsilon(x-y)}{\partial x_j}dy = \int f(x) \frac{\partial f_\varepsilon}{\partial x_j}, \quad f \in \tilde{H}^1(\Omega).
\]

In other words, distributional differentiation commutes with mollifying. Clearly, we have (by the Cauchy-Schwarz inequality)

\[
|f_\varepsilon(x)| \leq C_\varepsilon \int |f(y)|dy \leq C_\varepsilon' \left( \int |f(y)|^2dy \right)^{\frac{1}{2}}, \quad f \in F.
\]

Therefore, \( F_\varepsilon \) is uniformly bounded (on \( \mathbb{R}^n \)) for a fixed \( \varepsilon > 0 \). Further, by similar reasoning the derivatives \( \frac{\partial f_\varepsilon}{\partial x_j} = \left( \frac{\partial f}{\partial x_j} \right)_\varepsilon \) are uniformly bounded for \( f \in F \) and a fixed \( \varepsilon > 0 \). Therefore, \( F_\varepsilon \) is equicontinuous. By the Arzela-Ascoli theorem, \( F_\varepsilon \) is precompact in \( C(\mathbb{R}^n) \) (we may assume that \( F_\varepsilon \subset C(K) \), where \( K \) is a compact neighborhood of \( \Omega \) in \( \mathbb{R}^n \)). It follows that \( F_\varepsilon \) is precompact in \( L^2(\mathbb{R}^n) \) and \( F_{\varepsilon|\Omega} = \{ f_\varepsilon|\Omega, f_\varepsilon \in F_\varepsilon \} \) is precompact in \( L^2(\Omega) \).

It remains to verify that for any \( \delta > 0 \) there exists \( \varepsilon > 0 \), such that \( F \) belongs to the \( \delta \)-neighborhood of \( F_\varepsilon \). To this end, let us estimate the norm
of \( f - f_\varepsilon \) in \( L^2(\mathbb{R}^n) \). (This norm will be, for the time being, denoted by just \( \| \cdot \| \).) First, assume that \( f \in D(\Omega) \). We have

\[
f(x) - f_\varepsilon(x) = \int [f(x) - f(x-y)] \varphi_\varepsilon(y) dy = \int dy \int_0^1 dt \frac{d}{dt} [f(x) - f(x-ty)] \varphi_\varepsilon(y)
\]

\[
= \int_0^1 dt \int dy \sum_{j=1}^n y_j \frac{\partial f}{\partial x_j}(x-ty) \varphi_\varepsilon(y).
\]

Using Minkowski’s inequality (4.53) with \( p = 2 \) and the Cauchy-Schwarz inequality for finite sums, we obtain

\[
\| f - f_\varepsilon \| \leq \int_0^1 dt \int dy \sum_{j=1}^n |y_j| \left\| \frac{\partial f}{\partial x_j}(\cdot - ty) \right\| \varphi_\varepsilon(y)
\]

\[
= \int_0^1 dt \int dy \sum_{j=1}^n |y_j| \left\| \frac{\partial f(\cdot)}{\partial x_j} \right\| \varphi_\varepsilon(y) = \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\| \int \varphi_\varepsilon(y)|y_j|dy
\]

\[
\leq \delta_1 \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\| \leq \delta_1 \sqrt{n} \| f \|_1,
\]

where \( \delta_1 = \sup_{x \in \text{supp} \varphi_\varepsilon} \sum_{j=1}^n |x_j| \). Clearly, \( \delta_1 \leq C \varepsilon \), hence \( \delta_1 \to 0 \) as \( \varepsilon \to 0 \).

Since \( f \in \mathcal{F} \) means \( \| f \|_1 \leq 1 \), it follows that \( \| f - f_\varepsilon \| \leq \delta = \delta_1 \sqrt{n} \) for \( f \in \mathcal{F}, f \in D(\Omega) \).

In the resulting inequality

\[(8.10) \quad \| f - f_\varepsilon \| \leq C_1 \varepsilon \| f \|_1 \]

(which we proved for all \( f \in D(\Omega) \)) we can pass to the limit along a sequence \( \{ f_k \in D(\Omega) \} \), converging to \( f \in H^1(\Omega) \) in the \( H^1 \)-norm \( \| \cdot \|_1 \), to conclude that the inequality holds for all \( f \in H^1(\Omega) \). To make it more precise, we need the estimate \( \| f_\varepsilon \| \leq \| f \| \) which can be proved as follows:

\[
\| f_\varepsilon \| = \left\| \int f(x-y) \varphi_\varepsilon(y) dy \right\| \leq \int \| f(\cdot - y) \| \varphi_\varepsilon(y) dy = \| f \|,
\]

where we again used Minkowski’s inequality (4.53). If \( f_k \in D(\Omega), k = 1, 2, \ldots \), are chosen so that \( \| f - f_k \|_1 \leq 1/k \), hence \( \| f - f_k \| \leq 1/k \), then \( \| f_\varepsilon - (f_k)_\varepsilon \| = \|(f - f_k)_\varepsilon \| \leq 1/k \) and, therefore,

\[
\| f - f_\varepsilon \| \leq \| f - f_k \| + \| (f_k)_\varepsilon \| + \| f_\varepsilon - (f_k)_\varepsilon \| \leq C_1 \varepsilon \| f_k \|_1 + \frac{2}{k} \leq C_1 \varepsilon \| f \|_1 + \frac{C_1 + 2}{k}.
\]
In the resulting inequality we can pass to the limit as $k \to +\infty$, to conclude that (8.10) holds for $f \in H^1(\Omega)$ with the same constant $C_1$. It follows that $\mathcal{F}$ lies in $C_{1\varepsilon}$-neighborhood of $\mathcal{F}_\varepsilon$ which ends the proof of Theorem 8.7. □

On a compact manifold $M$ without boundary one can prove compactness of the embedding $H^s(M) \subset H^{s'}(M)$ for any $s, s' \in \mathbb{R}$ such that $s > s'$. The embedding $H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)$ is never compact whatever $s$ and $s'$.

8.3. Dirichlet’s integral. The Friedrichs inequality

Dirichlet’s integral is

$$D(u) = \int_\Omega \sum_{j=1}^n \left( \frac{\partial u}{\partial x_j} \right)^2 \, dx,$$

where $u \in H^1(\Omega)$.

One of the possible physical interpretations of this integral is the potential energy for small vibrations of a string (for $n = 1$), or a membrane (for $n = 2$), where $u(x)$ is the displacement (assumed to be small) from the equilibrium. (In case of the membrane we should assume that the membrane points only move orthogonally to the coordinate plane. Besides, we will assume that the boundary of the membrane is fixed, i.e. it can not move.)

Taking any displacement function $u$, let us try to understand when $u$ represents an equilibrium. In case of mechanical systems with finitely many degrees of freedom (see Sect.2.5) the equilibriums are exactly the critical (or stationary) points of the potential energy. It is natural to expect that the same is true for systems with infinitely many degrees of freedom, such as a string or a membrane. So the equilibrium property means vanishing of the variation of the Dirichlet integral under infinitesimal deformations preserving $u|_{\partial \Omega}$, i.e., the Euler equation for the functional $D(u)$. We will see that it is just the Laplace equation. For the string this had been actually verified in Section 2.1, where string’s Lagrangian was given. Let us verify this in the multidimensional case.

▼ Let us take, for simplicity, $u \in C^\infty(\bar{\Omega})$, $\delta u \in C^\infty(\bar{\Omega})$ and $\delta u|_{\partial \Omega} = 0$, where $\Omega$ is assumed to be bounded with a smooth boundary. Let us write explicitly the condition for $D(u)$ to be stationary, $\delta D(u) = 0$, where

$$\delta D(u) = \frac{d}{dt} D(u + t \delta u) \bigg|_{t=0}.$$
(See Sect. 2.5.) We have
\[
\delta D(u) = 2 \sum_{j=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} (\delta u) \, dx \\
= -2 \int_{\Omega} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \delta u \, dx = -2 \int_{\Omega} \Delta u(x) \cdot \delta u(x) \, dx.
\]
(The integral over \(\partial \Omega\) appearing due to integration by parts vanishes since \(\delta u|_{\partial \Omega} = 0\).) Thus, Dirichlet’s integral for \(u \in C^\infty(\bar{\Omega})\) and fixed \(u|_{\partial \Omega}\) is stationary if and only if \(\Delta u = 0\). Therefore, the Dirichlet problem for the Laplace equation can be considered as a minimization problem for the Dirichlet integral in the class of functions such that \(u|_{\partial \Omega} = \varphi\), where \(\varphi\) is a fixed function on \(\partial \Omega\).

The Laplace equation \(\Delta u(x) = 0\) is also easy to derive for \(u \in H^1(\Omega)\), if \(u\) is a stationary point of Dirichlet’s integral for permissible variations \(\delta u \in C^\infty_0(\Omega)\). The same argument fits, but integration by parts should be replaced by the definition of derivative.

Note that, in fact, the stationary points of Dirichlet’s integral are the points of minima (this is clear from what will follow). Therefore, Dirichlet’s problem may be considered as a minimization problem for Dirichlet’s integral in the class of functions with fixed boundary values. ▲

**Proposition 8.8.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\). Then there exists a constant \(C > 0\), such that

\[(8.11) \quad \int_{\Omega} |u(x)|^2 \, dx \leq C \int_{\Omega} \sum_{j=1}^{n} \left| \frac{\partial u(x)}{\partial x_j} \right|^2 \, dx, \quad u \in \bar{H}^1(\Omega).\]

This inequality is called the **Friedrichs inequality**.

**Proof.** Both parts of (8.11) continuously depend on \(u \in H^1(\Omega)\) in the norm of \(H^1(\Omega)\). Since \(\bar{H}^1(\Omega)\) is the closure of \(D(\Omega)\) in \(H^1(\Omega)\), it is clear that it suffices to prove (8.11) for \(u \in D(\Omega)\) with \(C\) independent of \(u\).

Let \(\Omega\) be contained in the strip \(\{x : 0 < x_n < R\}\) (this can always be achieved by taking a sufficiently large \(R\) and shifting \(\Omega\) in \(\mathbb{R}^n\) with the help of a translation that does not affect (8.11)). For \(u \in D(\Omega)\) we have

\[u(x) = u(x', x_n) = \int_{0}^{x_n} \frac{\partial u}{\partial x_n}(x', t) \, dt.\]
By the Cauchy–Schwarz inequality, this implies
\[ |u(x)|^2 \leq |x_n| \cdot \int_0^{x_n} \left| \frac{\partial u}{\partial x_n}(x', t) \right|^2 dt \leq R \int_0^R \left| \frac{\partial u(x)}{\partial x_n} \right|^2 dx_n. \]
Integrating this inequality over \( \Omega \), we get
\[ \int_{\Omega} |u(x)|^2 dx \leq R^2 \int_{\Omega} \left| \frac{\partial u}{\partial x_n} \right|^2 dx. \]
In particular, this implies that (8.11) holds for \( u \in D(\Omega) \) with a constant \( C = R^2 \).

The Friedrichs inequality implies that, for a bounded open set \( \Omega \), the norm \( \| \cdot \|_1 \) in \( \tilde{H}^1(\Omega) \) is equivalent to the norm defined by Dirichlet’s integral, i.e. \( u \mapsto D(u)^{1/2} \). Indeed,
\[ \|u\|^2_1 = \|u\|^2 + D(u), \quad u \in H^1(\Omega), \]
where \( \| \cdot \| \) is the norm in \( L^2(\Omega) \). The Friedrichs inequality means that \( \|u\|^2 \leq CD(u) \), for all \( u \in \tilde{H}^1(\Omega) \), implying
\[ D(u) \leq \|u\|^2_1 \leq C_1 D(u), \quad u \in \tilde{H}^1(\Omega), \]
as required.

Together with Dirichlet’s integral, we will use the corresponding inner product
\[ [u, v] = \int_{\Omega} \sum_{j=1}^n \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx, \quad u, v \in H^1(\Omega). \]
On \( \tilde{H}^1(\Omega) \), it defines a Hilbert space structure, since \( \tilde{H}^1(\Omega) \) is a closed subspace in \( H^1(\Omega) \) and \( \|u\|_1 \) is equivalent in \( H^1(\Omega) \) to the norm \( [u, u]^{1/2} = D(u)^{1/2} \).

8.4. Dirichlet’s problem (generalized solutions)

Consider two generalized settings of Dirichlet’s boundary value problem for the Laplace operator. In both cases, the existence and uniqueness of a solution easily follows, after the above preliminary argument, from general theorems of functional analysis.

1. The first of the settings contains, as a classical analogue, the following problem (the boundary value problem with zero boundary condition for the Poisson equation):
\[
\begin{cases}
\Delta u(x) = f(x), & x \in \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\]
Instead of \( u|_{\partial\Omega} = 0 \), write
\[
(8.13) \quad u \in \dot{H}^1(\Omega).
\]
Now let us try to make sense of \( \Delta u = f \). First, let \( u \in C^2(\bar{\Omega}), u|_{\partial\Omega} = 0 \). If \( v \in D(\Omega) \) then, integrating by parts, we get
\[
\int_{\Omega} (\Delta u(x)v(x)) \, dx = -\int_{\Omega} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} \, dx = -[u, v].
\]
If \( \Delta u = f \) this means that
\[
(8.14) \quad [u, v] = -(f, v),
\]
where \((\cdot, \cdot)\) is the inner product in \( L^2(\Omega) \). Now note that if both parts of \( (8.14) \) are considered as functionals of \( v \), then these functionals are linear and continuous with respect to the norm \( \| \cdot \|_1 \) in \( H^1(\Omega) \). Therefore, \( (8.14) \) holds when \( v \) belongs to the closure of \( D(\Omega) \) in \( H^1(\Omega) \), i.e., when \( v \in \dot{H}^1(\Omega) \).

Conversely, let \( \Omega \) be bounded and have a smooth boundary, \( u \in C^2(\bar{\Omega}) \cap \dot{H}^1(\Omega), f \in L^2(\Omega) \) and \( (8.14) \) holds for any \( v \in \dot{H}^1(\Omega) \). Let us prove that \( u \) is a solution of \( (8.12) \).

First, observe that \( u \in \dot{H}^1(\Omega) \) implies \( u|_{\partial\Omega} = 0 \). This follows from the fact that, due to Theorem 8.5, the map \( u \to u|_{\partial\Omega} \) extends to a continuous map \( H^1(\Omega) \to L^2(\partial\Omega) \) which maps all functions from \( D(\Omega) \) and, therefore, all functions from \( \dot{H}^1(\Omega) \) to zero. Note that here we only use, actually, the fact that
\[
u \in C^2(\bar{\Omega}) \cap \dot{H}^1(\Omega) \implies u|_{\partial\Omega} = 0.
\]
This fact can be obtained directly, avoiding Theorem 8.5 and its generalization to manifolds. Let us sketch the corresponding argument, omitting the details.

Using a partition of unity and locally rectifying the boundary with the help of a diffeomorphism, we can, by the same arguments as those used in the proof of the Friedrichs inequality, get
\[
\int_{\partial\Omega} |u(x)|^2 \, dS_x \leq C D(u)
\]
for \( u \in C^1(\bar{\Omega}) \) with the support in a small neighborhood of a fixed point of the boundary. Therefore, if \( u|_{\partial\Omega} \neq 0 \), then it is impossible to approximate \( u \) by functions from \( D(\Omega) \) with respect to the norm in \( H^1(\Omega) \). Thus, if \( u \in C^2(\bar{\Omega}) \cap \dot{H}^1(\Omega) \), then \( u|_{\partial\Omega} = 0 \).
Conversely, if \( u \in C^1(\Omega) \), then
\[
u|_{\partial \Omega} = 0 \implies u \in \dot{H}^1(\Omega).
\]
This may be proved as follows. The \( \varepsilon \)-mollified characteristic function of \( \Omega_{2\varepsilon} \), where
\[
\Omega_{\delta} = \{ x : x \in \Omega, \rho(x, \partial \Omega) \geq \delta \},
\]
is a function \( \chi_{\varepsilon} \in D(\Omega) \), such that \( \chi_{\varepsilon}(x) = 1 \) for \( x \in \Omega_{3\varepsilon} \) and \( |\partial^\alpha \chi_{\varepsilon}(x)| \leq C_\alpha \varepsilon^{-|\alpha|} \) (the constant \( C_\alpha \) does not depend on \( \varepsilon \)). Now, if \( u \in C^1(\Omega) \), then \( \chi_{\varepsilon} u \in \dot{H}^1(\Omega) \) and \( u|_{\partial \Omega} = 0 \) implies that if \( x \in \Omega \setminus \Omega_{3\varepsilon} \), then \( |u(x)| \leq C \varepsilon \), where \( C \) hereafter depends on \( u \) but does not depend on \( \varepsilon \). This enables us to estimate
\[
\|(u - \chi_{\varepsilon} u)^2\|_1^2 = \|(u - \chi_{\varepsilon} u)^2 + D(u - \chi_{\varepsilon} u)\|,
\]
where \( \| \cdot \| \) is the norm in \( L^2(\Omega) \) and \( D \) the Dirichlet integral. Since the Lebesgue measure of \( \Omega \setminus \Omega_{3\varepsilon} \) does not exceed \( C \varepsilon \), we have
\[
\|(u - \chi_{\varepsilon} u)^2\|_1^2 \leq C \varepsilon \sup_{\Omega \setminus \Omega_{3\varepsilon}} |u - \chi_{\varepsilon} u|^2 \leq C_1 \varepsilon
\]
and
\[
\begin{align*}
D(u - \chi_{\varepsilon} u) & \leq \text{mes}(\Omega \setminus \Omega_{3\varepsilon}) \cdot \sup_{\Omega \setminus \Omega_{3\varepsilon}} \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} (u - \chi_{\varepsilon} u) \right|^2 \\
& \leq C \varepsilon \sup_{\Omega \setminus \Omega_{3\varepsilon}} \sum_{j=1}^{n} \left[ \left| \frac{\partial u}{\partial x_j} \right|^2 |(1 - \chi_{\varepsilon})|^2 + |u|^2 \left| \frac{\partial}{\partial x_j} (1 - \chi_{\varepsilon}) \right|^2 \right] \leq C_1 \varepsilon,
\end{align*}
\]
where \( C_1 \) does not depend upon \( \varepsilon \) (but depends upon \( u \)). The product of \( u \) and \( \frac{\partial}{\partial x_j} (1 - \chi_{\varepsilon}) \) is bounded because in the \( 3\varepsilon \)-neighborhood of the boundary we have \( u = O(\varepsilon) \) whereas the derivative is \( O(\varepsilon^{-1}) \).

So, it is possible to approximate \( u \) with respect to the norm \( \| \cdot \|_1 \) by functions from \( C^1(\Omega) \) whose supports are compact in \( \Omega \). In turn, they can be easily approximated with respect to the norm \( \| \cdot \|_1 \) by functions from \( D(\Omega) \) with the help of the mollifying.

Thus, supposing \( u \in C^2(\Omega) \) for a bounded open set \( \Omega \) with a smooth boundary we get \( u|_{\partial \Omega} = 0 \) if and only if \( u \in \dot{H}^1(\Omega) \). This justifies the replacement of the boundary conditions (8.12) with (8.13).

Further, (8.14) holds for \( u \in \dot{H}^1(\Omega) \) and any \( v \in \dot{H}^1(\Omega) \) if and only if \( \Delta u = f \), where \( \Delta \) is understood in the distributional sense. Indeed, if (8.14) holds, then, taking \( v \in D(\Omega) \), we obtain by moving the derivatives from \( u \) to \( v \) (by use of the definition of the distributional derivatives) that
(u, ∆v) = (f, v), that ∆u = f, because v can be arbitrary function from D(Ω).

Conversely, if ∆u = f, u ∈ ℋ¹(Ω) and f ∈ L²(Ω), then (8.14) holds for v ∈ D(Ω), hence, by continuity, for any v ∈ ℋ¹(Ω).

All this justifies the following generalized setting of the problem (8.12):

• Let Ω be an arbitrary bounded open set in \( \mathbb{R}^n \), f ∈ L²(Ω). Find u ∈ ℋ¹(Ω), such that (8.14) holds for any v ∈ ℋ¹(Ω) (or, which is the same, for any v ∈ D(Ω)).

In this case u is called a generalized solution of the problem (8.12). We have seen above that if Ω is a bounded open set with a smooth boundary and u ∈ C²(Ω), then u is a generalized solution of (8.12) if and only if it is a solution of this problem in the classical sense.

Theorem 8.9. A generalized solution of (8.12) exists and is unique.

Proof. Rewrite (8.14) in the form

\[ [v, u] = -(v, f) \]

and consider the right hand side as a functional in v for v ∈ ℋ¹(Ω). This is a linear continuous functional. Its linearity is obvious and its continuity follows from the Cauchy–Schwarz inequality:

\[ |(v, f)| \leq \|f\| \cdot \|v\| \leq \|f\| \cdot \|v\|_1, \]

where \( \|\cdot\| \) is the norm in L²(Ω).

By the Riesz representation theorem we may (uniquely) write this functional in the form \([v, u]\), where u is a fixed element of ℋ¹(Ω). This proves the unique solvability of (8.12) (in the generalized setting).

The generalized setting of the Dirichlet problem leaves two important questions open:

1) the exact description of the class of all functions u which appear as the solutions when f runs through all possible functions from L²(Ω),

2) the existence of a more regular solution for a more regular f.

Observe that local regularity problems are easily solved by the remark that if \( \mathcal{E}(x) \) is a fundamental solution of the Laplace operator in \( \mathbb{R}^n \), then the convolution

\[ u_0(x) = (\mathcal{E} * f)(x) = \int_{\Omega} \mathcal{E}(x - y)f(y)dy \]
8.4. Dirichlet’s problem (generalized solutions)

(defined for almost all \( x \)) is a solution of \( \Delta u_0 = f \). Therefore, \( \Delta (u - u_0) = 0 \), i.e., \( u - u_0 \) is an analytic function in \( \Omega \). Therefore, the regularity of \( u \) coincides locally with that of \( u_0 \) and, therefore, can be easily described (e.g. if \( f \in C^\infty(\Omega) \), then \( u \in C^\infty(\Omega) \), as proved above).

The question of regularity near the boundary is more complicated. Moreover, for non-smooth boundary the answer is not completely clear even now. However, if \( \partial \Omega \) is sufficiently smooth (say, of class \( C^\infty \)) there are exact (though not easy to prove) theorems describing the regularity of \( u \) in terms of regularity of \( f \). We will formulate one of such theorems without proof.

**Theorem 8.10.** Let the boundary of \( \Omega \) be of class \( C^\infty \). If \( f \in H^s(\Omega) \), \( s \in \mathbb{Z}_+ \), then \( u \in H^{s+2}(\Omega) \cap \dot{H}^1(\Omega) \).

(For the proof see e.g. Gilbarg and Trudinger [8], Sect. 8.4.)

Conversely, it is clear that if \( u \in H^{s+2}(\Omega) \), then \( \Delta u = f \in H^s(\Omega) \). Therefore,

\[
\Delta : H^{s+2}(\Omega) \cap \dot{H}^1(\Omega) \rightarrow H^s(\Omega), \quad s \in \mathbb{Z}_+,
\]

is an isomorphism. In particular, in this case all generalized solutions (for \( f \in L^2(\Omega) \)) lie in \( H^2(\Omega) \cap \dot{H}^1(\Omega) \).

Another version of an exact description of the regularity, which we already discussed above, is Schauder’s Theorem 6.20 in Hölder functional spaces. There are numerous other descriptions, depending on the choice of functional spaces. Most of them are useful in miscellaneous problems of mathematical physics.

Note that the last statement fails in the usual classes \( C^k(\bar{\Omega}) \) even locally. Namely, \( f \in C(\bar{\Omega}) \) does not even imply \( u \in C^2(\bar{\Omega}) \).

2. Let us pass to the generalized setting of the usual Dirichlet problem for the Laplace equation:

\[
\begin{align*}
\Delta u &= 0 \\
 u|_{\partial \Omega} &= \varphi
\end{align*}
\]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^n \). First, there arises a question of interpretation of the boundary condition. We will do it as by assuming that there exists a function \( v \in H^1(\Omega) \), such that \( v|_{\partial \Omega} = \varphi \), and formulating the boundary condition for \( u \) as follows: \( u - v \in \dot{H}^1(\Omega) \). This implies \( u \in H^1(\Omega) \). Introducing \( v \) saves us from describing the smoothness of \( \varphi \) (which requires using Sobolev spaces with non-integer indices on \( \partial \Omega \) if \( \Omega \)
Let us give the generalized formulation of Dirichlet’s problem (8.15):

- Given a bounded domain \( \Omega \subset \mathbb{R}^n \) and \( v \in H^1(\Omega) \), find a function \( u \in H^1(\Omega) \), such that \( \Delta u = 0 \) and \( u - v \in \dot{H}^1(\Omega) \).

At the first sight this problem seems to be reducible to the above one if we seek \( u - v \in \dot{H}^1(\Omega) \) and put \( f = \Delta(u - v) = -\Delta v \). But we have no reason to think that \( \Delta v \in L^2(\Omega) \), and we did not consider a more general setting. Therefore, we will consider this problem independently of the first problem, especially because its solution is as simple as the solution of the first problem.

The reasoning above makes it clear that if \( \Omega \) has a smooth boundary, \( v \in C(\overline{\Omega}) \), \( v|_{\partial\Omega} = \varphi \), \( u \in C^2(\overline{\Omega}) \) and \( u \) is a generalized solution of (8.15), then \( u \) is a classical solution of this problem.

**Theorem 8.11.** A generalized solution \( u \) of Dirichlet’s problem exists and is unique for any bounded domain \( \Omega \subset \mathbb{R}^n \) and any \( v \in H^1(\Omega) \). This solution has strictly and globally minimal Dirichlet integral \( D(u) \) among all \( u \in H^1(\Omega) \) such that \( u - v \in \dot{H}^1(\Omega) \).

Conversely, if \( u \) is a critical (stationary) point of the Dirichlet integral in the class of all \( u \in H^1(\Omega) \) such that \( u - v \in \dot{H}^1(\Omega) \), then the Dirichlet integral actually reaches the strict and global minimum at \( u \) and \( u \) is a generalized solution of Dirichlet’s problem.

**Proof.** Let \( u \in H^1(\Omega) \). The equation \( \Delta u = 0 \) can be expressed in the form

\[
(u, \Delta w) = 0, \quad w \in D(\Omega),
\]

or, which is the same, in the form

\[
[u, w] = 0, \quad w \in D(\Omega).
\]

Since \( [u, w] \) is a continuous functional in \( w \) on \( H^1(\Omega) \), then, by continuity, we get

(8.16) \[
[u, w] = 0, \quad w \in \dot{H}^1(\Omega).
\]

This condition is equivalent to the Laplace equation \( \Delta u = 0 \).

Now we can invoke a geometric interpretation of the problem: we have to find \( u \in H^1(\Omega) \), which is orthogonal to \( H^1(\Omega) \) with respect to \( [\cdot, \cdot] \) and such that \( v - u \in \dot{H}^1(\Omega) \). Imagining for a moment that \( H^1(\Omega) \) is a Hilbert
space with respect to the inner product $\langle \cdot, \cdot \rangle$, we see that $u$ is the component in the decomposition $v = (v - u) + u$ in the direct orthogonal decomposition

$$H^1(\Omega) = \dot{H}^1(\Omega) + (\dot{H}^1(\Omega))^\perp.$$ 

It is well known from Functional Analysis that such a decomposition exists, is unique (in any Hilbert space and for any closed subspace of this Hilbert space), and besides, the perpendicular (to $\dot{H}^1(\Omega)$) component $u$ has the strictly minimal norm compared with all other decompositions $v = (v - w) + w$ with $v - w \in \dot{H}^1(\Omega)$. Vice versa, in this case, if $u$ is stationary, then it is orthogonal to $\dot{H}^1(\Omega)$.

However, the “inner product” $\langle \cdot, \cdot \rangle$ defines a Hilbert space structure on $\dot{H}^1(\Omega)$, but not on $H^1(\Omega)$ (it is not strictly positive; for example, $[1, 1] = 0$ whereas $1 \neq 0$). Therefore, the above argument should be made more precise. Denote $z = v - u \in \dot{H}^1(\Omega)$ so that $v = u + z$. It follows from (8.16) that

$$[w, v] = [w, z], \quad w \in \dot{H}^1(\Omega). \tag{8.17}$$

Now it is easy to prove the existence of solution. Namely, let us consider $[w, v]$ as a linear continuous functional in $w$ on $\dot{H}^1(\Omega)$. By the Riesz theorem it can be uniquely represented in the form $[w, z]$, where $z$ is a fixed element of $\dot{H}^1(\Omega)$. It remains to set $u = v - z$. The definition of $z$ implies (8.17) which in turn implies (8.16). The condition $u - v \in \dot{H}^1(\Omega)$ is obvious.

The uniqueness of the solution is clear, since if $u_1, u_2$ are two solutions, then $u = u_1 - u_2 \in \dot{H}^1(\Omega)$ satisfies (8.16), and this yields $u = 0$.

Let us now prove that the Dirichlet integral is minimal on the solution $u$ in the class of all $u_1 \in H^1(\Omega)$ such that $u_1 - v \in \dot{H}^1(\Omega)$. Set $z = u_1 - u$; hence, $u_1 = u + z$. Clearly, $z \in \dot{H}^1(\Omega)$ and (8.16) yields

$$D(u_1) = [u_1, u_1] = [u, u] + [z, z] = D(u) + D(z) \geq D(u),$$

where the equality is only attained for $D(z) = 0$, i.e., for $z = 0$ or, which is the same, for $u = u_1$.

Finally, let us verify that if Dirichlet’s integral is stationary on $u \in H^1(\Omega)$ in the class of all $u_1 \in H^1(\Omega)$ such that $u_1 - v \in \dot{H}^1(\Omega)$, then $u$ is a generalized solution of the Dirichlet problem (hence the Dirichlet integral reaches the strict global minimum at $u$). The stationarity property of the Dirichlet integral at $u$ means that

$$\frac{d}{dt}D(u + tz)|_{t=0} = 0, \quad z \in \dot{H}^1(\Omega).$$
But
\[ \left( \frac{d}{dt} D(u + tz) \right)_{t=0} = \left( \frac{d}{dt} [u + tz, u + tz] \right)_{t=0} = [u, z] + [z, u] = 2 \text{Re}[u, z]. \]

The stationarity property gives \( \text{Re}[u, z] = 0 \) for any \( z \in \dot{H}^1(\Omega) \). Replacing \( z \) by \( iz \), we get \( [u, z] = 0 \) for any \( z \in \dot{H}^1(\Omega) \) yielding (8.16). Theorem 8.11 is proved. □

Finally, we will give without proof a precise description of regularity of solutions from the data on \( \varphi \) in the boundary value problem (8.15) for a domain \( \Omega \) with a smooth boundary. It was already mentioned that if \( u \in H^s(\Omega) \), \( s \in \mathbb{Z}_+ \), \( s \geq 1 \), then \( u|_{\partial\Omega} \in H^{s-\frac{1}{2}}(\partial\Omega) \). It turns out that the converse also holds: if \( \varphi \in H^{s-\frac{1}{2}}(\partial\Omega) \), where \( s \in \mathbb{Z}_+, s \geq 1 \), then there exists a unique solution \( u \in H^s(\Omega) \) of the problem (8.15) (see e.g. Lions and Magenes [20]).

Similarly, if \( \varphi \in C^{m+\gamma}(\partial\Omega) \), where \( m \in \mathbb{Z}_+, \gamma \in (0, 1) \), then there exists a (unique) solution \( u \in C^{m+\gamma}(\overline{\Omega}) \).

Combining the described regularity properties of solutions of problems (8.12) and (8.15), it is easy to prove that the map
\[ u \mapsto \{ \Delta u, u|_{\Gamma} \} \]
can be extended to a topological isomorphism
\[ H^s(\Omega) \to H^{s-2}(\Omega) \oplus H^{s-\frac{1}{2}}(\partial\Omega), \quad s \in \mathbb{Z}_+, s \geq 2, \]
and also to a topological isomorphism
\[ C^{m+2+\gamma}(\overline{\Omega}) \to C^{m+\gamma}(\overline{\Omega}) \oplus C^{m+2+\gamma}(\partial\Omega), \quad m \in \mathbb{Z}_+, \gamma \in (0, 1). \]
(See e.g. the above quoted books Gilbarg and Trudinger [8], Ladyzhenskaya and Ural’tseva [17].)

Similar theorems can be proved also for general elliptic equations with general boundary conditions satisfying an algebraic condition called the ellipticity condition (or coercitivity, or Shapiro–Lopatinsky condition). For the general theory of elliptic boundary value problems, see e.g. books by Lions and Magenes [20], Hörmander [12], Ch.X, or [14], vol.3, Ch.20.

8.5. Problems

8.1. Using the Fourier method, solve the following Dirichlet problem: find a function \( u = u(x, y) \) such that \( \Delta u = 0 \) in the disc \( r \leq a \), where \( r = \sqrt{x^2 + y^2} \), \( u|_{r=a} = (x^4 - y^4)|_{r=a} \).
8.2. By the Fourier method solve the Dirichlet problem in the disk $r \leq a$ for a smooth boundary function $f(\varphi)$, where $\varphi$ is the polar angle, i.e., find a function $u = u(x, y)$ such that $\Delta u = 0$ in the disc $r \leq a$ and $u|_{r=a} = f(\varphi)$. Justify the solution. Derive for this case Poisson’s formula:

$$u(r, \varphi) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(a^2 - r^2)f(\alpha)d\alpha}{a^2 - 2ar\cos(\varphi - \alpha) + r^2}.$$  

8.3. Using the positivity of Poisson’s kernel

$$K(r, \varphi, \alpha) = \frac{a^2 - r^2}{2\pi[a^2 - 2ar\cos(\varphi - \alpha) + r^2]},$$

prove the maximum principle for the solutions of Dirichlet’s problem obtained in the preceding problem. Derive from the maximum principle the solvability of Dirichlet’s problem for any continuous function $f(\varphi)$.

8.4. Find a function $u = u(x, y)$ such that $\Delta u = x^2 - y^2$ for $r \leq a$ and $u|_{r=a} = 0$.

8.5. In the ring $0 < a < r < b < +\infty$, find a function $u = u(x, y)$ such that $\Delta u = 0$, $u|_{r=a} = 1$, $\frac{\partial u}{\partial r}|_{r=b} = (\cos \varphi)^2$.

8.6. By the Fourier method solve the Dirichlet problem for the Laplace equation in the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ with the boundary conditions of the form

$$u|_{x=0} = \varphi_0(y), \quad u|_{x=a} = \varphi_1(y), \quad (0 \leq y \leq b);$$
$$u|_{y=0} = \psi_0(x), \quad u|_{y=b} = \psi_1(x), \quad (0 \leq x \leq a),$$

where (the continuity condition for the boundary function)

$$\varphi_0(0) = \psi_0(0), \quad \varphi_0(b) = \psi_1(0), \quad \varphi_1(0) = \psi_0(a), \quad \varphi_1(b) = \psi_1(a).$$

Describe a way for solving the general problem, and then solve the problem in the following particular case:

$$\varphi_0(y) = Ay(b - y), \quad \psi_0(x) = B \sin \frac{\pi x}{a}, \quad \varphi_1(y) = \psi_1(x) = 0.$$

8.7. In the half-plane $\{(x, y) : y \geq 0\}$ solve Dirichlet’s problem for the Laplace equation in the class of bounded continuous functions. For this purpose use the Fourier transform with respect to $x$ to derive the following formula for the solutions which are in $L^2(\mathbb{R})$ with respect to $x$:

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-z)^2} \varphi(z) dz,$$
where $\varphi(x) = u(x, 0)$. Then investigate the case of a general bounded continuous function $\varphi$.

8.8. Investigate for which $n, s$ and $\alpha$ ($\alpha > -n$) the function $u(x) = |x|^\alpha$ belongs to $H^s(B_1)$, where $B_1$ is the open ball of radius 1 with the center at the origin of $\mathbb{R}^n$.

8.9. Does the function $u(x) \equiv 1$, where $x \in (-1, 1)$, belong to $\tilde{H}^1((-1, 1))$?

8.10. Prove that the inclusion $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $s > 0$, is not a compact operator.
Chapter 9

The eigenvalues and eigenfunctions of the Laplace operator

9.1. Symmetric and self-adjoint operators in Hilbert space

Let $\mathcal{H}$ be a Hilbert space. Recall that a linear operator in $\mathcal{H}$ is a collection of two objects:

a) a linear (but not necessarily closed) subspace $D(A) \subset \mathcal{H}$ called the domain of the operator $A$;

b) a linear map $A : D(A) \rightarrow \mathcal{H}$.

In this case we will write, by an abuse of notations, that a linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is defined, though, strictly speaking, there is no map $\mathcal{H} \rightarrow \mathcal{H}$, because in general $A$ may not be defined on the whole $\mathcal{H}$.

It is possible to add linear operators and take their products (compositions). Namely, given two linear operators $A_1, A_2$ in $\mathcal{H}$, we set

$$D(A_1 + A_2) = D(A_1) \cap D(A_2),$$

$$(A_1 + A_2)x = A_1x + A_2x \text{ for } x \in D(A_1 + A_2).$$

Further, set

$$D(A_1A_2) = \{ x : x \in D(A_2), A_2x \in D(A_1) \},$$

$$(A_1A_2)x = A_1(A_2x) \text{ for } x \in D(A_1A_2).$$
Define also
\[
\text{Ker } A = \{ x : x \in D(A), \ Ax = 0 \},
\]
\[
\text{Im } A = \{ x : x = Ay, \ y \in D(A) \}.
\]
If \( \text{Ker } A = 0 \), then the inverse operator \( A^{-1} \) is defined. Namely, set \( D(A^{-1}) = \text{Im } A \) and \( A^{-1}x = y \) for \( x = Ay \).

If there are two linear operators \( A, B \) in \( H \), such that \( D(A) \subset D(B) \) and \( Ax = Bx \) for \( x \in D(A) \), then we say that \( B \) is an extension of \( A \) and write \( A \subset B \).

An operator \( A \) is called symmetric if
\[
(Ax, y) = (x, Ay), \quad x, y \in D(A).
\]
In what follows, we will also introduce the notion of a self-adjoint operator (for unbounded operators symmetry and self-adjointness do not coincide!)

Let \( A \) be a linear operator such that \( D(A) \) is dense in \( H \). Then the adjoint operator \( A^* \) is defined via the identity
\[
(Ax, y) = (x, A^*y), \quad x \in D(A), \ y \in D(A^*).
\]
More precisely, let \( D(A^*) \) consist of \( y \in H \) such that there exists \( z \in H \) satisfying
\[
(Ax, y) = (x, z) \quad \text{for all } x \in D(A).
\]
Since \( D(A) \) is dense in \( H \), we see that \( z \) is uniquely defined and we set \( A^*y = z \).

If the domain of \( A \) is dense and \( A \) is symmetric, then, clearly, \( A \subset A^* \).

An operator \( A \) is self-adjoint if \( A^* = A \) (it is understood that the domain of \( A \) is dense and \( D(A^*) = D(A) \)).

Clearly, any self-adjoint operator is symmetric. The converse is not always true.

Define the graph of \( A \) as the set
\[
G_A = \{ \{ x, Ax \}, x \in D(A) \} \subset H \times H.
\]
An operator is closed if its graph is closed in \( H \times H \), i.e. if \( x_n \in D(A) \), \( x_n \to x \) and \( Ax_n \to y \) (the convergence is understood with respect to the norm in \( H \)) yield \( x \in D(A) \) and \( Ax = y \).

It is easy to see that every bounded everywhere defined linear operator is closed; more generally, a bounded linear operator \( A \) with a closed domain \( D(A) \) is closed. But generally closed operators can be unbounded.
Any self-adjoint operator is closed. A more general fact is true: the operator \( A^* \) is always closed for any \( A \).

Indeed, if \( y_n \in D(A^*) \), \( y_n \to y \), and \( A^*y_n \to z \) then passing to the limit as \( n \to \infty \) in the identity
\[
(Ax, y_n) = (x, A^*y_n), \quad x \in D(A),
\]
we get
\[
(Ax, y) = (x, z), \quad x \in D(A),
\]
yielding \( y \in D(A^*) \) and \( A^*y = z \).

If \( D(A) \) is a closed subspace in \( \mathcal{H} \) (in particular, if \( D(A) = \mathcal{H} \), i.e., \( A \) is defined everywhere) then the closedness of \( A \) is equivalent to its boundedness, due to the closed graph theorem (which we already used in Sect. 7.6; see also Reed and Simon [22], Theorem III.12). In particular, a self-adjoint everywhere defined operator \( A \) is necessarily bounded.

Let us indicate a way of constructing self-adjoint operators.

**Proposition 9.1.** Let \( A \) be a self-adjoint operator in \( \mathcal{H} \) and \( \text{Ker} \ A = 0 \). Then \( A^{-1} \) is self-adjoint.

**Proof.** Denote by \( E^\perp \) the orthogonal complement to \( E \subset \mathcal{H} \), namely
\[
E^\perp = \{ y : y \in \mathcal{H}, \ (x, y) = 0 \text{ for all } x \in E \}.
\]
Given a linear operator \( A \) with \( D(A) \) dense in \( \mathcal{H} \), we easily derive from the definition of \( A^* \) that
\[
\text{Ker} \ A^* = (\text{Im} \ A)^\perp.
\]
(9.2) Indeed, the equality \( (Ax, y) = 0 \) for all \( x \in D(A) \) means, due to (9.1), exactly that \( y \in D(A^*) \) and \( A^*y = 0 \).

Note that \( E^\perp = 0 \) if and only if the linear span of \( E \) is dense in \( \mathcal{H} \).

Thus if \( A \) is self-adjoint and \( \text{Ker} \ A = \{0\} \), then from (9.2) it follows that \( D(A^{-1}) = \text{Im} \ A \) is dense in \( \mathcal{H} \).

Therefore, \( A^{-1} \) and \( (A^{-1})^* \) are defined. It remains to verify that \( (A^{-1})^* = A^{-1} \).

We have \( y \in D((A^{-1})^*) \) and \( (A^{-1})^*y = z \) if and only if
\[
(A^{-1}x, y) = (x, z), \quad x \in \text{Im} \ A.
\]
(9.3) Setting \( x = Af, \ f \in D(A) \), we see that (9.3) is equivalent to
\[
(f, y) = (Af, z), \quad f \in D(A),
\]
yielding \( z \in D(A^*) \) and \( A^*z = y \). But, since \( A^* = A \), this means that \( z \in D(A) \) and \( Az = y \) which may be also rewritten as \( z = A^{-1}y \). We have proved that \( y \in D((A^{-1})^*) \) with \( (A^{-1})^*y = z \) are equivalent to \( y \in D((A^{-1})) \) with \( A^{-1}y = z \). Therefore, \( (A^{-1})^* = A^{-1} \) as required. □

**Corollary 9.2.** If \( B \) is a bounded, everywhere defined symmetric operator with \( \text{Ker} \ B = 0 \), then \( A = B^{-1} \) is self-adjoint.

**Elementary Spectral Theory**

The importance of self-adjoint operators follows from the spectral theorem. Before we formulate it, let us give an example of a self-adjoint operator.

Let \( M \) be a space with measure \( d\mu \), i.e., a set with a \( \sigma \)-algebra of its subsets and a countable-additive measure (with values in \([0, +\infty]\)) on it. Construct in a usual way the space \( L^2(M, d\mu) \) consisting of classes of measurable functions on \( M \), such that the square of their absolute value is integrable. (Two functions are in the same class if they coincide almost everywhere.) Then \( L^2(M, d\mu) \) is a Hilbert space. Let \( a(m) \) be a real-valued measurable and almost everywhere finite function on \( M \). The operator \( A \) of multiplication by \( a(m) \) is defined by the formula

\[
Af(m) = a(m)f(m),
\]

where \( f \in D(A) = \{ f : f \in L^2(M, d\mu), \, af \in L^2(M, d\mu) \} \).

This operator is always self-adjoint. Indeed, its symmetry is obvious and we have only to verify that \( D(A^*) = D(A) \).

Let \( u \in D(A^*) \), \( A^*u = v \). Let us check that \( u \in D(A) \) and \( a(m)u(m) = v(m) \) for almost all \( m \). Consider the set

\[
M_N = \{ m : m \in M, \, |a(m)| \leq N \},
\]

and let \( \chi_N(m) = 1 \) for \( m \in M_N \) and \( \chi_N(m) = 0 \) for \( m \notin M_N \). The identity \( A^*u = v \) means that

\[
\int_M a(m)f(m)\overline{(u(m))}d\mu = \int_M f(m)\overline{(v(m))}d\mu, \quad \text{for any } f \in D(A).
\]

Note that if \( g \in L^2(M, d\mu) \), then \( \chi_Ng \in D(A) \), implying

\[
\int_{M_N} a(m)g(m)\overline{(u(m))}d\mu = \int_{M_N} g(m)\overline{(v(m))}d\mu, \quad \text{for any } g \in L^2(M_N, d\mu).
\]

But it is clear that

\[
(au)|_{M_N} \in L^2(M_N, d\mu)
\]

since \( a \) is bounded on \( M_N \). Therefore, the arbitrariness of \( g \) implies \( au|_{M_N} = v|_{M_N} \) (almost everywhere). Therefore, if \( \hat{M} = \bigcup_{N=1}^{\infty} M_N \), then \( au|_{\hat{M}} = v|_{\hat{M}} \).
9.1. Symmetric and self-adjoint operators

Since \( a(m) \) is almost everywhere finite, then the measure of \( M \setminus \hat{M} \) is equal to 0. Hence, \( a(m)u(m) = v(m) \) almost everywhere. In particular, \( a(m)u(m) \in L^2(M, d\mu) \), i.e., \( u \in D(A) \) and \( au = v \), as required. □

An important example: consider the Hilbert space of sequences

\[ l^2 = \{ x = \{ x_1, x_2, \ldots \} : x_j \in \mathbb{C}, \sum_{j=1}^{\infty} |x_j|^2 < +\infty \}. \]

For any sequence of real numbers \( \lambda_1, \lambda_2, \ldots \), consider the operator \( A \) in \( l^2 \) which is defined as follows:

\[ D(A) = \{ \{ x_1, x_2, \ldots \} : x_j \in \mathbb{C}, \sum_{j=1}^{\infty} |\lambda_j|^2 |x_j|^2 < +\infty \}, \]

\[ A\{ x_1, x_2, \ldots \} = \{ \lambda_1 x_1, \lambda_2 x_2, \ldots \}. \]

In this example \( M \) is a countable set and the measure of each point is equal to 1.

Operators \( A_1 : \mathcal{H}_1 \to \mathcal{H}_1 \) and \( A_2 : \mathcal{H}_2 \to \mathcal{H}_2 \) are called unitarily equivalent if there exists a unitary (i.e., invertible and inner product preserving) operator \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) such that \( U^{-1} A_2 U = A_1 \). (Recall that the equality of operators by definition implies the coincidence of their domains.)

In other words, in these notations the unitary equivalence of \( A_1 \) and \( A_2 \) means the existence of a unitary operator \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) such that the diagram

\[ \begin{array}{ccc}
\mathcal{H}_1 & \xrightarrow{A_1} & \mathcal{H}_1 \\
\downarrow U & & \downarrow U \\
\mathcal{H}_2 & \xrightarrow{A_2} & \mathcal{H}_2
\end{array} \]

is commutative.

The following theorem is one of the most important results in functional analysis.

**The Spectral Theorem.** Any self-adjoint operator in a Hilbert space is unitarily equivalent to an operator of multiplication by a measurable real-valued function in a space \( L^2(M, d\mu) \).

Proofs can be found, e.g., in Reed and Simon [22], Chapters VII and VIII, or in Berezin and Shubin [4], Appendix 1.

Note that \( A \) is unitarily equivalent to the above described operator with real eigenvalues in \( l^2 \) if and only if \( \mathcal{H} \) is separable and there is an orthonormal
basis $\varphi_1, \varphi_2, \ldots$ in $\mathcal{H}$ consisting of eigenvectors of $A$ with real eigenvalues $\lambda_1, \lambda_2, \ldots$. In this case

$$D(A) = \left\{ \varphi = \sum_{j=1}^{\infty} c_j \varphi_j : \sum_{j=1}^{\infty} |c_j|^2 < +\infty, \sum_{j=1}^{\infty} |\lambda_j|^2 |c_j|^2 < +\infty \right\}.$$ 

If $|\lambda_j| \to +\infty$ as $j \to +\infty$, then we say that the spectrum of $A$ is discrete. If $\text{Ker} \, A = 0$, then the spectrum of $A$ is discrete if and only if $A^{-1}$ is everywhere defined and compact. This follows from the Hilbert-Schmidt theorem (already used in Ch. 3), which states that a compact self-adjoint operator with real eigenvalues has an orthogonal basis of eigenvectors with real eigenvalues $\mu_1, \mu_2, \ldots$, where $\mu_j \to 0$ as $j \to +\infty$.

For a self-adjoint operator $A$ the discreteness of spectrum is equivalent to the requirement that there exists $\lambda_0 \in \mathbb{R}$ such that the operator $(A - \lambda_0 I)^{-1}$ exists, is everywhere defined and compact. In this case, the discreteness of spectrum of $A$ is equivalent to the compactness of $(A - \lambda_0 I)^{-1}$.

**Important remark:** In the subsection above we mainly considered operators with purely discrete spectrum. But a self-adjoint operator $A$ may have no non-vanishing eigenvectors. (See e.g. the multiplication by the function $a(x) = x, x \in [0, 1]$, in the Hilbert space $L^2([0, 1])$.)

### 9.2. The Friedrichs extension

Let us give an example of a symmetric but not self-adjoint operator. Let $\Delta$ be the Laplace operator, $\Omega$ a bounded domain in $\mathbb{R}^n$. In $L^2(\Omega)$, consider an operator $A_0$ with a domain $D(A_0) = \mathcal{D}(\Omega)$ mapping $\varphi \in \mathcal{D}(\Omega)$ to $(-\Delta \varphi)$. An elementary integration by parts shows that $A_0$ is symmetric, i.e.,

$$(A_0 \varphi, \psi) = (\varphi, A_0 \psi), \quad \varphi, \psi \in \mathcal{D}(\Omega),$$

where parentheses denote the inner product in $L^2(\Omega)$. This operator is not even closed. Indeed, it is easy to see that if $\varphi \in C^2(\Omega)$ and supp $\varphi$ is a compact subset in $\Omega$, then there exists a sequence $\varphi_k \in \mathcal{D}(\Omega)$ (easily obtained, e.g. by mollifying) such that $\varphi_k \to \varphi$ and $\Delta \varphi_k \to \Delta \varphi$ with respect to the norm in $L^2(\Omega)$. Taking $\varphi \not\in \mathcal{D}(\Omega)$, we see that $A_0$ is not closed. However, it is closable, in the sense that we may define a closed operator $\tilde{A}_0$ whose graph is the closure of the graph of $A_0$ (in $L^2(\Omega) \times L^2(\Omega)$). This means that if $\varphi \in L^2(\Omega)$ and there exists a sequence $\varphi_k \in \mathcal{D}(\Omega)$ such that $\varphi_k \to \varphi$ and $\Delta \varphi_k \to f$ in $L^2(\Omega)$, then, by definition, $\varphi \in D(A_0)$ and
\[ \tilde{A}_0 \varphi = f. \] Clearly, in this case we will again have
\[ (\tilde{A}_0 \varphi, \psi) = (\varphi, A_0 \psi), \quad \psi \in D(\Omega). \]
This, in particular, implies that \( \tilde{A}_0 \varphi \) is well defined (does not depend on the choice of the sequence \( \{ \varphi_k \} \) and \( \tilde{A}_0 \subset A_0^\ast \).

Incidentally, it is easy to see that \( (\tilde{A}_0)^\ast = A_0^* \) (passing to the limit in the identity defining \( A_0^\ast \)).

It is interesting to understand what \( A_0^* \) is. By definition, \( D(A_0^\ast) \) consists of \( u \in L^2(\Omega) \) such that there exists \( v \in L^2(\Omega) \) such that
\[ (v, \psi) = (u, (-\Delta) \psi), \quad \psi \in D(\Omega). \]
But this means that \( -\Delta u = v \) in the distributional sense. Therefore
\[ D(A_0^\ast) = \{ u : u \in L^2(\Omega), \Delta u \in L^2(\Omega) \}. \tag{9.4} \]
It is not trivial to describe \( D(\tilde{A}_0) \). However, it is easy to see that \( D(\tilde{A}_0) \subset \tilde{H}^1(\Omega) \). Indeed, \( (A_0 u, u) = D(u) \) (the Dirichlet integral of \( u \)) for \( u \in D(\Omega) \). Therefore, if \( \varphi_k \in D(\Omega), k = 1, 2, \ldots, \varphi_k \to \varphi \) and \( A_0 \varphi_k \to \tilde{A}_0 \varphi \) in \( L^2(\Omega) \), then \( (A_0(\varphi_k - \varphi_l), \varphi_k - \varphi_l) \to 0 \) as \( k, l \to +\infty \) implying \( \varphi_k \to \varphi \) in \( H^1(\Omega) \); hence, \( \varphi \in \tilde{H}^1(\Omega) \).

So, we have proved that \( D(\tilde{A}_0) \subset \tilde{H}^1(\Omega) \). At the same time, it follows from (9.4) that \( D(A_0^\ast) \supset H^2(\Omega) \), implying \( A_0^\ast \neq \tilde{A}_0 \). For example, if \( \Omega \) has a smooth boundary and \( u \in C^2(\Omega) \), then \( u \in H^2(\Omega) \), hence \( u \in D(A_0^\ast) \); but if we also assume \( u|_{\partial \Omega} \neq 0 \), then \( u \notin \tilde{H}^1(\Omega) \); hence, \( u \notin D(\tilde{A}_0) \). For the general bounded \( \Omega \) it is easy to see that \( u \equiv 1 \) is in \( H^2(\Omega) \), hence in \( D(A_0^\ast) \), but not in \( \tilde{H}^1(\Omega) \) due to Friedrichs’ inequality (8.11), hence not in \( D(\tilde{A}_0) \).

Therefore, \( \tilde{A}_0 \) is symmetric and closed but not self-adjoint.

There exists a natural construction of a self-adjoint extension of any semibounded symmetric operator called the Friedrichs extension. We have actually applied this construction to get a generalized solution of the problem \( \Delta u = f, \; u|_{\partial \Omega} = 0 \). Now we will describe the Friedrichs extension in a more general abstract setting.

A symmetric operator \( A_0 : \mathcal{H} \to \mathcal{H} \) is semibounded below if there exists a constant \( C \) such that
\[ (A_0 \varphi, \varphi) \geq -C(\varphi, \varphi), \quad \varphi \in D(A_0). \tag{9.5} \]
Adding \( (C + \varepsilon)I \), where \( \varepsilon > 0 \), to \( A_0 \), we get a new operator (denote it again by \( A_0 \)) such that
\[ (A_0 \varphi, \varphi) \geq \varepsilon(\varphi, \varphi), \quad \varphi \in D(A_0). \tag{9.6} \]
We will assume from the very beginning that this stronger estimate (9.6) holds. From the point of view of the eigenvalue problem the addition of 

\((C + \varepsilon)I\) is of no importance, because it only shifts eigenvalues without affecting eigenfunctions. Note, that the Friedrichs inequality shows that in 

our main example above the estimate (9.6) is already satisfied.

Set 

\[ [u, v] = (A_0 u, v), \quad u, v \in D(A_0). \]

This inner product defines a pre-Hilbert structure on \(D(A_0)\) (the positive definiteness follows from (9.6)). Denote by \(\| \cdot \|_1\) the corresponding norm (i.e. \(\| u \|_1 = [u, u]^{1/2}\)). Then the convergence with respect to \(\| \cdot \|_1\) implies convergence with respect to the norm \(\| \cdot \|\) in \(\mathcal{H}\). In particular, if a sequence \(\varphi_k \in D(A_0)\) is a Cauchy sequence with respect to \(\| \cdot \|_1\), then it is a Cauchy sequence with respect to \(\| \cdot \|\); hence, \(\varphi_k\) converges in \(\mathcal{H}\).

Denote by \(\mathcal{H}_1\) the completion of \(D(A_0)\) with respect to \(\| \cdot \|_1\) (in the example, \(\mathcal{H}_1 = \tilde{H}^1(\Omega)\)). The inequality (9.6) and the above arguments show the existence of a natural linear map \(j : \mathcal{H}_1 \to \mathcal{H}\). Namely, for any \(g \in \mathcal{H}_1\), its image \(g^* = jg \in \mathcal{H}\) is the limit in \(\mathcal{H}\) of a sequence \(\{\varphi_k\}\), \(\varphi_k \in D(A_0)\), converging to \(g\) in \(\mathcal{H}_1\). Let us prove that this map is injective.

Indeed, assume that \(g^* = 0\). This means that for the sequence \(\{\varphi_k\}\) above we have \(\varphi_k \to g\) in \(\mathcal{H}_1\) and \(\varphi_k \to 0\) in \(\mathcal{H}\). Then by continuity of the inner products,

\[
\|g\|^2 = \lim_{k,l \to \infty} [\varphi_k, \varphi_l] = \lim_{k,l \to \infty} (A_0 \varphi_k, \varphi_l) = \lim_{k \to \infty} \left\{ \lim_{l \to \infty} (A_0 \varphi_k, \varphi_l) \right\} = 0,
\]

hence \(g = 0\) which proves the injectivity of \(j\).

Therefore, there is a natural imbedding \(\mathcal{H}_1 \subset \mathcal{H}\), so we will identify the elements of \(\mathcal{H}_1\) with the corresponding elements of \(\mathcal{H}\). Note, that (9.6) implies that

\[
(9.7) \quad \|u\|^2 \leq \varepsilon^{-1} \|u\|_1^2, \quad u \in \mathcal{H}_1,
\]

which means that the norm of the imbedding operator is not greater than \(\varepsilon^{-1/2}\).

Now set 

\[ D(A) = \mathcal{H}_1 \cap D(A_0^*) \]

and 

\[ A = A_0^*|_{D(A)}. \]

We get an operator \(A : \mathcal{H} \to \mathcal{H}\) called the **Friedrichs extension of \(A_0\)**.
9.2. The Friedrichs extension

If for only a weaker estimate (9.5) is satisfied for $A_0$, then the construction of its Friedrichs extension should be modified as follows: we should take the Friedrichs extension of $A_0 + (C + \varepsilon)I$ and subtract $(C + \varepsilon)I$ from the result. ▲

**Theorem 9.3.** 1. The Friedrichs extension of a semibounded operator $A_0$ is a self-adjoint operator $A : \mathcal{H} \rightarrow \mathcal{H}$. It is again semibounded below and has the same lower bound as $A_0$. For example, if $A_0$ satisfies (9.6), then

\[ (Au, u) \geq \varepsilon (u, u), \quad u \in D(A), \]

where $\varepsilon > 0$ is the same constant as in the similar estimate (9.6) for $A_0$.

2. Assuming (9.6) is satisfied, the operator $A^{-1}$ is everywhere defined and bounded linear operator in $\mathcal{H}$, with the operator norm $\| A^{-1} \| \leq \varepsilon^{-1}$. Moreover, $A^{-1}$ maps $\mathcal{H}$ to $\mathcal{H}_1$ as a bounded linear operator with the same norm not greater than $\varepsilon^{-1/2}$, that is

\[ \| A^{-1} f \|_1 \leq \varepsilon^{-1/2} \| f \|, \quad f \in \mathcal{H}, \]

where $\| \cdot \|$ and $\| \cdot \|_1$ are the norms in $\mathcal{H}$ and $\mathcal{H}_1$ respectively.

**Proof.** 1. Let us verify that $A^{-1}$ exists, is everywhere defined, bounded and symmetric. This implies that $A$ is self-adjoint (see Corollary 9.2).

First, verify that

\[ (Au, v) = [u, v], \quad u, v \in D(A). \]

Indeed, this is true for $u, v \in D(A_0)$. By continuity this is also true for $u \in D(A_0), v \in \mathcal{H}_1$.

Now, make use of the identity

\[ (A_0 u, v) = (u, A_0^* v), \quad u \in D(A_0), v \in D(A_0^*). \]

In particular, this is true for $u \in D(A_0), v \in D(A)$. Then $A_0 u = Au, A_0^* v = Av$, and therefore, we have

\[ (Au, v) = (u, Av) = [u, v], \quad u \in D(A_0), v \in D(A). \]

Taking here 2nd and 3rd terms, we may pass to the limit with respect to $u$ (if the limit is taken in $\mathcal{H}_1$). We get

\[ (u, Av) = [u, v], \quad u \in \mathcal{H}_1, v \in D(A). \]

In particular, this holds for $u \in D(A), v \in D(A)$. Interchanging $u$ and $v$, we get (9.10), implying

\[ (Au, v) = (u, Av) = [u, v], \quad u, v \in D(A). \]
Therefore, $A$ is symmetric and
\begin{equation}
(Au,u) = [u,u] \geq \varepsilon(u,u), \quad u \in D(A),
\end{equation}
where $\varepsilon > 0$ is the same as in (9.6).

It follows from (9.13) that $\text{Ker} A = 0$ and $A^{-1}$ is well-defined. The symmetry of $A$ implies that of $A^{-1}$. It follows from (9.13) that $A^{-1}$ is bounded. Indeed, if $u \in D(A)$, then
\begin{equation*}
\|u\|^2 \leq \varepsilon^{-1}(Au,u) \leq \varepsilon^{-1}\|u\| \cdot \|Au\|,
\end{equation*}
implying
\begin{equation*}
\|u\| \leq \varepsilon^{-1}\|Au\|, \quad u \in D(A).
\end{equation*}
Setting $v = Au$, we get
\begin{equation*}
\|A^{-1}v\| \leq \varepsilon^{-1}\|v\|, \quad v \in D(A^{-1}),
\end{equation*}
which means the boundedness of $A^{-1}$ together with the desired estimate of $\|A^{-1}\|$ in $\mathcal{H}$.

Let us prove that $D(A^{-1}) = \mathcal{H}$, i.e. $\text{Im} A = \mathcal{H}$. We need to prove that if $f \in \mathcal{H}$, then there exists $u \in D(A)$, such that $Au = f$. But this means that $u \in \mathcal{H}_1 \cap D(A_0^*)$ and $A_0^*u = f$, i.e.,
\begin{equation}
(A_0\varphi, u) = (\varphi, f), \quad \varphi \in D(A_0).
\end{equation}
Taking (9.12) into account, we may rewrite this in the form
\begin{equation}
[\varphi, u] = (\varphi, f), \quad \varphi \in D(A_0).
\end{equation}
Now, by the Riesz theorem about the structure of continuous linear functionals on a Hilbert space, for any $f \in \mathcal{H}$ there exists $u \in \mathcal{H}_1$ such that (9.15) holds. It remains to verify that $u \in D(A_0^*)$. But this is clear because, due to (9.11), we may rewrite (9.15) in the form (9.14).

2. Clearly, $A^{-1}$ maps $\mathcal{H}$ to $\mathcal{H}_1$ because $D(A) \subset \mathcal{H}_1$. Returning to (9.13) and using (9.7), we obtain for any $u \in D(A)$
\begin{equation*}
\|u\|^2_1 = (Au,u) \leq \|Au\| \cdot \|u\| \leq \varepsilon^{1/2}\|Au\| \cdot \|u\|_1,
\end{equation*}
which immediately implies (9.9) if we take $f = Au$. □

Now, let us pass to the model example.

Let $A_0$ be the operator $(-\Delta)$ on $D(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$. As we have already seen, the adjoint operator $A_0^*$ maps $u$ to $(-\Delta u)$ if $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$ (here $\Delta$ is understood in the distributional sense). Due to the Friedrichs inequality, $A_0$ is positive, i.e., (9.6) holds. Let
us construct the Friedrichs extension of $A$. This is an operator in $L^2(\Omega)$, such that
\[ A_0 \subset A \subset A_0^* \]
and
\[ D(A) = \dot{H}^1(\Omega) \cap D(A_0^*), \]
i.e.
\[ D(A) = \{ u : u \in \dot{H}^1(\Omega), \Delta u \in L^2(\Omega) \}. \]
By Theorem 9.3, it follows that $A$ is self-adjoint and positive. It is usually called the self-adjoint operator in $L^2(\Omega)$ defined by $(-\Delta)$ and the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. The operator $A^{-1}$ is everywhere defined and bounded.

9.3. Discreteness of spectrum for the Laplace operator in a bounded domain

Let $\Omega$ be a bounded open subset in $\mathbb{R}^n$.

**Theorem 9.4.** 1) The spectrum of the operator $A$ defined in $L^2(\Omega)$ by $(-\Delta)$ and the Dirichlet boundary condition, is discrete. More precisely, there exists an orthonormal basis of eigenvectors $\psi_j$, $j = 1, 2, \ldots$, such that $\psi_j \in \dot{H}^1(\Omega)$ and $(-\Delta)\psi_j = \lambda_j \psi_j$ in $L^2(\Omega)$, where $\lambda_j \to +\infty$ as $j \to +\infty$.

2) All eigenvalues of $A$ are strictly positive.

**Proof.** Applying Theorem 9.3 with $\mathcal{H} = L^2(\Omega)$ and $A_0 = -\Delta$ in $L^2(\Omega)$ with the domain $D(A_0) = \mathcal{D}(\Omega)$, we obtain $A$ as the Friedrichs extension of the operator $A_0$. In this construction we get $\mathcal{H}_1 = \dot{H}^1(\Omega)$, and $A$ has a bounded, everywhere defined inverse operator $A^{-1}$ in $L^2(\Omega)$, which is actually a composition of a bounded operator from $L^2(\Omega)$ to $\dot{H}^1(\Omega)$ and the imbedding operator of $\dot{H}^1(\Omega)$ into $L^2(\Omega)$. Since the imbedding is a compact operator due to Theorem 8.7, the operator $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact self-adjoint operator. By the Hilbert-Schmidt Theorem (see e.g. Reed-Simon [22], Theorem VI.16) it has an orthonormal basis of eigenfunctions in $L^2(\Omega)$, with real eigenvalues $\mu_j$, $j = 1, 2, \ldots$, such that $\mu_j \to 0$ as $j \to \infty$.

Clearly, 0 can not be an eigenvalue of $A^{-1}$. Therefore, the eigenvalues $\lambda_j$ of $A$ are $\lambda_j = \mu_j^{-1}$, with the same eigenfunctions. Therefore, $\lambda_j \to +\infty$ as $j \to \infty$, and the eigenvalues $\lambda_j$ are all isolated and have finite multiplicities. □
9.4. Fundamental solution of the Helmholtz operator and the analyticity of eigenfunctions of the Laplace operator at the interior points. Bessel’s equation

We would like to prove analyticity of eigenfunctions of the Laplace operator in the interior points of the domain. Since eigenvalues are positive, it suffices to prove analyticity of any solution \( u \in D'(\Omega) \) of the equation

\[
\Delta u + k^2 u = 0, \quad k > 0,
\]

called the *Helmholtz equation*. First, let us find a fundamental solution of the Helmholtz operator \( \Delta + k^2 \) in an explicit or asymptotic form. Denote this fundamental solution by \( E_n^{(k)}(x) \). If it turns out that \( E_n^{(k)}(x) \) is analytic for \( x \neq 0 \), hence any solution \( u \in D'(\Omega) \) of the equation (9.16) is analytic in \( \Omega \); cf. Theorem 5.15.

Since \( \Delta + k^2 \) commutes with rotations, it is natural to seek a spherically symmetric fundamental solution, that is one which is invariant under rotations. Using operation of averaging over all rotations, we can construct a spherically symmetric fundamental solution starting from an arbitrary one. (The averaging should be done over the group \( SO(n) \) of all orientation preserving rotations in \( \mathbb{R}^n \), with respect to the Haar measure on this group, that is, the invariant measure on \( SO(n) \) – see e.g. Rudin \[24\], Ch. 5.)

For a function on \( \mathbb{R}^n \), the spherical symmetry means that this function has the form \( E_n^{(k)}(x) = f(r) \), where \( r = |x| \), for \( x \neq 0 \). Assuming \( f \) to be smooth \( (C^2) \) for \( r \neq 0 \), we easily obtain that \( f(r) \) satisfies the ordinary differential equation

\[
f''(r) + \frac{n-1}{r}f'(r) + k^2 f(r) = 0
\]

(see how to compute \( \Delta f(r) \) in Example 4.10). Clearly, to get a fundamental solution we should find a solution \( f(r) \) with a singularity as \( r \to 0^+ \) which is similar to the fundamental solution \( E_n(x) = E_n^{(0)}(x) \) of the Laplace operator. For instance, if it turns out that we have found a solution \( f(r) \) of the equation (9.17) such that

\[
f(r) = E_n(r)g(r), \quad g \in C^2([0, +\infty)), \quad g(0) = 1,
\]

then the verbatim repetition of the verification that \( \Delta E_n(x) = \delta(x) \) yields \( (\Delta + k^2)f(|x|) = \delta(x) \). We will explain the precise meaning of these words later in this Section.
9.4. Fundamental solution of the Helmholtz operator

The required smoothness of $f$ can be easily deduced from the smoothness of the fundamental solution of the Helmholtz operator, which holds because this operator is elliptic. However, since we did not prove the desired smoothness, we can avoid using it, instead deducing the smoothness from the direct construction by separation of variables, which we sketch below. ▲

▼ We will reduce (9.17) to the Bessel equation. This may be done even for a more general equation

\begin{equation}
(9.18) \quad f''(r) + \frac{\alpha}{r} f'(r) + \left( k^2 + \frac{\beta}{r^2} \right) f(r) = 0,
\end{equation}

where $\alpha, \beta, k$ are arbitrary real (or complex) constants. First, let us rewrite (9.18) in the form

\begin{equation}
r^2 f''(r) + \alpha r f'(r) + (\beta + k^2 r^2) f(r) = 0.
\end{equation}

Here the first three terms of the left hand side above constitute Euler’s operator

\[ L = r^2 \frac{d^2}{dr^2} + \alpha r \frac{d}{dr} + \beta, \]

applied to $f$. The Euler operator commutes with the homotheties of the real line (i.e. with the transformations $x \mapsto \lambda x$, with $\lambda \in \mathbb{R} \setminus 0$, acting upon functions on $\mathbb{R}$ by the change of variables), and $r^\kappa$ for any $\kappa \in \mathbb{C}$ are its eigenfunctions. (It is also useful to notice that the change of variable $r = e^t$ transforms Euler’s operator into an operator commuting with all translations with respect to $t$, i.e., into an operator with constant coefficients.)

We can start with the following change of the unknown function in (9.18):

\[ f(r) = r^\kappa z(r), \]

which leads to the equation for $z(r)$:

\begin{equation}
(9.19) \quad z''(r) + \frac{\alpha + 2\kappa}{r} z'(r) + \left[ k^2 + \frac{\kappa(\kappa + \alpha - 1) + \beta}{r^2} \right] z(r) = 0.
\end{equation}

Here the parameter $\kappa$ is at our disposal. The first idea is to set $\kappa = -\alpha/2$ in order to cancel $z'(r)$. Then we get

\begin{equation}
(9.20) \quad z''(r) + \left[ k^2 + \frac{c}{r^2} \right] z(r) = 0.
\end{equation}

It is easy to derive from (9.20) that for $k \neq 0$ the solutions $z(r)$ of this equation behave, for $r \to +\infty$, as solutions of the equation

\[ u''(r) + k^2 u(r) = 0. \]
More precisely, let, for example, \( k > 0 \) (this case is most important to us). Then there exist solutions \( z_1(r) \) and \( z_2(r) \) of (9.20) whose asymptotics as \( r \to +\infty \) are

\[
(9.21) \quad z_1(r) = \sin kr + O \left( \frac{1}{r} \right), \quad z_2(r) = \cos kr + O \left( \frac{1}{r} \right).
\]

This can be proved by passing to an integral equation as we did when seeking asymptotics of eigenfunctions and eigenvalues of the Sturm-Liouville operator. Namely, let us rewrite (9.20) in the form

\[
z''(r) + k^2 z(r) = -c r^2 z(r)
\]

and seek \( z(r) \) in the form

\[
z(r) = C_1(r) \cos kr + C_2(r) \sin kr.
\]

The variation of parameters method for functions \( C_j(r) \) yields that we can take \( C_1(r), C_2(r) \) satisfying

\[
\begin{align*}
C_1'(r) \cos kr + C_2'(r) \sin kr &= 0, \\
-kC_1'(r) \sin kr + kC_2'(r) \cos kr &= -\frac{c}{r^2} z(r),
\end{align*}
\]

which imply

\[
C_1(r) = A + \int_{r_0}^r \frac{c}{\rho^2} \frac{\sin k\rho}{k} z(\rho) d\rho, \\
C_2(r) = B - \int_{r_0}^r \frac{c}{\rho^2} \frac{\cos k\rho}{k} z(\rho) d\rho.
\]

We wish that \( C_1(r) \to A \) and \( C_2(r) \to B \) as \( r \to +\infty \). But then it is natural to take \( r_0 = +\infty \) and write

\[
(9.22) \quad z(r) = A \cos kr + B \sin kr + \frac{c}{k} \int_r^\infty \frac{1}{\rho^2} \sin k(r - \rho) z(\rho) d\rho,
\]

which is an integral equation for \( z(r) \). Let us rewrite it in the form

\[
(I - T) z = A \cos kr + B \sin kr,
\]

where \( T \) is an integral operator mapping \( z(r) \) to the last summand in (9.22). Consider this equation in the Banach space \( C_b([r_0, \infty)) \) of continuous bounded functions on \([r_0, \infty)\), with the norm \( \|z\| = \sup_{r \geq r_0} |z(r)| \). It is clear that

\[
\|Tz\| \leq \frac{|c|}{|k|} \int_{r_0}^\infty \frac{1}{\rho^2} d\rho \|z\| = \frac{c}{|k|r_0} \|z\|.
\]

If \( r_0 \) is sufficiently large, then \( \|T\| < 1 \), so the operator \( I - T \) is invertible, because

\[
(I - T)^{-1} = I + T + T^2 + \ldots,
\]
9.4. Fundamental solution of the Helmholtz operator

where the series converges with respect to the operator norm in \(C_b([r_0, \infty))\). Therefore, the integral equation (9.22) has a unique solution in \(C_b([r_0, \infty))\). But its continuous bounded solutions are solutions of (9.20) and if \(z(r)\) is such a solution, then

\[
|z(r) - A \cos kr - B \sin kr| \leq C \int_r^\infty \frac{1}{\rho^2} d\rho = \frac{C}{r},
\]
i.e.

\[
(9.23) \quad z(r) = A \cos kr + B \sin kr + O \left(\frac{1}{r}\right) \text{ as } r \to +\infty.
\]

Thus, for any \(A\) and \(B\), the equation (9.20) has a solution with the asymptotic (9.23). In particular, there are solutions \(z_1(r)\) and \(z_2(r)\) with asymptotics (9.21). Clearly, these solutions are linearly independent. The asymptotic (9.23) may also be expressed in the form

\[
z(r) = A \sin (k(r - r_0)) + O \left(\frac{1}{r}\right),
\]
which makes it clear that if \(z(r)\) is real (this is so, for instance, if all constants are real), then \(z(r)\) has infinitely many zeros with approximately the same behavior as that of zeros of \(\sin (k(r - r_0))\), as \(r \to +\infty\).

**Bessel equation and general cylindric functions.**

Now, return to (9.19) and choose the parameter \(\kappa\) so as to get \(\alpha + 2\kappa = 1\). Then we get the equation for \(z(r)\):

\[
(9.24) \quad z''(r) + \frac{1}{r} z'(r) + \left[ k^2 - \frac{\nu^2}{r^2} \right] z(r) = 0,
\]
where \(\nu^2 = \kappa^2 - \beta\). (In particular, we can take \(\nu = \pm \kappa\) in the case \(\beta = 0\), which was our starting point; generally, \(\nu\) may be not real.) Put \(r = k^{-1}x\) or \(x = kr\). Introducing \(x\) as a new independent variable, we get instead of (9.24) the following equation for \(y(x) = z(k^{-1}x)\) (or \(z(x) = y(kx)\)):

\[
(9.25) \quad y''(x) + \frac{1}{x} y'(x) + \left( 1 - \frac{\nu^2}{x^2} \right) y(x) = 0.
\]

(One should not confuse the real variable \(x > 0\) here with the \(x \in \mathbb{R}^n\) used earlier.)

The equation (9.25) is called the **Bessel equation** and its solutions are called **cylindric functions.** ▲
Cylindric functions appear when we try to apply the separation of variables (or the Fourier method) to find eigenfunctions of the Laplace operator in a disc and also solutions of the Dirichlet problem in a cylinder (over a disc) of finite or infinite height (this accounts for the term “cylindric functions”).

Eliminating, as above, \( y'(x) \) in (9.25), we see that the asymptotic of any cylindric function as \( x \to +\infty \) has the form

\[
y(x) = \frac{A}{\sqrt{x}} \sin(x-x_0) + O\left(\frac{1}{x}\right).
\]

Incidentally, substituting all parameters, we see that (9.19) in our case (for \( \alpha = 1, \kappa = -1/2, k = 1, \beta = -\nu^2 \)) takes the form

\[
z''(r) + \left[1 + \frac{1}{4} - \frac{\nu^2}{r^2}\right] z(r) = 0,
\]

which makes it clear that for \( \nu = \pm \frac{1}{2} \) the Bessel equation is explicitly solvable and the solutions are of the form coinciding with the first term in (9.26), i.e.,

\[
y(x) = \frac{1}{\sqrt{x}}(A \cos x + B \sin x).
\]

In particular, for \( n = 3 \) we can solve explicitly the equation (9.17) that arose from the Helmholtz equation. Namely, setting \( \alpha = 2, \beta = 0, \kappa = -1, \) we see that (9.19) takes the form \( z'' + k^2 z = 0 \), implying that for \( n = 3 \) the spherically symmetric solutions of the Helmholtz equation are of the form

\[
f(x) = \frac{1}{r}(A \cos kr + B \sin kr), \quad r = |x|.
\]

We get the fundamental solution if \( A = \frac{1}{4\pi} \). In particular, functions

\[
\frac{e^{ikr}}{4\pi r}, \quad \frac{e^{-ikr}}{4\pi r}, \quad \frac{\cos kr}{4\pi r}, \quad \frac{\sin kr}{r},
\]

are fundamental solutions, and the function \( u = \frac{\sin kr}{r} \) is a solution of the homogeneous equation

\[
(\Delta + k^2)u(x) = 0, \quad x \in \mathbb{R}^3.
\]

Let us return to the general equation (9.25). We should describe somehow the behavior of its solutions as \( x \to 0^+ \). The best method for this is to try to find the solution in the form of a series in \( x \). Since \( \frac{\nu^2}{x^2} \) plays a more important role than 1, as \( x \to 0^+ \), we may expect that \( y(x) \) behaves at zero as a solution of Euler’s equation obtained from (9.25), if 1 is removed from the coefficient by \( y(x) \). But solutions of Euler’s equation are linear
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combinations of functions $x^\sigma$ and, perhaps, $x^\sigma \ln x$ for appropriate $\sigma$. In our case, substituting $x^\sigma$ into Euler’s equation, we get $\sigma = \pm \nu$. Therefore, it is natural to expect that we may find a pair of solutions of (9.25) that behave as $x^{\pm \nu}$ as $x \to +0$. This is, however, true up to some corrections only.

For the time being, let us seek solutions $y(x)$ of (9.25) in the form

\begin{equation}
(9.27) \quad y(x) = x^\sigma (a_0 + a_1 x + a_2 x^2 + \ldots) = a_0 x^\sigma + a_1 x^{\sigma+1} + \ldots.
\end{equation}

Substituting this series in (9.25) and equating coefficients of all powers of $x$ to zero, we get

\begin{align*}
2^0 \sigma (\sigma - 1) + a_0 \sigma - a_0 \nu^2 &= 0 \\
2^1 (\sigma + 1) + a_1 (\sigma + 1) - a_1 \nu^2 &= 0, \\
& \cdots \\
2^k (\sigma + k)(\sigma + k - 1) + a_k (\sigma + k) + a_{k-2} - a_k \nu^2 &= 0, \quad k = 2, 3, \ldots
\end{align*}

or

\begin{align*}
2^0 (\sigma^2 - \nu^2) &= 0, \\
2^1 [(\sigma + 1)^2 - \nu^2] &= 0, \\
2^k [(\sigma + k)^2 - \nu^2] + a_{k-2} &= 0, \quad k = 2, 3, \ldots
\end{align*}

Without loss of generality, we may assume that $a_0 \neq 0$ (for $a_0 = 0$ we may replace $\sigma$ by $\sigma + k$, where $k$ is the subscript number of the first non-zero coefficient $a_k$, and then reenumerate the coefficients). But then (9.28) implies $\sigma = \pm \nu$. If $\nu$ is neither integer nor half-integer, then (9.29) and (9.30) imply

\begin{align*}
a_1 = a_3 = \ldots = 0, \\
a_k &= -\frac{a_{k-2}}{(\sigma + k)^2 - \nu^2} = -\frac{a_{k-2}}{(\sigma + \nu + k)(\sigma - \nu + k)} = -\frac{a_{k-2}}{k(k + 2\sigma)}.
\end{align*}

If $k = 2m$, then this yields

\begin{align*}
a_{2m} &= -a_{2m-2} \frac{1}{2^2 m(m + \sigma)} = \cdots = \frac{(-1)^m a_0}{2^{2m} m! (m + \sigma)(m + \sigma - 1) \ldots (\sigma + 1)}.
\end{align*}

Set $a_0 = \frac{1}{\Gamma(\sigma + 1)}$, where $\Gamma$ denotes the Euler $\Gamma$-function. Then, using the identity $s \Gamma(s) = \Gamma(s + 1)$, we get

\begin{equation}
a_{2m} = \frac{(-1)^m}{2^{2m+\sigma} m! \Gamma(m + \sigma + 1)}.
\end{equation}
The corresponding series for \( \sigma = \nu \) is denoted by \( J_\nu(x) \) and is called the Bessel or cylindric function of the 1st kind. Thus,

\[
J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left( \frac{x}{2} \right)^{2k+\nu}.
\]

(9.31)

It is easy to see that this series converges for all complex \( x \neq 0 \).

All these calculations make sense also for an integer or half-integer \( \sigma = \nu \), if \( \Re \nu \geq 0 \). We could assume that \( \Re \nu \geq 0 \) without loss of generality, because only \( \nu^2 \) enters the equation. But it makes sense to consider \( \nu \) with \( \Re \nu < 0 \) in the formulas for the solutions, to get the second solution supplementing the one given by (9.31). (In other words, we could take \( \sigma = -\nu \), returning to (9.27).) For \( \nu = -1, -2, \ldots \), the series (9.31) may be defined by continuity, since \( \Gamma(z) \) has no zeros but only poles at \( z = 0, -1, -2, \ldots \).

So, the Bessel equation (9.25) for \( \Re \nu \geq 0 \) always has a solution of the form \( J_\nu(x) = x^\nu g_\nu(x) \), where \( g_\nu(x) \) is an entire analytic function of \( x \), such that \( g_\nu(0) = \frac{1}{\Gamma(\nu+1)} \neq 0 \).

We will neither seek the second solution (in the cases when we did not do this yet) nor investigate other properties of cylindric functions. We will only remark that in reference books and textbooks on special functions, properties of cylindric functions are discussed with the same minute details, as the properties of trigonometric functions are discussed in textbooks on trigonometry. Cylindric functions are tabulated and are a part of standard mathematics software, such as Maple, Mathematica and MATLAB. The same applies to many other special functions. Many of them are, as cylindric functions, solutions of certain second order linear ordinary differential equations.

For more details about the cylindric functions we refer the reader to the books by Lebedev [18], Whittaker and Watson [34]. ▲

▼ Let us return to Helmholtz’s equation. We have seen that (9.17) reduces to Bessel’s equation. Indeed, it is of the form (9.18) with \( \alpha = n - 1 \), \( \beta = 0 \). Let us pass to (9.19) and set \( \kappa = \frac{1-\alpha}{2} = \frac{2-n}{2} \). Then we get (9.24), where

\[
\nu^2 = \frac{n-2}{2} \left[ n - 2 + \frac{2-n}{2} \right] = \left( \frac{n-2}{2} \right)^2,
\]
implying \( \nu = \pm \frac{n-2}{2} \). Therefore, in particular, we get a solution of (9.17) of the form

\[
f(r) = r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(kr).
\]

Actually, this solution does not have any singularities (it expands in a series in powers of \( r^2 \)).

We may also consider the solution

\[
f(r) = r^{\frac{2-n}{2}} J_{\frac{2-n}{2}}(kr).
\]

If \( n \) is odd then, multiplying this function by a constant, we may get a fundamental solution for Helmholtz’s operator (this solution is, actually, an elementary function). If \( n \) is even, then, together with \( J_{\frac{n-2}{2}}(x) \), we should use the second solution of Bessel’s equation (it is called a cylindric function of the second kind or Neumann’s function and its expansion contains \( \ln x \) as \( x \to 0^+ \)). It can also be expressed in terms of \( J_{\frac{n-2}{2}}(x) \) via a quadrature with the help of a well known Wronskian trick (which gives for the second solution a 1st order linear differential equation). In any case, we see that there exists a fundamental solution \( e_n^{(b)}(r) \), analytic in \( r \) for \( r \neq 0 \). This implies that all eigenfunctions of the Laplace operator are analytic at interior points of the domain.

Note, however, that there is a general theorem which guarantees the analyticity of any solution of an elliptic equation with analytic coefficients. Solutions of an elliptic equation with smooth \((C^\infty)\) coefficients are also smooth.

9.5. Variational principle. The behavior of eigenvalues under variation of the domain. Estimates of eigenvalues

Let \( A \) be a self-adjoint semibounded below operator with discrete spectrum in a Hilbert space \( \mathcal{H} \). Let \( \varphi_1, \varphi_2, \ldots \) be a complete orthonormal system of eigenvectors of \( A \) with the eigenvalues \( \lambda_1, \lambda_2, \ldots \), i.e., \( A\varphi_j = \lambda_j \varphi_j \) for \( j = 1, 2, \ldots \). The discreteness of the spectrum means that \( \lambda_j \to +\infty \) as \( j \to +\infty \). Therefore, we may assume that eigenvalues increase with \( j \), i.e.,

\[
\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots,
\]

and we will do this in what follows. Instead of the set of eigenvalues, it is convenient to consider a (non-strictly) increasing function \( N(\lambda) \) defined for \( \lambda \in \mathbb{R} \) as the number of eigenvalues not exceeding \( \lambda \). This function takes only non-negative integer values, is constant between eigenvalues, and at eigenvalues its jumps are equal to the multiplicities of these eigenvalues. It is continuous from the right, i.e. \( N(\lambda) = N(\lambda + 0) \).
The eigenvalues $\lambda_j$ are easily recovered from $N(\lambda)$. Namely, if $\lambda$ is a real number, then it is an eigenvalue of multiplicity $N(\lambda + 0) - N(\lambda - 0)$ (if $N(\lambda + 0) - N(\lambda - 0) = 0$, then $\lambda$ is not an eigenvalue). If $N(\lambda - 0) < j \leq N(\lambda + 0)$ and $j$ is a nonnegative integer, then $\lambda_j = \lambda$. In short, $\lambda_j$ is uniquely defined from the condition $N(\lambda_j - 0) < j \leq N(\lambda_j + 0)$.

The following proposition is important.

**Proposition 9.5.** (Glazman’s lemma)

\[
N(\lambda) = \sup_{L \subset D(A)} \dim L, \quad (Au, u) \leq \lambda(u, u) \quad u \in L
\]

where $L$ denotes a finite dimensional linear subspace of $D(A)$ such that $(Au, u) \leq \lambda(u, u)$ for all $u \in L$. (Under the “sup” sign stands the indication that supremum is taken over all $L$ such that $(Au, u) \leq \lambda(u, u)$ for all $u \in L$.)

If $A$ is the Friedrichs extension of $A_0$ and $\lambda$ is not an eigenvalue then $L \subset D(A)$ may be replaced by a stronger condition $L \subset D(A_0)$.

**Proof.** Let $L_\lambda$ be a finite-dimensional subspace in $\mathcal{H}$ spanned by all eigenvectors $\varphi_j$ with eigenvalues $\lambda_j \leq \lambda$. Clearly, $L_\lambda \subset D(A)$ and $A - \lambda I$ has only non-positive eigenvalues on $L_\lambda$; therefore, it is non-positive itself, i.e., $(Au, u) \leq \lambda(u, u)$, $u \in L_\lambda$. In particular, we may take $L = L_\lambda$ in the right-hand side of (9.32). But $\dim L_\lambda = N(\lambda)$, implying that the left hand side of (9.32) is not greater than the right hand side.

Let us verify that the left-hand side of (9.32) is not less than the right-hand side. Let $L \subset D(A)$ and $(Au, u) \leq \lambda(u, u)$, $u \in L$. We wish to prove that $\dim L \leq N(\lambda)$. Fix such a subspace $L$ and set $M_\lambda = L_\lambda^\perp$ (the orthogonal complement to $L_\lambda$). Clearly, if $u \in M_\lambda$, then $u$ can be expanded with respect to the eigenfunctions $\varphi_j$ with the eigenvalues $\lambda_j > \lambda$. Therefore, if $u \in D(A) \cap M_\lambda$ and $u \neq 0$, then $(Au, u) > \lambda(u, u)$ implying $L \cap M_\lambda = 0$.

Now, consider the projection $E_\lambda : \mathcal{H} \rightarrow \mathcal{H}$ onto $L_\lambda$ parallel to $M_\lambda$, i.e., if $u = v + w$, where $v \in L_\lambda$, $w \in M_\lambda$, then $E_\lambda u = v$. Clearly, $\text{Ker} \ E_\lambda = M_\lambda$.

Now, consider the operator $E_\lambda|_L : L \rightarrow L_\lambda$. Since $(\text{Ker} \ E_\lambda) \cap L = 0$, then $E_\lambda|_L$ is injective. Therefore, $\dim L \leq \dim L_\lambda = N(\lambda)$, as required.

It remains to verify the last statement of the proposition. Without loss of generality, we may assume that $(Au, u) \geq (u, u)$, for $u \in D(A_0)$. Let us make use of the Hilbert space $\mathcal{H}_1$, the completion of $D(A_0)$ with respect to the norm defined by the inner product $[u, v] = (A_0 u, v)$, where $u, v \in D(A_0)$. 

Instead of \((Au, u)\) in (9.32), we may write \([u, u]\). Let \(\|\cdot\|_1\) be the norm in \(H_1\). Let \(L\) be any finite dimensional subspace of \(D(A)\). In particular, \(L \subset H_1\). In \(L\), choose an orthonormal (with respect to the inner product in \(H_1\)) basis \(e_1, \ldots, e_p\) and let vectors \(f_1, \ldots, f_p \in D(A_0)\) be such that \(\|e_j - f_j\|_1 < \delta\), where \(\delta > 0\) is sufficiently small. Let \(L_1\) be a subspace spanned by \(f_1, \ldots, f_p\).

It is easy to see that \(\dim L_1 = \dim L = p\) for a small \(\delta\). Indeed, if \(\sum_{j=1}^{p} c_j f_j = 0\) with \(c_j \in \mathbb{C}\), then by taking the inner product with \(f_k\), we get

\[
\sum_{j=1}^{p} c_j (f_j, f_k) = 0, \quad k = 1, \ldots, p.
\]

But \((f_j, f_k) = \delta_{jk} + \varepsilon_{jk}\), where \(\varepsilon_{jk} \to 0\) as \(\delta \to 0\). Therefore, the matrix \((\delta_{jk} + \varepsilon_{jk})\), being close to the identity matrix, is invertible for a small \(\delta\).

Therefore, \(c_j = 0\) and \(f_1, \ldots, f_p\) are linearly independent, i.e., \(\dim L_1 = p\).

Further, if \([u, u] \leq \lambda (u, u)\) for all \(u \in L\), then \([v, v] \leq (\lambda + \varepsilon)(v, v)\), \(v \in L_1\), with \(\varepsilon = \varepsilon(\delta)\), such that \(\varepsilon(\delta) \to 0\) as \(\delta \to 0\). Indeed, if \(v = \sum_{j=1}^{p} c_j f_j\), then \(u = \sum_{j=1}^{p} c_j e_j\) is close to \(v\) with respect to \(\|\cdot\|_1\), implying the required statement.

Now, set

\[
N_1(\lambda) = \sup_{L_1 \subset D(A_0), \ (Au, u) \leq \lambda (u, u), \ u \in L_1} \dim L_1.
\]

Then from the above argument, it follows that \(N(\lambda - \varepsilon) \leq N_1(\lambda) \leq N(\lambda)\) for any \(\lambda \in \mathbb{R}\) and \(\varepsilon > 0\). If \(\lambda\) is not an eigenvalue, then \(N(\lambda)\) is continuous at \(\lambda\) implying \(N_1(\lambda) = N(\lambda)\). □

Remark. Obviously, \(N(\lambda)\) is recovered from \(N_1(\lambda)\) (namely, \(N(\lambda) = N_1(\lambda + 0)\)).

Another description of the eigenvalues in terms of the quadratic form of \(A\) is sometimes more convenient. Namely, the following statement holds.

**Proposition 9.6.** (Courant’s variational principle)

\[
\lambda_1 = \min_{\varphi \in D(A) \setminus \{0\}} \frac{(A\varphi, \varphi)}{(\varphi, \varphi)},
\]

\[
\lambda_{j+1} = \sup_{L \subset D(A), \dim L = j} \inf_{\varphi \in (L^\perp \setminus \{0\}) \cap D(A)} \frac{(A\varphi, \varphi)}{(\varphi, \varphi)}, \quad j = 1, 2, \ldots.
\]
If $A$ is the Friedrichs extension of $A_0$, then, by replacing $\min$ by $\inf$ in (9.33), we may write $\varphi \in D(A_0) \setminus \{0\}$ and $L \subset D(A_0)$, instead of $\varphi \in D(A) \setminus \{0\}$ and $L \subset D(A)$, respectively.

**Proof.** We will use the same notations as in the proof of Proposition 9.5. Let us prove (9.33) first. If $\varphi = \sum_{j=1}^{\infty} c_j \varphi_j$ for $\varphi \in D(A)$ (i.e., if $\sum_{j=1}^{\infty} |c_j|^2 \lambda_j^2 < +\infty$), then $(A\varphi, \varphi) = \sum_{j=1}^{\infty} \lambda_j |c_j|^2$ and $(\varphi, \varphi) = \sum_{j=1}^{\infty} |c_j|^2$.

Therefore, $(A\varphi, \varphi) = \sum_{j=1}^{\infty} \lambda_j |c_j|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |c_j|^2 = \lambda_1 (\varphi, \varphi)$, and the equality is attained at $\varphi = \varphi_1$. This proves (9.33).

Let us prove (9.34). Let $\Phi_j$ be the subspace spanned by $\varphi_1, \ldots, \varphi_j$. Take $L = \Phi_j$. We see that the right-hand side of (9.34) is not smaller than the left hand side.

Now, we should verify that the right hand side is not greater than the left hand side, i.e., if $L \subset D(A)$ and $\dim L = j$, then there exists a non-zero vector $\varphi \in L_+ \cap D(A)$ such that $(A\varphi, \varphi) \leq \lambda_{j+1}(\varphi, \varphi)$. But, as in the proof of Proposition 9.5, it is easy to verify that $\Phi_{j+1} \cap L_+ \neq 0$ (if we had $\Phi_{j+1} \cap L_+ = 0$, then the orthogonal projection onto $L$ would be an injective map of $\Phi_{j+1}$ in $L$, contradicting to the fact that $\dim \Phi_{j+1} = j + 1 > j = \dim L$). Therefore, we may take $\varphi \in \Phi_{j+1} \cap L_+, \varphi \neq 0$, and then it is clear that $(A\varphi, \varphi) \leq \lambda_{j+1}(\varphi, \varphi)$.

The last statement of Proposition 9.6 is proved as the last part of Proposition 9.5. □

Let us return to the eigenvalues of $(-\Delta)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ (with the Dirichlet boundary conditions). Denote them by $\lambda_j(\Omega)$, $j = 1, 2, \ldots$, and their distribution function, $N(\lambda)$, by $N_\Omega(\lambda)$. In this case $D(A_0) = D(\Omega)$. Now let $\Omega_1$ and $\Omega_2$ be two bounded domains such that $\Omega_1 \subset \Omega_2$. Then $D(\Omega_1) \subset D(\Omega_2)$, and Proposition 9.5 implies that

(9.35) \[ N_{\Omega_1}(\lambda) \leq N_{\Omega_2}(\lambda); \]

therefore,

\[ \lambda_j(\Omega_1) \geq \lambda_j(\Omega_2), \quad j = 1, 2, \ldots. \]

The eigenfunctions and eigenvalues may be found explicitly if $\Omega$ is a rectangular parallelepiped. Suppose $\Omega$ is of the form

\[ \Omega = (0, a_1) \times \ldots \times (0, a_n). \]
It is easy to verify that the eigenfunctions can be taken in the form
\[ \varphi_{k_1, \ldots, k_n} = C_{k_1, \ldots, k_n} \sin \frac{k_1 \pi x_1}{a_1} \cdots \sin \frac{k_n \pi x_n}{a_n}, \]
where \( k_1, \ldots, k_n \) are (strictly) positive integers, \( C_{k_1, \ldots, k_n} \) are normalizing constants. The eigenvalues are of the form
\[ \lambda_{k_1, \ldots, k_n} = \left( \frac{k_1 \pi}{a_1} \right)^2 + \cdots + \left( \frac{k_n \pi}{a_n} \right)^2. \]
The function \( N_\Omega(\lambda) \) in our case is equal to the number of points of the form \( (\frac{k_1 \pi}{a_1}, \ldots, \frac{k_n \pi}{a_n}) \) belonging to the closed ball of radius \( \sqrt{\lambda} \) with the center at the origin. If \( k_i \) are allowed to be any integers, we see that points \( (\frac{k_1 \pi}{a_1}, \ldots, \frac{k_n \pi}{a_n}) \) run over a lattice in \( \mathbb{R}^n \) obtained by obvious homotheties of an integer lattice. It is easy to see that \( N_\Omega(\lambda) \) for large \( \lambda \) has a two-sided estimate in terms of the volume of a ball of radius \( \sqrt{\lambda} \), i.e.,
\[ (9.36) \quad C^{-1} \lambda^{n/2} \leq N_\Omega(\lambda) \leq C \lambda^{n/2}, \]
where \( C > 0 \).

Now note that for any non-empty bounded open set \( \Omega \subset \mathbb{R}^n \) we may take cubes \( Q_1, Q_2 \) such that \( Q_1 \subset \Omega \subset Q_2 \). Then \( N_{Q_1}(\lambda) \leq N_\Omega(\lambda) \leq N_{Q_2}(\lambda) \). It follows that (9.36) holds for any such \( \Omega \).

Making these arguments more precise, it is possible to prove the following asymptotic formula by H.Weyl:
\[ N_\Omega(\lambda) \sim (2\pi)^{-n} V_n \cdot \text{mes}(\Omega) \cdot \lambda^{n/2}, \quad \text{as} \quad \lambda \to +\infty, \]
where \( V_n \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( \text{mes}(\Omega) \) is the Lebesgue measure of \( \Omega \), see e.g. Courant, Hilbert [5], v.1.

9.6. Problems

9.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary. Let \( L \) be a self-adjoint operator in \( L^2(\Omega) \) determined by the Laplace operator in \( \Omega \) with the Dirichlet boundary conditions. The Green function of the Laplace operator in \( \Omega \) is the Schwartz kernel of \( L^{-1} \), i.e., a distribution \( G(x, y) \), \( x, y \in \Omega \), such that
\[ L^{-1}f(x) = \int_\Omega G(x, y)f(y)dy. \]
Prove that
\[ \Delta_x G(x, y) = \delta(x - y), \]
\[ G|_{x \in \partial \Omega} = 0, \]
and $G$ is uniquely defined by these conditions. Give a physical interpretation of the Green function.

9.2. Find out what kind of singularity the Green function $G(x, y)$ has at $x = y \in \Omega$.

9.3. Prove that the Green function is symmetric, i.e.,

$$G(x, y) = G(y, x), \quad x, y \in \Omega.$$

9.4. Prove that the solution of the problem $\Delta u = f$, $u|_{\partial \Omega} = \varphi$ in the domain $\Omega$ is expressed via the Green function by the formula

$$u(x) = \int_{\Omega} G(x, y)f(y)dy - \int_{\partial \Omega} \varphi(y) \frac{\partial G(x, y)}{\partial n_y} dS_y,$$

where $n_y$ is the exterior normal to the boundary at $y$ and $dS_y$ is the area element of the boundary at $y$.

9.5. The Green function of the Laplace operator in an unbounded domain $\Omega \subset \mathbb{R}^n$ is a function $G(x, y)$, $x, y \in \Omega$, such that $\Delta_x G(x, y) = \delta(x - y)$, $G|_{x \in \partial \Omega} = 0$ and $G(x, y) \to 0$ as $|x| \to +\infty$ (for every fixed $y \in \Omega$) if $n \geq 3$; $G(x, y)$ is bounded for $|x| \to +\infty$ for every fixed $y \in \Omega$ if $n = 2$.

Find the Green function for the half-space $x_n \geq 0$.

9.6. Using the result of the above problem, write a formula for the solution of Dirichlet’s problem in the half-space $x_n \geq 0$ for $n \geq 3$. Solve this Dirichlet’s problem also with the help of the Fourier transform with respect to $x' = (x_1, \ldots, x_{n-1})$ and compare the results.

9.7. Find the Green function of a disc in $\mathbb{R}^2$ and a ball in $\mathbb{R}^n$, $n \geq 3$.

9.8. Write a formula for the solution of the Dirichlet problem in a disc and a ball. Obtain in this way the Poisson formula for $n = 2$ from Problem 8.2.

9.9. Find the Green function of a half-ball.

9.10. Find the Green function for a quarter of $\mathbb{R}^3$, i.e., for $\{(x_1, x_2, x_3) : x_2 > 0, x_3 > 0\} \subset \mathbb{R}^3$. 

9.11. The Bessel function $J_\nu(x)$ is defined for $x > 0$ and any $\nu \in \mathbb{C}$, by the series

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$ 

Prove that if $n \in \mathbb{Z}_+$, then $J_{-n}(x) = (-1)^n J_n(x)$.

9.12. Prove that if $n \in \mathbb{Z}_+$, then the Bessel equation $y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$ has for a solution not only $J_n(x)$ but also a function defined by the series

$$(\ln x) \cdot x^{-n} \sum_{k=0}^{\infty} c_k x^k + \sum_{k=0}^{\infty} d_k x^k, \quad c_0 \neq 0.$$  NOT CORRECT

9.13. Prove that the Laurent expansion of $e^{\frac{x}{2}(t-\frac{1}{t})}$ with respect to $t$ (for $t \neq 0, \infty$) has the form

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$ 

9.14. Prove that the expansion of $e^{ix\sin \varphi}$ into the Fourier series of $\varphi$ is of the form

$$e^{ix\sin \varphi} = J_0(x) + 2 \sum_{n=1}^{\infty} [J_{2n}(x) \cos 2n \varphi + iJ_{2n-1}(x) \sin(2n-1)\varphi].$$

9.15. Find a complete orthonormal system of the eigenvalues and eigenfunctions of the Laplace operator in a rectangle.

9.16. Let $k \in \mathbb{Z}_+$ and $\alpha_{k,1} < \alpha_{k,2} < \ldots$ be all the zeros of $J_k(x)$ for $x > 0$. Prove that

$$\int_0^R J_k \left(\frac{\alpha_{k,n}}{R} r\right) J_k \left(\frac{\alpha_{k,m}}{R} r\right) r dr = 0, \quad \text{for } m \neq n.$$ 

9.17. Using the result of Problem 9.16, find a complete orthonormal system of eigenfunctions of the Laplace operator in a disc.

9.18. Describe a way to use the Fourier method to solve the Dirichlet problem in a straight cylinder (of finite height) with a disc as the base.
Chapter 10

The wave equation

10.1. Physical problems leading to the wave equation

There are many physical problems leading to the wave equation

\[ u_{tt} = a^2 \Delta u, \]

where \( u = u(t, x), \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ \Delta \) is the Laplacian with respect to \( x \). The following more general non-homogeneous equation is also often encountered:

\[ u_{tt} = a^2 \Delta u + f(t, x). \]

We have already seen that small oscillations of the string satisfy (10.1) (for \( n = 1 \)), and in the presence of external forces they satisfy (10.2). It can be shown that small oscillations of a membrane satisfy similar equations, if we denote by \( u = u(t, x) \), for \( x \in \mathbb{R}^2 \), the vertical displacement of the membrane from the (horizontal) equilibrium position. Similarly, under small oscillations of a gas (acoustic wave) its parameters (e.g. pressure, density, displacement of gas particles) satisfy (10.1) with \( n = 3 \).

\[ ▼ \] One of the most important fields, where equations of the form (10.1) and (10.2) play a leading role, is classical electrodynamics. Before explaining this in detail, let us recall some notations from vector analysis. (See e.g. Ch. 10 in [23].)

Let \( \mathbf{F} = \mathbf{F}(x) = (F_1(x), F_2(x), F_3(x)) \) be a vector field defined in an open set \( \Omega \subset \mathbb{R}^3 \). Here \( x = (x_1, x_2, x_3) \in \Omega \), and we will assume that \( \mathbf{F} \) is sufficiently smooth, so that all derivatives of the components \( F_j \), which we need, are continuous. It is usually enough to assume that \( \mathbf{F} \in C^2(\Omega) \) (which
means that every component $F_j$ is in $C^2(\Omega)$. If $f = f(x)$ is a sufficiently smooth function on $\Omega$, then its gradient
\[
\text{grad } f = \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)
\]
is an example of a vector field in $\Omega$. Here it is convenient to consider $\nabla$ as a vector
\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)
\]
whose components are operators acting, say, in $C^\infty(\Omega)$, or, as operators from $C^{k+1}(\Omega)$ to $C^k(\Omega)$, $k = 0, 1, \ldots$.

The divergence of a vector field $\mathbf{F}$ is a scalar function
\[
\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.
\]

For a vector field $\mathbf{F}$ in $\Omega$ its curl is another vector field
\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left( \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right).
\]

The following identities, which hold for every smooth scalar function $f$ and vector field $\mathbf{F}$, are easy to verify:

(10.3) \quad \text{div grad } f = \nabla \cdot \nabla f = \Delta f,

(10.4) \quad \text{curl grad } f = \nabla \times \nabla f = 0,

(10.5) \quad \text{div curl } \mathbf{F} = 0,

(10.6) \quad \text{curl(curl } \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) = \text{grad div } \mathbf{F} - \Delta \mathbf{F},

where in the last term the Laplacian $\Delta$ is applied to every component of $\mathbf{F}$.

Vector analysis is easier to understand if we use the language of differential forms. To this end, with each vector field $\mathbf{F}$ we associate two differential forms
\[
\lambda_{\mathbf{F}} = F_1 dx_1 + F_2 dx_2 + F_3 dx_3,
\]
and
\[ \omega_{\mathbf{F}} = F_1 \, dx_2 \wedge dx_3 + F_2 \, dx_3 \wedge dx_1 + F_3 \, dx_1 \wedge dx_2. \]

It is easy to see that \( d\lambda_{\mathbf{F}} = \omega_{\text{curl}\, \mathbf{F}} \), which can serve as a definition of \( \text{curl}\, \mathbf{F} \). Also,
\[ d\omega_{\mathbf{F}} = \text{div} \, \mathbf{F} \, dx_1 \wedge dx_2 \wedge dx_3. \]

In particular, the identities (10.4) and (10.5) immediately follow from the well known relation \( d^2 = 0 \) for the de Rham differential \( d \) on differential forms.

In what follows we will deal with functions and vector fields which additionally depend upon the time variable \( t \). In this case all operations \( \text{grad}, \text{div}, \text{curl} \) should be applied with respect to the space variables \( x \) (for every fixed \( t \)). ▲

\[ \nabla \text{ The equations of classical electrodynamics (Maxwell’s equations) are } \]
\[
\begin{align*}
\text{(M.1)} & \quad \text{div} \, \mathbf{E} = \frac{\rho}{\varepsilon_0} \\
\text{(M.2)} & \quad \text{curl} \, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\text{(M.3)} & \quad \text{div} \, \mathbf{B} = 0 \\
\text{(M.4)} & \quad c^2 \, \text{curl} \, \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0} + \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Here \( \mathbf{E}, \mathbf{B} \) are vectors of electric and magnetic fields, respectively (three-dimensional vectors depending on \( t \) and \( x \in \mathbb{R}^3 \)), \( \rho \) a scalar function of \( t \) and \( x \) (density of the electric charge), \( \mathbf{j} \) a 3-dimensional vector of density of the electric current, also depending on \( t \) and \( x \) (if charges with density \( \rho \) at a given point and time move with velocity \( \mathbf{v} \), then \( \mathbf{j} = \rho \mathbf{v} \)). Numbers \( c, \varepsilon_0 \) are universal constants depending on the choice of units: the speed of light and the dielectric permeability of vacuum.

For applications of the Maxwell equations it is necessary to complement them with the formula for the Lorentz force acting on a moving charge. This force is of the form
\[ \text{(M.5)} \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \]

where \( q \) is the charge, \( \mathbf{v} \) its velocity. Formula (M.5) may be used for experimental measurement of \( \mathbf{E} \) and \( \mathbf{B} \) (or, say, to define them as physical quantities).
10. The wave equation

The discussion of experimental facts on which equations (M.1)-(M.5) are based, may be found in textbooks on physics (see e.g. Feynmann, Leighton, Sands [7], vol. 2, Ch. 18).

Let us rewrite the Maxwell equations, introducing scalar and vector potentials \( \varphi \) and \( \mathbf{A} \). For simplicity assume that the fields \( \mathbf{E} \) and \( \mathbf{B} \) are defined and sufficiently smooth in the whole space \( \mathbb{R}^4_{t,x} \). By the Poincaré lemma, it follows from (M.3) that \( \mathbf{B} = \text{curl} \mathbf{A} \), where \( \mathbf{A} \) is a vector function of \( t \) and \( x \) determined up to a field \( \mathbf{A}_0 \) such that \( \text{curl} \mathbf{A}_0 = 0 \), hence \( \mathbf{A}_0 = \text{grad} \psi \), where \( \psi \) is a scalar function. Substituting \( \mathbf{B} = \text{curl} \mathbf{A} \) into (M.2), we get \( \text{curl}(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \). Hence, using the Poincaré lemma again, we see that \( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad} \varphi \), where \( \varphi \) is a scalar function. Therefore, instead of (M.2) and (M.3), we may write

\[
\begin{align*}
  (M.6) \quad & \mathbf{B} = \text{curl} \mathbf{A} \\
  (M.7) \quad & \mathbf{E} = -\text{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t}
\end{align*}
\]

The vector-function \( \mathbf{A} \) is called the vector potential, and the scalar function \( \varphi \) is called the scalar potential. Note that equations (M.6) and (M.7) do not uniquely determine potentials \( \mathbf{A} \) and \( \varphi \). Without affecting \( \mathbf{E} \) and \( \mathbf{B} \), we may replace \( \mathbf{A} \) and \( \varphi \) by

\[
(M.8) \quad \mathbf{A}' = \mathbf{A} + \text{grad} \psi, \quad \varphi' = \varphi - \frac{\partial \psi}{\partial t}
\]

(this transformation of potentials is called a gauge transformation). Transformations (M.8) may be used to get simpler equations for potentials. For instance, using (M.8) we may get any value of \( \text{div} \mathbf{A}' = \text{div} \mathbf{A} + \text{div} \text{grad} \psi = \text{div} \mathbf{A} + \Delta \psi \), since solving Poisson’s equation \( \Delta \psi = \alpha(t,x) \) we may get any value of \( \Delta \psi \), hence, of \( \text{div} \mathbf{A} \).

Let us derive equations for \( \mathbf{A} \) and \( \varphi \) using two remaining Maxwell’s equations. Substituting the expression for \( \mathbf{E} \) in terms of the potentials into (M.1), we get

\[
(M.9) \quad -\Delta \varphi - \frac{\partial}{\partial t} \text{div} \mathbf{A} = \frac{\rho}{\varepsilon_0}.
\]

Substituting \( \mathbf{E} \) and \( \mathbf{B} \) in (M.4), we get

\[
c^2 \text{curl curl} \mathbf{A} - \frac{\partial}{\partial t} \left( -\text{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\mathbf{j}}{\varepsilon_0}.
\]
Using (10.6), we obtain
\begin{equation}
\tag{M.10}
-c^2 \Delta A + c^2 \text{grad} \text{(div} A\text{)} + \frac{\partial}{\partial t} \text{grad} \varphi + \frac{\partial^2 A}{\partial t^2} = \frac{j}{\varepsilon_0}.
\end{equation}

Now, choose \text{div} A with the help of the gauge transformation such that
\begin{equation}
\tag{M.11}
\text{div} A = -\frac{1}{c^2} \frac{\partial \varphi}{\partial t}.
\end{equation}

More precisely, let first some potentials \varphi' and \textbf{A}' be given. We want to find the function \psi in the gauge transformation (M.8) so that \varphi and \textbf{A} satisfy (M.11). This yields for \psi an equation of the form
\begin{equation}
\tag{M.12}
\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi = \theta(t, x),
\end{equation}
where \theta(t, x) is a known function. This is a non-homogeneous wave equation. Suppose that we have solved this equation and, finding \psi, got \varphi and \textbf{A} satisfying (M.11). Then the second and the third terms in (M.10) cancel and we get
\begin{equation}
\tag{M.13}
\frac{\partial^2 \textbf{A}}{\partial t^2} - c^2 \Delta \textbf{A} = \frac{j}{\varepsilon_0},
\end{equation}
and (M.9) takes the form
\begin{equation}
\tag{M.14}
\frac{\partial^2 \varphi}{\partial t^2} - c^2 \Delta \varphi = \frac{c^2 \rho}{\varepsilon_0}.
\end{equation}

Therefore, we may assume that \varphi and \textbf{A} satisfy non-homogeneous wave equations (M.13), (M.14), which, together with (M.11), are equivalent to the Maxwell equations. The gauge condition (M.11) is called the Lorentz gauge condition.

Denote by \Box the wave operator (or d’Alembertian)
\begin{equation}
\tag{M.15}
\Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta
\end{equation}
and let \Box^{-1} be the convolution with a fundamental solution of this operator. Then \Box^{-1} commutes with differentiations. Suppose \Box^{-1} j and \Box^{-1} \rho make sense. Then the potentials \textbf{A} = \Box^{-1} \frac{j}{\varepsilon_0} and \varphi = \Box^{-1} \frac{c^2 \rho}{\varepsilon_0} satisfy (M.13) and (M.14).

Will the Lorentz gauge condition be satisfied for this particular choice of the potentials? Substituting the found \textbf{A} and \varphi into (M.11) we see that this condition is satisfied if and only if
\begin{equation}
\tag{M.16}
\frac{\partial \rho}{\partial t} + \text{div} \; \textbf{j} = 0.
\end{equation}
But this condition means the conservation of charge. Therefore, if the conservation of charge is observed, then a solution of the Maxwell equations may be found if we can solve a non-homogeneous wave equation.

Reducing Maxwell’s equations to the form (M.13), (M.14) shows that Maxwell’s equations are invariant with respect to Lorentz transformations (linear transformations of the space-time $\mathbb{R}^4_{t,x}$ preserving the Minkowski metric $c^2 dt^2 - dx^2 - dy^2 - dz^2$). Moreover, they are invariant under the transformations from the Poincaré group, which is generated by the Lorentz transformations and translations.

In the empty space (in the absence of currents and charges) the potentials $\varphi$ and $A$ satisfy the wave equation of the form (10.1) for $n = 3$ and $\alpha = c$. This implies that all components of $E$ and $B$ satisfy the same equation. ▲

### 10.2. Plane, spherical and cylindric waves

A plane wave is a solution of (10.1) which, for a fixed $t$, is constant on each hyperplane which is parallel to a given one. By a rotation in $x$-space we may make this family of planes to be of the form $x_1 = \text{const}$. Then a plane wave is a solution of (10.1) depending only on $t$ and $x_1$. But then it is a solution of the one-dimensional wave equation $u_{tt} = \alpha^2 u_{x_1x_1}$, i.e., is of the form

$$u(t, x) = f(x_1 - at) + g(x_1 + at),$$

where $f, g$ are arbitrary functions. Before the rotation, each of the summands had clearly been of the form $f(r \cdot x - at)$, where $r \in \mathbb{R}^n$, $|r| = 1$. Another form of expression: $f(k \cdot x - \omega t)$, where $k \in \mathbb{R}^n$, but may have an arbitrary length. Such a form is convenient to make the argument of $f$ dimensionless, and is most often used in physics and mechanics. If the dimension of $t$ is [second] and that of $x$ is [meter], then that of $\omega$ is $[\text{sec}^{-1}]$ and that of $k$ is $[\text{m}^{-1}]$.

Substituting $f(k \cdot x - \omega t)$ to (10.1), we see that $\omega^2 = \alpha^2 |k|^2$ or $\alpha = \frac{\omega}{|k|}$. Clearly, the speed of the plane wave front (the plane where $f$ takes a prescribed value) is equal to $\alpha$ and the equation of such a front is $k \cdot x - \omega t = \text{const}$, where $k$ and $\omega$ satisfy $\omega^2 - \alpha^2 |k|^2 = 0$. An important example of a plane wave is $e^{i(k \cdot x - \omega t)}$. (Up to now we only considered real solutions in this section, and we will do this in the future, but we write this exponent assuming that either real or imaginary part is considered. They will also be solutions of the wave equation.) In this case each fixed point $x$ oscillates
The vector \( k \) is often called \textit{wave vector}. The relation \( \omega^2 = a^2|k|^2 \) is the \textit{dispersion law} for the considered waves (in physics more general dispersion laws of the form \( \omega = \omega(k) \) are encountered). Many solutions of the wave equation may be actually obtained as a superposition of plane waves with different \( k \). We will, however, obtain the most important types of such waves directly.

In what follows, we will assume that \( n = 3 \), and study spherical waves, solutions of (10.1) depending only on \( t \) and \( r \), where \( r = |x| \). Thus, let \( u = u(t,r) \), where \( r = |x| \). Then (10.1) can be rewritten in the form

\[
(10.7) \quad \frac{1}{a^2} u_{tt} = u_{rr} + \frac{2}{r} u_r.
\]

Let us multiply (10.7) by \( r \) and use the identity

\[
r u_{rr} + 2 u_r = \frac{\partial^2}{\partial r^2}(ru).
\]

Then, clearly, (10.7) becomes

\[
\frac{1}{a^2} (ru)_{tt} = (ru)_{rr},
\]

implying \( ru(t,r) = f(r - at) + g(r + at) \) and

\[
(10.8) \quad u(t,r) = \frac{f(r - at)}{r} + \frac{g(r + at)}{r}.
\]

This is the general form of spherical waves. The wave \( r^{-1}f(r - at) \) diverges from the origin \( 0 \in \mathbb{R}^3 \), as from its source, and the wave \( r^{-1}g(r + at) \), conversely, converges to the origin (coming from infinity). In electrodynamics one usually considers the wave coming out of a source only, discarding the second summand in (10.8) by physical considerations. In this case the function \( f(r) \) characterizes properties of the source. The front of the emanating spherical wave is the sphere \( r - at = \text{const} \). It is clear that the speed of the front is \( a \), as before.

Let us pass to \textit{cylindric waves} – solutions of (10.1) depending only on \( t \) and the distance to the \( x_3 \)-axis. A cylindric wave only depends, actually, on \( t, x_1, x_2 \) (and even only on \( t \) and \( \rho = \sqrt{x_1^2 + x_2^2} \)) so that it is a solution of the wave equation (10.1) with \( n = 2 \). It is convenient, however, to consider it as a solution of the wave equation with \( n = 3 \), but independent of \( x_3 \).

One of the methods to construct cylindric waves is as follows: take a superposition of identical spherical waves with sources at all points of the
$x_3$ axis (or converging to all points of the $x_3$ axis). Let $e_3$ be the unit vector of the $x_3$-axis. Then we get a cylindric wave, by setting

$$v(t, x) = \int_{-\infty}^{\infty} f\left(\frac{|x - z e_3|}{|x - z e_3|} - at\right) dz + \int_{-\infty}^{\infty} g\left(\frac{|x - z e_3|}{|x - z e_3|} + at\right) dz. \tag{10.9}$$

Set

$$r = |x - z e_3| = \sqrt{\rho^2 + (x_3 - z)^2}, \quad \rho = \sqrt{x_1^2 + x_2^2}.$$ 

Clearly, $v(t, x)$ does not depend on $x_3$ and only depends on $t$ and $\rho$. Therefore, we may assume that $x_3 = 0$ in (10.9). Then the integrands are even functions of $z$, and it suffices to calculate the integrals over $(0, +\infty)$. For the variable of integration take $r$ instead of $z$, so that

$$dr = \frac{z}{r} dz, \quad dz = \frac{r}{z} dr = \frac{r}{\sqrt{r^2 - \rho^2}} dr.$$

Since $r$ runs from $\rho$ to $\infty$ (for $z \in [0, \infty)$), we get

$$v(t, \rho) = 2 \int_{\rho}^{\infty} \frac{f(r - at)}{\sqrt{r^2 - \rho^2}} dr + 2 \int_{\rho}^{\infty} \frac{g(r + at)}{\sqrt{r^2 - \rho^2}} dr$$

or

$$v(t, \rho) = 2 \int_{\rho - at}^{\infty} \frac{f(\xi) d\xi}{\sqrt{(\xi + at)^2 - \rho^2}} + 2 \int_{\rho + at}^{\infty} \frac{g(\xi) d\xi}{\sqrt{(\xi - at)^2 - \rho^2}}. \tag{10.10}$$

All this makes sense when the integrals converge. For instance, if $f, g$ are continuous functions of $\xi$, then for the convergence it suffices that

$$\int_{M}^{\infty} \frac{|f(\xi)| d\xi}{|\xi|} < +\infty, \quad \int_{M}^{\infty} \frac{|g(\xi)| d\xi}{|\xi|} < +\infty, \quad \text{for } M > 0.$$

It can be shown that (10.10) is the general form of the cylindric waves.

Let us briefly describe another method of constructing cylindric waves. Seeking a cylindric wave of the form $v(t, \rho) = e^{i\omega t} f(\rho)$, we easily see that $f(\rho)$ should satisfy the equation

$$f'' + \frac{1}{\rho} f' + k^2 f = 0, \quad k = \frac{\omega}{a},$$

which reduces to the Bessel equation with $\nu = 0$ (see Section 9). Then we can take a superposition of such waves (they are called monochromatic waves) by integrating over $\omega$. 
10.3. The wave equation as a Hamiltonian system

In Section 2.1 we have already discussed for \( n = 1 \) the possibility of expressing the wave equation as a Lagrange equation of a system with an infinite number of degrees of freedom. Now, let us briefly perform the same for \( n = 3 \), introducing the corresponding quantities by analogy with the one-dimensional case. Besides, we will discuss a possibility to pass to the Hamiltonian formalism. For simplicity, we will only consider real-valued functions \( u(t, x) \), which are smooth functions of \( t \) with values in \( S(\mathbb{R}^3) \), i.e., we will analyze the equation (10.1) in the class of rapidly decaying functions with respect to \( x \). Here temporarily we will denote by \( S(\mathbb{R}^3) \) the class of real-valued functions satisfying the same restrictions as in the definition of \( S(\mathbb{R}^3) \) which was previously formulated. We prefer to do this rather than complicate notations, and we hope that this will not lead to a confusion. All other functions in this section will be also assumed real-valued.

Let us introduce terminology similar to one used in classical mechanics.

The space \( M = S(\mathbb{R}^3) \) will be referred to as the configuration space. An element \( u = u(x) \in S(\mathbb{R}^3) \) may be considered as a “set of coordinates” of the three-dimensional system, where \( x \) is the “label” of the coordinate and \( u(x) \) is the coordinate itself with the index \( x \). The tangent bundle on \( M \) is the direct product \( TM = S(\mathbb{R}^3) \times S(\mathbb{R}^3) \). As usual, the set of pairs \( \{(u_0, v) \in TM \} \) for a fixed \( u_0 \in M \) is denoted by \( TM_{u_0} \) and called the tangent space at \( u_0 \) (this space is canonically isomorphic to \( M \) as in the case when \( M \) is a finite dimensional vector space). Elements of \( TM \) are called tangent vectors.

A path in \( M \) is a function \( u(t, x) \), \( t \in (a, b) \), infinitely differentiable with respect to \( t \) with values in \( S(\mathbb{R}^3) \). The velocity of \( u(t, x) \) at \( t_0 \) is the tangent vector \( \{u(t_0, x), \dot{u}(t_0, x)\} \), where \( \dot{u} = \frac{\partial u}{\partial t} \).

The kinetic energy is the function \( K \) on the tangent bundle introduced by the formula

\[
K(\{u, v\}) = \frac{1}{2} \int_{\mathbb{R}^3} v^2(x) dx.
\]

Given a path \( u(t, x) \) in \( M \) then, taking for each \( t \) a tangent vector \( \{u, \dot{u}\} \) and the value of the kinetic energy at it, we get the function of time

\[
K(u) = \frac{1}{2} \int_{\mathbb{R}^3} [\dot{u}(t, x)]^2 dx,
\]

which we will call the kinetic energy along the path \( u \).
The potential energy is the following function on $M$:
\[ U(u) = \frac{a^2}{2} \int_{\mathbb{R}^3} |u_x(x)|^2 dx, \]
where $u_x = \text{grad } u(x) = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right)$.

If there is a path in $M$, then the potential energy along this path is a function of time. With the help of the canonical projection $TM \rightarrow M$, the function $U$ is lifted to a function on $TM$, again denoted by $U$. Therefore, $U(\{u,v\}) = U(u)$.

The Lagrangian or the Lagrange function is the function $L = K - U$ on $TM$. The action along the path $u(t,x)$, $t \in [t_0, t_1]$, is the integral along this path:
\[ S = S[u] = \int_{t_0}^{t_1} L dt, \]
where
\[ L = L(\{u(t,x), \dot{u}(t,x)\}) = \frac{1}{2} \int_{\mathbb{R}^3} [\dot{u}(t,x)]^2 dx - \frac{a^2}{2} \int_{\mathbb{R}^3} |u_x(x)|^2 dx. \]

The wave equation (10.1) may be expressed in the form $\delta S = 0$, where $\delta S$ is the variation (or the differential) of the functional $S$ taken along paths $u(t,x)$ with fixed source $u(t_0,x)$ and the target $u(t_1,x)$. Indeed, if $\delta u(t,x)$ is an admissible variation of the path, i.e., a smooth function of $t \in [t_0, t_1]$ with values in $S(\mathbb{R}^3)$, such that $\delta u(t_0,x) = \delta u(t_1,x) = 0$, then for such a variation of the path, $\delta S$ takes the form
\[ \delta S = \frac{d}{d\varepsilon} S[u + \varepsilon(\delta u)]|_{\varepsilon=0} = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \dot{u}(\delta u) dx dt - \frac{a^2}{2} \int_{t_0}^{t_1} \int_{\mathbb{R}^3} u_x \cdot \delta u_x dx dt \]
\[ = - \int \int \frac{\partial^2 u}{\partial t^2} \delta u dx dt + a^2 \int \int (\Delta u) \delta u dx dt \]
\[ = \int \int (-\Box u) \delta u dx dt, \]
where $\Box = \frac{\partial^2}{\partial t^2} - a^2 \Delta_x$ is the d’Alembertian.

It is clear now that $\delta S = 0$ is equivalent to $\Box u = 0$, i.e., to the wave equation (10.1).

To pass to the Hamiltonian formalism, we should now introduce the cotangent bundle $T^*M$. But in our case each $TM_{u_0}$ is endowed with the inner product determined by the quadratic form $2K$ which makes it natural to set $T^*M_{u_0} = TM_{u_0}$ and $T^*M = TM$.

The Hamiltonian or energy is the following function on $TM$:
\[ H = H(\{u,v\}) = K + U = \frac{1}{2} \int_{\mathbb{R}^3} v^2(x) dx + \frac{a^2}{2} \int_{\mathbb{R}^3} |u_x(x)|^2 dx. \]
10.3. The wave equation as a Hamiltonian system

Given a path \( u(t, x) \), we see that \( H(\{u, \dot{u}\}) \) is a function of time along this path.

**Proposition 10.1.** (The energy conservation law) *Energy is constant along any path \( u(t, x) \) satisfying \( \Box u = 0 \).*

**Proof.** Along the path \( u(t, x) \), we have

\[
\frac{dH}{dt} = \int_{\mathbb{R}^3} \dot{u} \dddot{u} dx + a^2 \int_{\mathbb{R}^3} u_x \cdot \dddot{u}_x dx = \int_{\mathbb{R}^3} \dddot{u} \dddot{u} dx - a^2 \int_{\mathbb{R}^3} \Delta u \cdot \dddot{u} dx = \int_{\mathbb{R}^3} (\Box u) \cdot \dddot{u} dx = 0,
\]
as required. \( \square \)

**Corollary 10.2.** For any \( \varphi, \psi \in S(\mathbb{R}^3) \) there exists at most one path \( u(t, x) \) satisfying \( \Box u = 0 \) and initial conditions

\[ u(t_0, x) = \varphi(x), \quad \dot{u}(t_0, x) = \psi(x). \]

**Proof.** It suffices to verify that if \( \varphi \equiv \psi \equiv 0 \), then \( u \equiv 0 \). But this follows immediately from \( H = \text{const} = 0 \) along this path. \( \square \)

The Hamiltonian expression of equations uses also the symplectic form on \( T^*M \). To explain and motivate further calculations, let us start with a toy model case of a particle which moves along the line \( \mathbb{R} \) in a potential field given by a potential energy \( U = U(x) \) (see Appendix to Chapter 2). Here \( M = \mathbb{R}, \ T M = \mathbb{R} \times \mathbb{R}, \ T^* M = \mathbb{R} \times \mathbb{R} \), the equation of motion is \( m \ddot{x} = -U'(x) \), and the energy is \( H(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + U(x) \). Introducing the momentum \( p = m \dot{x} \) and using \( q \) instead of \( x \) (as physicists do) we can rewrite the Hamiltonian in the form

\[
H(q, p) = \frac{p^2}{2m} + U(q)
\]
and then the equation of motion is equivalent to a *Hamiltonian system*

\[
\begin{cases}
\dot{q} = \frac{\partial H}{\partial p} \\
\dot{p} = -\frac{\partial H}{\partial q}.
\end{cases}
\]

Let us agree that when we talk about a point in coordinates \( q, p \), then it is understood that we mean a point in \( T^*M \), and we will simply write \( \{q, p\} \in T^* M \).

Another important object (in this toy model) is the symplectic 2-form \( \omega = dp \wedge dq \) on \( T^* M \). It can be also considered as a skew-symmetric form on each tangent space \( T_{q,p}(T^* M) \), which in this case can identified with the
set of 4-tuples \( \{q,p,X_q,X_p\} \), where all entries are reals, and \( \{X_q,X_p\} \) are the natural coordinates along the fiber of \( T(T^*M) \) over the point \( \{q,p\} \):

\[
\omega(\{q,p,X_q,X_p\},\{q,p,Y_q,Y_p\}) = X_p Y_q - X_q Y_p.
\]

Now with any vector field \( X \) on \( T^*M \) we can associate a 1-form \( \alpha_X \) on \( T^*M \) given by the relation

\[
\alpha_X(Y) = \omega(Y,X).
\]

Note that the form \( \omega \) is non-degenerate in the following sense: if \( X \) is a tangent vector to \( T^*M \) at \( z \in T^*M \), such that \( \omega(X,Y) = 0 \) for all \( Y \in T_z(T^*M) \), then \( X = 0 \). Therefore, the vector field \( X \) is uniquely defined by its 1-form \( \alpha_X \). So we will denote by \( X_\alpha \) the vector field corresponding to any 1-form \( \alpha \) on \( T^*M \), i.e., the relations \( \alpha = \alpha_X \) and \( X = X_\alpha \) are equivalent.

Let us try to find \( X = X_{dH} \) where \( H \) is a smooth function on \( T^*M \) (a Hamiltonian). By definition we should have for any vector field \( Y \) on \( T^*M \)

\[
\frac{\partial H}{\partial q} Y_q + \frac{\partial H}{\partial p} Y_p = X_q Y_p - X_p Y_q,
\]

hence

\[
X_\alpha(Y) = \omega(Y,X).
\]

Note that this argument works for any smooth Hamiltonian \( H \), not only for the specific one given by (10.12).

The arguments and calculations above can be easily extended to the case of motion in \( M = \mathbb{R}^n \). In this case we should allow \( x, p, q \) to be vectors in \( \mathbb{R}^n \), \( x = (x_1, \ldots, x_n), p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \), the potential energy \( U = U(x) \) be a smooth scalar (real-valued) function on \( \mathbb{R}^n \), then take the Hamiltonian

\[
H(q,p) = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + U(q),
\]

where \( q = x \). Here \( H(q,p) \) should be considered as a function on \( T^*M = \mathbb{R}^{2n} \). Now the equations of motion

\[
m_j \ddot{x}_j = - \frac{\partial U}{\partial x_j}, \quad j = 1, \ldots, n,
\]

again can be rewritten as a Hamiltonian system (10.13), where \( \partial H/\partial p \) and \( \partial H/\partial q \) should be understood as gradients with respect to variables \( p \) and \( q \).
respectively, for example,

\[ \frac{\partial H}{\partial p} = \left( \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n} \right). \]

Instead of the form \( dp \wedge dq \) we should take the 2-form

\[ (10.16) \quad \omega = \sum_{j=1}^{n} dp_j \wedge dq_j. \]

which is usually called \textit{canonical symplectic form} on \( T^*M \). It is non-degenerate in the sense explained above. Therefore, (10.14) again defines a one-one correspondence between vector-fields and 1-forms on \( T^*M \). Then again it is easy to check that the 1-form \( dH \) corresponds to the vector field which generates the Hamiltonian system (10.13).

The reader may wonder why do we need fancy notations like \( M, T^*M \) for usual vector spaces \( \mathbb{R}^n, \mathbb{R}^{2n} \). The point is that in fact \( M \) can be an arbitrary smooth manifold. In fact, choosing local coordinates \( x = (x_1, \ldots, x_n) \) in an open set \( G \subset M \), we can define natural coordinates \( q, p \) in the cotangent bundle \( T^*M \), so that \( q, p \) represents the cotangent vector \( pdq = p_1dq_1 + \cdots + p_ndq_n \) at \( q = x \). Then the 1-form \( pdq \) is well defined on \( M \), i.e. does not depend on the choice of local coordinates. (We leave the proof to the reader as an easy exercise.) Therefore, the 2-form \( \omega = d(pdq) \) is also well defined on \( M \).

In even more general context, a 2-form \( \omega \) on a manifold \( Z \) is called \textit{symplectic} if it is closed and non-degenerate. Then the same machinery works. So, starting with a Hamiltonian \( H \) (a smooth function on \( Z \)), we can take the 1-form \( dH \) and the corresponding vector field \( X_H \). The system of ODE

\[ (10.17) \quad \dot{z} = X_H(z) \]

is called a \textit{Hamiltonian system with the Hamiltonian} \( H \). Locally, it is nothing new: by the Darboux theorem we can choose local coordinates so that \( \omega \) is given by (10.16) in these coordinates. Therefore, the Hamiltonian system (10.17) has the canonical form (10.13). More details on symplectic structure, Hamiltonian systems and related topics can be found in [3].

Now let us make a leap to infinite-dimensional spaces, namely, return to \( M = S(\mathbb{R}^2) \).

The symplectic form should be defined on each tangent space to \( T^*M \) at a fixed point. In our case this tangent space is naturally identified with
$T^*M = TM$ itself, so we have $T(T^*M) = T^*M \times T^*M = TM \times TM$. An element of $T(T^*M)$ should be expressed as a four-tuple $\{u, v, \delta u, \delta v\}$ consisting of functions belonging to $S(\mathbb{R}^n)$. The symplectic form itself depends only on $\delta u, \delta v$. For brevity let us set $\delta u = \alpha$, $\delta v = \beta$. Then define the symplectic form to be

$$[(u, v, \alpha, \beta), (u, v, \alpha_1, \beta_1)] = [\{\alpha, \beta\}, \{\alpha_1, \beta_1\}] = \int_{\mathbb{R}^3} (\alpha \beta_1 - \beta \alpha_1) dx.$$ 

(This is an analogue of the canonical 2-form $\omega = \sum_{j=1}^n dp_j \wedge dq_j$, where summation is replaced by integration, though we use brackets instead of $\omega$ to simplify notations.)

Choosing an arbitrary $\{u, v\} \in T^*M$, let us take variation of $H = H(\{u, v\})$ (which replaces the differential in the infinite-dimensional case) in the direction of a vector $\{u, v, \delta u, \delta v\} \in T(T^*M)$. In our case $u, v, \delta u, \delta v$ are arbitrary functions from $S(\mathbb{R}^3)$. We obtain, differentiating the expression of $H$ in (10.11) and integrating by parts,

$$\delta H(\delta u, \delta v) = \int_{\mathbb{R}^3} \left( a^2 \frac{\delta H}{\delta u_x} \cdot \delta u_x + v \delta v \right) dx = \int_{\mathbb{R}^3} (-a^2 \Delta u \delta u + v \delta v) dx = \int_{\mathbb{R}^3} (\alpha \delta u + \beta \delta v) dx,$$

where $\alpha = -a^2 \Delta u$ and $\beta = v$. Now, if we take another vector $Y = \{u, v, \alpha_1, \beta_1\} \in T(T^*M)$ (with the same $u, v$), and denote by $X_H$ Hamiltonian vector field with the Hamiltonian $H$, then we should have $[Y, X_H] = \delta H(Y)$, which gives

$$\int_{\mathbb{R}^3} (\alpha_1 \beta - \beta_1 \alpha) dx = \int_{\mathbb{R}^3} (\alpha_1 v + \beta_1 a^2 \Delta u) dx = \int_{\mathbb{R}^3} (\alpha_1 X_\alpha + \beta_1 X_\beta) dx,$$

where $\{u, v, X_\alpha, X_\beta\}$ is the vector of the Hamiltonian vector field on $T^*M$. Since $\alpha_1$ and $\beta_1$ are arbitrary, we should have $X_\alpha = v$, $X_\beta = a^2 \Delta u$, and the corresponding differential equation on $T^*M$ has the form

$$\begin{cases}
\dot{u} = v \\
\dot{v} = a^2 \Delta u,
\end{cases}$$

which is equivalent to the wave equation (10.1) for $u$. So we established that the wave equation can be rewritten as a Hamiltonian system with the Hamiltonian (10.11) and the symplectic form given by (10.18).

Later we will use the fact that the flow of our Hamiltonian system preserves the symplectic form (or, as one says, the transformations in this flow are canonical). This is true for general Hamiltonian systems (see e.g. [3]) but
we will not prove this. Instead we will directly establish the corresponding analytical fact for the wave equation.

**Proposition 10.3.** Let \(u(t,x)\) and \(v(t,x)\) be two paths. Consider the following function of \(t\):

\[
[u,v] = \int_{\mathbb{R}^3} (\dot{u} \hat{v} - \dot{v} \hat{u}) dx.
\]

If \(\Box u = \Box v = 0\), then \([u,v] = \text{const}\).

**Proof.** We have

\[
\frac{d}{dt} [u,v] = \int_{\mathbb{R}^3} \frac{d}{dt} (\dot{u} \hat{v} - \dot{v} \hat{u}) dx = \int_{\mathbb{R}^3} (\ddot{u} \hat{v} - \ddot{v} \hat{u}) dx = a^2 \int_{\mathbb{R}^3} (u \Delta v - \Delta u \cdot v) dx = 0,
\]

as required. \(\square\)

### 10.4. A spherical wave caused by an instant flash and a solution of the Cauchy problem for the 3-dimensional wave equation

Let us return to spherical waves and consider the diverging wave

(10.19) \(f(r-at)/r, \quad r = |x|\).

The function \(f(-at)\) characterizes the intensity of the source (located at \(x = 0\)) at the moment \(t\). It is interesting to take the wave emitted by an instant flash at the source. This corresponds to \(f(\xi) = \delta(\xi)\). We get the wave

(10.20) \(\delta(r-at)/r = \delta(r-at)/(at), \quad r = |x|\),

whose meaning is to be clarified.

It is possible to make sense to (10.20) in many equivalent ways. We will consider it as a distribution with respect to \(x\) depending on \(t\) as a parameter.

First, consider the wave (10.19) and set \(f = f_k\), where \(f_k(\xi) \in C^\infty_0(\mathbb{R}^1)\), \(f_k(\xi) \geq 0, f_k(\xi) = 0\) for \(|\xi| \geq 1/k\) and \(\int f_k(\xi) d\xi = 1\), so that \(f_k(\xi) \rightarrow \delta(\xi)\) as \(k \rightarrow +\infty\). Now, set

(10.21) \(\delta(r-at) = \lim_{k \rightarrow \infty} f_k(r-at)\),

if this limit exists in \(\mathcal{D}'(\mathbb{R}^3)\) (or, which is the same, in \(\mathcal{E}'(\mathbb{R}^3)\), since supports of all functions \(f_k(r-at)\) belong to a fixed compact provided \(t\) belongs to a fixed finite interval of \(\mathbb{R}\)). Clearly, this limit exists for \(t < 0\) and is 0, i.e.,
The wave equation

\[ \delta(r - at) = 0 \] for \( t < 0 \). We will see that the limit exists also for \( t > 0 \). Then this limit is the distribution \( \delta(r - at) \in \mathcal{E}'(\mathbb{R}^3) \) and, obviously,

\[ \text{supp} \, \delta(r - at) = \{ x : |x| = at \}. \]

Since \( 1/r = 1/|x| \) is a \( C^\infty \)-function in a neighborhood of \( \text{supp} \, \delta(r - at) \) for \( t \neq 0 \), then the distribution \( \frac{\delta(r - at)}{r} \) is well-defined and

\[ \frac{\delta(r - at)}{r} = \lim_{k \to \infty} \frac{f_k(r - at)}{r}, \quad t \neq 0. \]

Let us prove the existence of the limit (10.21) for \( t > 0 \) and compute it. Let \( \varphi \in \mathcal{D}(\mathbb{R}^3) \). Let us express the integral \( \langle f_k(r - at), \varphi \rangle \) in polar coordinates

\[ \langle f_k(r - at), \varphi \rangle = \int_{\mathbb{R}^3} f_k(|x| - at) \varphi(x) dx = \]

\[ = \int_0^\infty \left( \int_{|x|=r} f_k(r - at) \varphi(x) dS_r \right) dr = \]

\[ = \int_0^\infty f_k(r - at) \left( \int_{|x|=r} \varphi(x) dS_r \right) dr, \]

where \( dS_r \) is the area element of the sphere of radius \( r \). Clearly, \( \int_{|x|=r} \varphi(x) dS_r \) is an infinitely differentiable function of \( r \) for \( r > 0 \). Since \( \lim_{k \to \infty} f_k(\xi) = \delta(\xi) \), we obviously get

\[ \lim_{k \to \infty} \langle f_k(r - at), \varphi \rangle = \int_{|x|=at} \varphi(x) dS_{at} \]

and, therefore, the limit of (10.21) exists and

\[ \langle \delta(r - at), \varphi \rangle = \int_{|x|=at} \varphi(x) dS_{at}. \]

Passing to the integration over the unit sphere we get

\[ \langle \delta(r - at), \varphi \rangle = a^2t^2 \int_{|x'|=1} \varphi((at)x') dS_{1}, \]

implying

\[ \left\langle \frac{\delta(r - at)}{r}, \varphi \right\rangle = at \int_{|x'|=1} \varphi((at)x') dS_{1}. \]

It is worthwhile to study the dependence on the parameter \( t \). It is clear from (10.23) that

\[ \lim_{t \to 0} \frac{\delta(r - at)}{r} = 0 \]
and, by continuity, we set $\frac{\delta(r-at)}{r}|_{t=0} = 0$. Finally, it is obvious from (10.23) that $\frac{\delta(r-at)}{r}$ is infinitely differentiable with respect to $t$ for $t > 0$ (in the weak topology). Derivatives with respect to $t$ are easy to compute. For instance,

$$
\langle \frac{\partial}{\partial t} \frac{\delta(r-at)}{r}, \varphi \rangle = \frac{d}{dt} \langle \frac{\delta(r-at)}{r}, \varphi \rangle = a \int_{|x'| = 1} \varphi((at)x')dS_1 + a^2 t \sum_{j=1}^{3} \int_{|x'| = 1} x_j \frac{\partial \varphi}{\partial x_j}((at)x')dS_1.
$$

(10.25)

This, in particular, implies that

$$
\lim_{t \to +0} \frac{\partial}{\partial t} \frac{\delta(r-at)}{r} = 4\pi a \delta(x).
$$

(the first of integrals in (10.25) tends to $4\pi a \varphi(0)$ and the second one to $0$).

Finally, by (10.22) it is clear that

$$
\Box \frac{\delta(r-at)}{r} = 0, \ t > 0.
$$

Therefore, the distribution $\frac{\delta(r-at)}{r}$ is a solution of the wave equation for $t > 0$ with the initial conditions

$$
\frac{\delta(r-at)}{r} \bigg|_{t=0} = 0, \ \frac{\partial}{\partial t} \frac{\delta(r-at)}{r} \bigg|_{t=0} = 4\pi a \delta(x).
$$

(10.26)

The converging wave $\frac{\delta(r+at)}{r}$ is similarly constructed. It is equal to $0$ for $t > 0$ and for $t < 0$ satisfies the wave equation with the initial conditions

$$
\frac{\delta(r+at)}{r} \bigg|_{t=0} = 0, \ \frac{\partial}{\partial t} \frac{\delta(r+at)}{r} \bigg|_{t=0} = -4\pi a \delta(x),
$$

(10.27)

(this is clear, for instance, because $\frac{\delta(r+at)}{r}$ is obtained from $\frac{\delta(r-at)}{r}$ by replacing $t$ with $-t$). Besides, we can perform translations with respect to the spatial and time variables and consider waves

$$
\frac{\delta(|x-x_0|-a(t-t_0))}{|x-x_0|}, \ \frac{\delta(|x-x_0|+a(t-t_0))}{|x-x_0|},
$$

which are also solutions of the wave equation with initial values (for $t = t_0$) obtained by replacing $\delta(x)$ with $\delta(x-x_0)$ in (10.26) and (10.27).

Now, we want to apply the invariance of the symplectic form (Proposition 10.3) to an arbitrary solution $u(t, x)$ of the equation $\Box u = 0$ and to the converging wave

$$
v(t, x) = \frac{\delta(|x-x_0|+a(t-t_0))}{|x-x_0|}.
$$
We will assume that $u \in C^1([0, t_0] \times \mathbb{R}^3)$ and $\square u = 0$ holds (where $\square$ is understood in the distributional sense). Then we can define

$$[u, v] = \int_{\mathbb{R}^3} (\dot{u} \psi - \dot{v} \psi) dx = \left\langle \frac{\partial}{\partial t} v(t, x), u(t, x) \right\rangle - \left\langle v(t, x), \frac{\partial}{\partial t} u(t, x) \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the value of the distribution of the variable $x$ at a test function ($t$ is considered as a parameter) which makes sense for $u \in C^1$ (not only for $u \in C^2_0$) due to explicit formulas (10.23), (10.25). Now, we use $[u, v] = \text{const}$, which holds for $u \in C^2$ and is proved similar to Proposition 10.3 or by approximating smooth solutions of the wave equation with compact (with respect to $x$) supports (this can be performed, for instance, with the help of mollifying). Actually, we only have to verify that $\frac{d}{dt}[u, v] = [\dot{u}, v] + [u, \dot{v}]$, which is clear, for example, from explicit formulas (10.23), (10.25) which define the functionals $v$ and $\dot{v}$.

Now, let us explicitly write the relation

(10.28) $[u, v]|_{t=0} = [u, v]|_{t=t_0}$.

Set $u|_{t=0} = \varphi(x)$, $\frac{\partial u}{\partial t}|_{t=0} = \psi(x)$. Then

$$[u, v]|_{t=0} = \left. \frac{\partial}{\partial t} \left\langle \frac{\delta(|x-x_0| + a(t-t_0))}{|x-x_0|}, \varphi(x) \right\rangle \right|_{t=0} - \left. \left\langle \frac{\delta(|x-x_0| + a(t-t_0))}{|x-x_0|}, \psi(x) \right\rangle \right|_{t=0}
= - \left. \frac{\partial}{\partial t_0} \left\langle \frac{\delta(|x-x_0| - at_0)}{|x-x_0|}, \varphi(x) \right\rangle - \left\langle \frac{\delta(|x-x_0| - at_0)}{|x-x_0|}, \psi(x) \right\rangle \right|_{t=0}
= - \frac{1}{at_0} \int_{|x-x_0|=at_0} \psi(x) dS_{at_0} - \frac{\partial}{\partial t_0} \left( \frac{1}{at_0} \int_{|x-x_0|=at_0} \varphi(x) dS_{at_0} \right).$$

Further, by (10.27), we have

$$[u, v]|_{t=t_0} = -4\pi a (\delta(x-x_0), u(t_0, x)) = -4\pi a u(t_0, x_0).$$

Therefore, (10.28) can be expressed in the form

$$-4\pi a u(t_0, x_0) = - \frac{1}{at_0} \int_{|x-x_0|=at_0} \psi(x) dS_{at_0} - \frac{\partial}{\partial t_0} \left( \frac{1}{at_0} \int_{|x-x_0|=at_0} \varphi(x) dS_{at_0} \right).$$

Replacing $t_0, x_0, x$ by $t, x, y$ we get

(10.29) $u(t, x) = \frac{\partial}{\partial t} \left( \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \varphi(y) dS_{at} \right) + \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \psi(y) dS_{at}$,

where $dS_{at}$ is the area element of the sphere $\{y : |y-x| = at\}$.
We obtained the formula for the solution of the Cauchy problem

\[(10.30) \quad \Box u = 0, \quad u|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \psi(x).\]

Formula (10.29) is called the Kirchhoff formula. Let us derive some corollaries of it.

1) The solution of the Cauchy problem is unique and continuously depends on initial data in appropriate topology (e.g. if $\psi$ continuously varies in $C(\mathbb{R}^3)$ and $\varphi$ continuously varies in $C^1(\mathbb{R}^3)$, then $u(t, x)$ continuously varies in $C(\mathbb{R}^3)$ for each $t > 0$).

2) The value $u(t, x)$ only depends on the initial data near the sphere $\{y : |y - x| = at\}$.

Suppose that for $t = 0$ the initial value differs from 0 in a small neighborhood of one point $x_0$ only. But then at $t$ the perturbation differs from 0 in a small neighborhood of the sphere $\{x : |x - x_0| = at\}$ only. This means that a perturbation spreads at a speed $a$ and vanishes after a short time at each observation point. Therefore, the spreading wave has a leading edge and a trailing one.

A strictly localized initial state is observed later at another point as a phenomenon, which is also strictly localized. This phenomenon is called the Huygens principle.

Since Huygens’ principle holds for the wave equation for $n = 3$, we may transmit information with the help of sound or light.

As we will see later, the Huygens principle does not hold for $n = 2$ (once started, oscillations never cease, e.g. waves on the water). Therefore, the residents of the Flatland (the flat 2-dimensional world described by E.A.Abbott in 1884) would find it very difficult to pass information through their space (the plane).

We have not yet proved the existence of the solution for the Cauchy problem (10.30). The simplest way to do this is to take the function $u(t, x)$ constructed by the Kirchhoff formula (10.29) and verify that it is a solution of (10.30). The fulfillment of $\Box u = 0$ may be deduced from the fact that the right-hand side of (10.29) is the sum of two convolutions $\varphi \ast \frac{\partial}{\partial t} \frac{1}{4\pi} \frac{\delta(r-at)}{r}$ and $\psi \ast \frac{1}{4\pi} \frac{\delta(r-at)}{r}$ (taken with respect to $x \in \mathbb{R}^3$ with $t$ as a parameter) of $\varphi$ and $\psi$ with distributions of $x$ depending smoothly on $t$ and satisfying in a natural sense the wave equation (for $t > 0$).
We will use more convenient arguments. In order not to bother with distributions depending on a parameter, it is convenient to consider \( \frac{\delta(r-at)}{r} \) as a distribution on \( \mathbb{R}^4_{t,x} \) by setting
\[
\langle \delta(r-at), \varphi(t,x) \rangle_{t,x} = \int_0^\infty \frac{1}{at} dt \int_{|x|=at} \varphi(t,x) dS_{at}
\]
This formula implies in particular, that the functional on \( \mathcal{D}(\mathbb{R}^4) \) determined by the left hand side is a distribution which we will again denote by \( \frac{\delta(r-at)}{r} \). Therefore, we may assume that \( \frac{\delta(r-at)}{r} \in \mathcal{D}'(\mathbb{R}^4) \).

Clearly, \( \text{supp} \frac{\delta(r-at)}{r} = K^+ \), where \( K^+ \) is the upper sheet of the light cone \( \{ (t,x) : |x|^2 - a^2t^2 = 0 \} \); namely,
\[
K^+ = \{ (t,x) : |x| = at \} = \{ (t,x) : |x|^2 - a^2t^2 = 0, \ t \geq 0 \}.
\]
It is also easy to verify that \( \Box \frac{\delta(r-at)}{r} = 0 \) for \( |t| + |x| \neq 0 \). Indeed, \( \frac{\delta(r-at)}{r} = 0 \) outside \( K^+ \); therefore, it suffices to check that \( \Box \frac{\delta(r-at)}{r} = 0 \) for \( t > 0 \). But this is clear, for example, from (10.22) which is true also when the limit is understood in \( \mathcal{D}'(\{ (t,x) : t > \varepsilon > 0 \}) \) for any \( \varepsilon > 0 \).

It is easy to prove that the right-hand side of (10.29) can be written in the form
\[
(10.31) \quad u = [\psi(x) \otimes \delta(t)] * \frac{1}{4\pi a^2 t} \frac{\delta(r-at)}{r} + [\varphi(x) \otimes \delta(t)] * \frac{\partial}{\partial t} \frac{1}{4\pi a^2 t} \frac{\delta(r-at)}{r}
\]
and then \( \Box u = 0 \) obviously holds due to the properties of convolutions.

The initial conditions in (10.30) may be also deduced from (10.31); we will, however, deduce them directly from the structure of formula (10.29). Note that Kirchhoff’s formula is of the form
\[
u = u_\psi + \frac{\partial}{\partial t} u_\varphi,
\]
where
\[
u_\psi(t,x) = \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \psi(y) dS_{at},
\]
and \( u_\varphi \) is a similar expression obtained by replacing \( \psi \) by \( \varphi \). This structure of Kirchhoff’s formula is not accidental. Indeed, suppose we have proved that \( u_\psi \) is a solution of the Cauchy problem
\[
(10.32) \quad \Box u_\psi = 0, \quad u_\psi|_{t=0} = 0, \quad \frac{\partial u_\psi}{\partial t} \bigg|_{t=0} = \psi.
\]
Let us prove then that \( v_\varphi = \frac{\partial}{\partial t}u_\varphi \) is a solution of the Cauchy problem

\[
\Box v_\varphi = 0, \quad v_\varphi|_{t=0} = \varphi, \quad \frac{\partial v_\varphi}{\partial t}|_{t=0} = 0.
\]

The equation \( \Box v_\varphi = 0 \) obviously holds and so does \( v_\varphi|_{t=0} = \varphi \) due to the second equation in (10.32). It remains to verify the second equality in (10.33). Assuming that \( u_\varphi \in C^2 \) for \( t \geq 0 \), we get

\[
\frac{\partial v_\varphi}{\partial t}|_{t=0} = \frac{\partial^2 u_\varphi}{\partial t^2}|_{t=0} = a^2 \Delta u_\varphi|_{t=0} = a^2 \Delta (u_\varphi|_{t=0}) = 0,
\]

as required. The condition \( u_\varphi \in C^2(t \geq 0) \) holds, for example, for \( \varphi \in C^2(\mathbb{R}^3) \), as is clear for example from the expression

\[
u_\varphi(t, x) = \frac{t}{4\pi} \int_{|y'|=1} \psi(x + aty')dS_1.
\]

This expression makes (10.32) obvious for \( \psi \in C^1(\mathbb{R}^3) \) also.

Thus, if \( \psi \in C^1(\mathbb{R}^3), \varphi \in C^2(\mathbb{R}^3) \) then Kirchhoff’s formula (10.29) gives a solution of the Cauchy problem (10.30) (the equation \( \Box u = 0 \) holds in the distributional sense for \( t > 0 \)). If we additionally assume that \( \psi \in C^2(\mathbb{R}^3) \) and \( \varphi \in C^3(\mathbb{R}^3) \), then \( u \in C^2(\mathbb{R}^3) \) and \( \Box u = 0 \) holds in the classical sense. Thus the Cauchy problem (10.30) is uniquely solvable. The Kirchhoff’s formula makes it also clear that it is well-posed.

10.5. The fundamental solution for the three-dimensional wave operator and a solution of the non-homogeneous wave equation

Set

\[
\mathcal{E}_3(t, x) = \frac{1}{4\pi a} \delta(|x| - at) \in \mathcal{D}'(\mathbb{R}^4).
\]

**Theorem 10.4.** The distribution \( \mathcal{E}_3(t, x) \) satisfies \( \Box \mathcal{E}_3(t, x) = \delta(t, x) \), i.e., it is a fundamental solution for the d’Alembert operator.

**Proof.** The statement actually follows from the fact that

\[
\Box \mathcal{E}_3(t, x) = 0 \quad \text{for } |t| + |x| \neq 0, \quad \mathcal{E}_3(t, x) = 0 \quad \text{for } t < 0,
\]

\[
\mathcal{E}_3(+0, x) = 0, \quad \frac{\partial \mathcal{E}_3}{\partial t}(+0, x) = \delta(x).
\]

The function \( \mathcal{E}_3(t, x) \) may be considered as a smooth function of \( t \) (for \( t \neq 0 \)) with values in \( \mathcal{D}'(\mathbb{R}^3) \) and for \( t = 0 \) it is continuous while \( \frac{\partial \mathcal{E}_3}{\partial t} \) has a jump equal to \( \delta(x) \). Therefore, applying \( \Box = \frac{\partial^2}{\partial t^2} - a^2 \Delta_x \) to \( \mathcal{E}_3 \), we get \( \delta(t) \otimes \delta(x) = \delta(t, x) \).
10. The wave equation

It is difficult to make the above arguments entirely rigorous (if we try to do this, we face the necessity of a cumbersome struggle with the topology in $\mathcal{D}'(\mathbb{R}^3)$). We may, however, consider all the above as a heuristic hint and directly verify that $\Box \mathcal{E}_3 = \delta$. This can be done similarly to the proof for the heat equation (see the proof of Theorem 7.5) and it is left to the reader as an exercise. □

Now, using $\mathcal{E}_3(t, x)$ we can solve the non-homogeneous wave equation $\Box u = f$ by writing

(10.34) \[ u = \mathcal{E}_3 * f, \]

if the convolution in the right-hand side makes sense. The convolution (10.34) is called the \textit{retarded potential}. In a more detailed expression, the retarded potential is of the form

(10.35) \[
\begin{align*}
  u(t, x) &= \frac{1}{4\pi a} \int_0^\infty d\tau \int_{|y|=a\tau} f(t-\tau, x-y) \frac{dS_{a\tau}}{|y|} \\
  &= \frac{1}{4\pi a^2} \int_0^\infty \frac{d\tau}{\tau} \int_{|y|=a\tau} f(t-\tau, x-y) dS_{a\tau}
\end{align*}
\]

(here the integration in the second integrals is done over $y$).

With the help of $\mathcal{E}_3(t, x)$, we may obtain the Kirchhoff formula (10.29) for the solution of the Cauchy problem (10.30) in a different way. Namely, let a solution $u(t, x)$ of the Cauchy problem (10.30) be given for $t > 0$. Extend it by zero onto the half-space $\{t : t < 0\}$ and consider the obtained distribution $u \in \mathcal{D}'(\mathbb{R}^4)$. Clearly,

(10.36) \[ \Box u = \delta'(t) \otimes \varphi(x) + \delta(t) \otimes \psi(x). \]

A solution of this equation vanishing for $t < 0$ may be found in the form of a retarded potential (10.34) taking $f$ equal to the right-hand side of (10.36). Clearly, we again get the Kirchhoff formula.

Note also that the retarded potential $u(t, x)$ obtained via (10.35) depends only on values $f(t', x')$ for $t' \leq t$ and $|x' - x| = a(t - t')$, i.e., on values of $f$ at points of the lower part of the light cone with the vertex at $(t, x)$ (the word “lower” here is understood with the reference to the direction of the $t$-axis). This is what the term “retarded potential” really means. This property of the retarded potential is ensured by the fact that $\text{supp} \mathcal{E}_3 \subset K^+$. □

A fundamental solution for $\Box$ with this property is unique. Even a stronger result holds:
Theorem 10.5. There exists exactly one fundamental solution for $\Box$ with the support belonging to the half-space $\{(t, x) : t \geq 0\}$, namely, $\mathcal{E}_3(t, x)$.

Proof If there were two such solutions, then their difference $u(t, x) \in \mathcal{D}'(\mathbb{R}^4)$ would have satisfied the wave equation $\Box u = 0$ and would have vanished for $t < 0$. If $u$ were a smooth function, then the uniqueness of the solution of the Cauchy problem would imply $u \equiv 0$ (Cauchy initial values vanish for $t = t_0 < 0$). We can, however, make $u$ smooth by mollifying.

Let $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^4)$, $\text{supp} \varphi_\varepsilon \subset \{(t, x) : |t| + |x| \leq \varepsilon\}$, $\varphi_\varepsilon \geq 0$ and $\int \varphi_\varepsilon dtdx = 1$, so that $\varphi_\varepsilon \to \delta(t, x)$ in $\mathcal{D}'(\mathbb{R}^4)$ as $\varepsilon \to +0$. Then $\lim_{\varepsilon \to +0} (u \ast \varphi_\varepsilon) = u$ in $\mathcal{D}'(\mathbb{R}^4)$ (see Proposition 5.7). But $u \ast \varphi_\varepsilon \in C^\infty(\mathbb{R}^4)$ and

$$\Box(u \ast \varphi_\varepsilon) = (\Box u) \ast \varphi_\varepsilon = 0,$$

and also

$$\text{supp}(u \ast \varphi_\varepsilon) \subset \text{supp} u + \text{supp} \varphi_\varepsilon \subset \{(t, x) : t \geq -\varepsilon\}.$$

Therefore, due to the uniqueness of the smooth solution of the Cauchy problem for the wave equation, $u \ast \varphi_\varepsilon \equiv 0$ for any $\varepsilon > 0$, implying $u \equiv 0$, as required. $\Box$

The importance of Theorem 10.5 stems from the fact that among all the fundamental solutions it chooses the only one which satisfies the causality principle, which claims that it is impossible to transmit information to the “past”. Therefore, in electrodynamics, to solve equations of the form $\Box u = f$, satisfied by the scalar and vector potentials, one uses just this fundamental solution. It seems that the Nature itself has chosen the solution satisfying the causality principle among all the solutions of the equation $\Box u = f$ (within the limits of the current accuracy of experiments).

10.6. The two-dimensional wave equation (the descent method)

Let us solve the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right), \quad u = u(t, x_1, x_2),$$

with the initial conditions

$$u|_{t=0} = \varphi(x_1, x_2), \quad \frac{\partial u}{\partial t} |_{t=0} = \psi(x_1, x_2).$$

The idea of the solution (the descent method) is very simple: introduce an additional variable $x_3$ and find the solution of the Cauchy problem for the
three-dimensional wave equation \( u_{tt} = a^2 \Delta u \) (where \( \Delta \) is the Laplacian with respect to \( x_1, x_2, x_3 \)), with the initial conditions (10.38), which are independent of \( x_3 \). Then the solution \( u(t, x_1, x_2, x_3) \) does not actually depend on \( x_3 \), since the function \( u_z(t, x_1, x_2, x_3) = u(t, x_1, x_2, x_3 + z) \) is a solution of the same equation \( u_{tt} = a^2 \Delta u \) for any \( z \) with the same initial conditions (10.38); therefore, the uniqueness theorem for the solution of the Cauchy problem for the three-dimensional wave equation implies that \( u_z \) does not depend on \( z \), i.e., \( u \) does not depend on \( x_3 \). Therefore the solution of the Cauchy problem (10.37)–(10.38) does exist (e.g. for any \( \varphi \in C^2(\mathbb{R}^2), \psi \in C^1(\mathbb{R}^2) \)). It is unique due to the uniqueness theorem concerning the three-dimensional case since the solution of (10.37)–(10.38) can also be regarded as a solution of the three-dimensional Cauchy problem.

Now, let us express \( u(t, x_1, x_2) \) via the Kirchhoff formula

\[
(10.39) \quad u(t, x) = \frac{\partial}{\partial t} \left( \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \varphi(y_1, y_2) dS_{at} \right) + \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \psi(y_1, y_2) dS_{at},
\]

where \( y = (y_1, y_2, y_3) \), \( dS_{at} \) is the area element of the sphere \( \{ y : y \in \mathbb{R}^3, |y-x| = at \} \) and \( x = (x_1, x_2) = (x_1, x_2, 0) \) (we identify the points of \( \mathbb{R}^2 \) with the points of \( \mathbb{R}^3 \) whose third coordinate is 0; this coordinate might have any other value since it does not affect anything). Let us rewrite the second summand in formula (10.39), which we will denote \( u_\psi \):

\[
(10.40) \quad u_\psi = \frac{1}{4\pi a^2 t} \int_{|y-x|=at} \psi(y_1, y_2) dS_{at}
\]

(the first summand in (10.39) is of the form \( \frac{\partial}{\partial t} u_\varphi \)).

Consider the sphere in \( \mathbb{R}^3 \) over which we integrate in (10.40). This is the sphere with the center at \( x \) of radius \( at \) (see Fig. 1). We are to integrate

![Figure 1. To the descent method](image-url)
10.6. The two-dimensional wave equation (the descent method)

a function, which is independent upon $y_3$, over the sphere. This actually means that we integrate twice over the projection of the sphere onto the plane $y_3 = 0$. Let $dy_1 dy_2$ be the Lebesgue measure of this plane, $dS_{at}$ the area element of the sphere at $y$ whose projection is equal to $dy_1 dy_2$. Clearly, $dy_1 dy_2 = |\cos \alpha(y)| dS_{at}$, where $\alpha(y)$ is the angle between the normal to the sphere and the $y_3$-axis. But the normal is proportional to the vector $y - x = (y_1 - x_1, y_2 - x_2, y_3)$ of length $at$. Let us integrate over the upper half-sphere ($y_3 > 0$) and then double the result. Then $|y - x| = at$ implies

$$y_3 = \sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2},$$

$$\cos \alpha(y) = \frac{y_3}{at} = \frac{1}{at} \sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2},$$

$$dS_{at} = \frac{dy_1 dy_2}{\cos \alpha(y)} = \frac{at dy_1 dy_2}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}}.$$

Therefore, we can rewrite (10.40) in the form

$$u_{\psi}(t, x) = \frac{1}{2\pi a} \int_{|y-x|\leq at} \frac{\psi(y)dy}{\sqrt{a^2 t^2 - |y-x|^2}},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $dy = dy_1 dy_2$. The solution of the Cauchy problem (10.37)–(10.38) is given by the formula

(10.41)

$$u(t, x) = \frac{\partial}{\partial t} \left( \frac{1}{2\pi a} \int_{|y-x|\leq at} \frac{\varphi(y)dy}{\sqrt{a^2 t^2 - |y-x|^2}} \right) + \frac{1}{2\pi a} \int_{|y-x|\leq at} \frac{\psi(y)dy}{\sqrt{a^2 t^2 - |y-x|^2}},$$

known as Poisson’s formula.

It is clear from Poisson’s formula that the value of the solution $u(t, x)$ at $x$ for $n = 2$ depends upon initial values $\varphi(y)$ and $\psi(y)$ in the disc $\{y : |y-x| \leq at\}$ and not only on its boundary (recall that for $n = 3$ it was sufficient to know initial values on the sphere $\{y : |y-x| = at\}$). In particular, if the supports on $\varphi, \psi$ are near origin then the solution in a neighborhood of $x$ is always non-zero after a certain moment. Therefore, a localized perturbation is not seen as a localized one from another point, i.e., the wave does not go tracelessly, but leaves an aftereffect, though decreasing with time as $\frac{1}{t}$, as is clear from (10.41). In other words, the Huygens principle fails for $n = 2$.

The method of descent enables us also to derive d’Alembert’s formula from Poisson’s formula, the former determining the solution of the Cauchy problem for the one-dimensional wave equation. However, we have already obtained this formula by another method.
Let us also find a fundamental solution for the two-dimensional wave operator. As in the three-dimensional case, we need to solve the Cauchy problem with the initial conditions

\[ u|_{t=0} = 0, \quad u_t|_{t=0} = \delta(x). \]

But it is clear from Poisson’s formula that such a solution \( \mathcal{E}_2(t, x) \) has the form

\[
\mathcal{E}_2(t, x) = \frac{\theta(at - |x|)}{2\pi a \sqrt{a^2 t^2 - |x|^2}}, \quad x \in \mathbb{R}^2.
\]

Now, it is easy to verify directly that \( \mathcal{E}_2(t, x) \) is locally integrable and is a fundamental solution for the two-dimensional wave operator. The latter fact is verified in exactly the same way as in the three-dimensional case and we leave this to the reader as an exercise.

Finally, d’Alembert’s formula makes it clear that a fundamental solution for the one-dimensional wave operator \( \partial^2_t - a^2 \partial^2_x \) is of the form

\[
\mathcal{E}_1(t, x) = \frac{1}{2a} \theta(at - |x|).
\]

10.7. Problems

10.1. By separation of variables find cylindric waves in \( \mathbb{R}^3 \).

10.2. Solve the “explosion-of-a-ball problem” in the three-dimensional space: find \( u = u(t, x), \ x \in \mathbb{R}^3 \), if

\[ u_{tt} = a^2 \Delta u, \quad u|_{t=0} = \varphi, \quad u_t|_{t=0} = 0, \]

where \( \varphi \) is the characteristic function of the ball \( \{x : |x| \leq R\} \). Draw a cartoon describing the behavior of \( u(t, x) \) as a function of \( t \) and \( |x| \).

10.3. The same as in the above problem but with different initial conditions: \( u|_{t=0} = 0, \ u_t|_{t=0} = \psi \), where \( \psi \) is the characteristic function of the ball \( \{x : |x| \leq R\} \).

10.4. Using the result of Problem 5.1 c), write, with the help of the Fourier transform, a formula for the solution of the Cauchy problem for the three-dimensional wave equation. (Derive once more Kirchhoff’s formula in this way.)

10.5. With the help of the Fourier transform with respect to \( x \), solve the Cauchy problem for the wave equation

\[ u_{tt} = a^2 \Delta u, \quad u = u(t, x), \quad x \in \mathbb{R}^n. \]
Write the fundamental solution for the $n$-dimensional d’Alembertian $\Box = \frac{\partial^2}{\partial t^2} - a^2 \Delta$ in the form of an integral and prove that its singularities belong to the light cone $\{(t, x) : |x|^2 = a^2 t^2\}$.

10.6. Describe location of the singularities of the fundamental solution for the operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - 4 \frac{\partial^2}{\partial y^2}$.

10.7. Let $u(t, x)$ be a solution of the Cauchy problem of the equation $u_{tt} = \Delta u$, $x = (x_1, x_2) \in \mathbb{R}^2$ with the initial conditions $u|_{t=0} = \varphi$, $u_t|_{t=0} = \psi$.

a) Let $\varphi, \psi$ be known in the rectangle $x_1 \in [0, a]$, $x_2 \in [0, b]$. Where can $u$ be determined? Draw the domain in $\mathbb{R}^3_{t,x_1,x_2}$ in which $u$ can be determined (for $t > 0$).

b) Let the supports of $\varphi, \psi$ belong to the rectangle $x_1 \in [0, a]$, $x_2 \in [0, b]$. Where is $u \neq 0$ for $t > 0$? Draw this domain in $\mathbb{R}^3_{t,x_1,x_2}$.

10.8. Solve Problem 10.7 for $x \in \mathbb{R}^3$ with the rectangle replaced by a rectangular parallelepiped. Instead of the domain in $\mathbb{R}^4_{t,x}$, draw its section by the hyperplane $t = 1$. 


Chapter 11

Properties of the potentials and their calculations

We have already met the potentials in this book. In Remark 4.13 we discussed the physical meaning of the fundamental solution of the Laplace operator in $\mathbb{R}^3$ and $\mathbb{R}^2$. It is explained there that the fundamental solution $E_3(x)$ of the Laplace operator in $\mathbb{R}^3$ has the meaning of the potential of a point charge, and the fundamental solution $E_2(x)$ of the Laplace operator in $\mathbb{R}^2$ has the meaning of the potential of a thin uniformly charged string. It is impossible to define the potential of the string as an ordinary integral (it diverges). We indicated two methods for defining this potential: its recovery from the strength of the field (which is minus the gradient of the potential), and making the divergent integral meaningful by renormalization of the charge. Both methods define the potential of the string up to a constant, which suffices since in the long run we are only interested in the strength of the field which is physically observable, whereas the potential is an auxiliary mathematical object (at least in the classical electrodynamics and gravitation theory).

In Example 5.2 we introduced distributions, called single and double layers on the plane $t = 0$ and on an arbitrary surface, and explained their physical meaning. In Example 5.6 we introduced the potentials of the simple
and double layers as convolutions of the fundamental solution $\mathcal{E}_n(x)$ with the single and double layers on the surface.

In Section 10.1 the role of the scalar and vector potentials in electrodynamics was clarified.

Here we will discuss in detail properties of potentials (and also how to define and calculate them). However, before we start getting acquainted with the details, we advise the reader to reread the above indicated passages.

Potentials will be understood with sufficient versatility. When the preliminary definitions that we begin with become unsuitable, we will modify them at our convenience trying all the time to preserve physical observables.

### 11.1. Definitions of potentials

Let $\mathcal{E}_n(x)$ be the standard fundamental solution for $\Delta$ in $\mathbb{R}^n$;

\[
\mathcal{E}_n(x) = \frac{1}{(2-n)\sigma_{n-1}}|x|^{2-n}, n \geq 3,
\]

where $\sigma_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$;

\[
\mathcal{E}_2(x) = \frac{1}{2\pi} \ln |x|.
\]

In particular, the case $n = 3$ is the most important one:

\[
\mathcal{E}_3(x) = -\frac{1}{4\pi|x|}.
\]

$\mathcal{E}_3(x)$ is the potential of a point charge, which is equal to $+1$ for a proper system of units, and $\mathcal{E}_2(x)$ is the potential of an infinite uniformly charged string (see e.g. Feynman, Leighton, Sands [7], vol. 2, ch. 4).

All potentials have the form

\[
(11.1) \quad -\mathcal{E}_n * f
\]

for different distributions $f \in \mathcal{E}'(\mathbb{R}^n)$ and, therefore, for $n = 3$ and 2 have the meaning of potentials of some distributed systems of charges. Every potential $u$ of the form (11.1) satisfies in $\mathcal{D}'(\mathbb{R}^n)$ the Poisson equation

\[
\Delta u = -f.
\]

Now we will describe the potentials which we need, turn by turn.

a) The volume potential of charges distributed with density $\rho(x)$ is the integral

\[
(11.2) \quad u(x) = -\int_{\mathbb{R}^n} \mathcal{E}_n(x-y)\rho(y)dy.
\]
We will always suppose that \( \rho(y) \) is a locally integrable function with compact support and \( \rho(y) \) is piecewise smooth, i.e., \( \rho \) is a \( C^\infty \)-function outside of a finitely many smooth hypersurfaces in \( \mathbb{R}^n \) on which it may have jumps. All the derivatives \( \partial^\alpha \rho \) are supposed to satisfy the same condition (\( \partial^\alpha \rho \) is taken here outside of the hypersurfaces). An example of an admissible function \( \rho(y) \) is the characteristic function of an arbitrary bounded domain with smooth boundary.

The most important property of the volume potential is that it satisfies the equation
\[
\Delta u = -\rho
\]
understood in the distributional sense; this follows from the equalities \( u = -\mathcal{E}_n * \rho \) and \( \Delta \mathcal{E}_n(x) = \delta(x) \).

b) The single layer potential is the integral
\[
(11.3) \quad u(x) = -\int_\Gamma \mathcal{E}_n(x - y)\sigma(y)dS_y,
\]
where \( \Gamma \) is a smooth hypersurface in \( \mathbb{R}^n \), \( \sigma \) is a \( C^\infty \) function on \( \Gamma \) and \( dS_y \) the area element on \( \Gamma \). For the time being, we assume that \( \sigma \) has a compact support (in the sequel, we will encounter examples when this does not hold). This potential satisfies
\[
\Delta u = -\sigma \delta_\Gamma,
\]
where \( \sigma \delta_\Gamma \) is the distribution with the support on \( \Gamma \) defined in Example 5.2.

c) The double layer potential is the integral
\[
(11.4) \quad u(x) = \int_\Gamma \frac{\partial \mathcal{E}_n(x - y)}{\partial \vec{n}_y} \alpha(y)dS_y,
\]
where \( \vec{n}_y \) is the outward (or in some other way chosen) normal to \( \Gamma \) at \( y \), and the remaining notations are the same as in the preceding case. The potential of the double layer satisfies
\[
\Delta u = -\frac{\partial}{\partial \vec{n}} (\alpha(y)\delta_\Gamma),
\]
where \( \frac{\partial}{\partial \vec{n}} (\alpha \delta_\Gamma) \) — the “double layer” — is the distribution defined in Example 5.2. In this case we assume for the time being that \( \alpha \) has a compact support and that \( \alpha \in C^\infty(\Gamma) \).

It is easy to see that the potentials (11.2)–(11.4) are defined and infinitely differentiable on \( \mathbb{R}^n \setminus \Gamma \) (where in case of the volume potential (11.2), \( \Gamma \) is understood as the union of surfaces where \( \rho \) or its derivatives have jumps).
We will understand these potentials everywhere in $\mathbb{R}^n$ as distributions obtained as convolutions (11.1), where $f$ for the volume potential is the same as $\rho$ in (11.2); for the case of the single layer potential $f = \sigma \delta_\Gamma$, where
\[
\langle \sigma \delta_\Gamma, \varphi \rangle = \int_{\Gamma} \sigma(x) \varphi(x) dS_x, \quad \varphi \in \mathcal{D}(\mathbb{R}^n),
\]
whereas for the case of the double layer potential
\[
f = \frac{\partial}{\partial n}(\alpha \delta_\Gamma),
\]
where
\[
\left\langle \frac{\partial}{\partial n}(\alpha \delta_\Gamma), \varphi \right\rangle = -\int_{\Gamma} \alpha(x) \frac{\partial \varphi(x)}{\partial n_x} dS_x.
\]
Clearly, the convolutions $u = -\mathcal{E}_n * f$ in all the above cases coincide for $x \in \mathbb{R}^n \setminus \Gamma$ with the expressions given by the integrals (11.2)–(11.4).

The physical meaning of the potentials (11.2)–(11.4) was described in Chapter 5.

11.2. Functions smooth up to $\Gamma$ from each side, and their derivatives

Let $\Omega$ be a domain in $\mathbb{R}^n$, $\Gamma$ a smooth hypersurface in $\Omega$, i.e., a closed in $\Omega$ submanifold of codimension 1, and $u \in C^\infty(\Omega \setminus \Gamma)$. We will say that $u$ is smooth up to $\Gamma$ from each side if all the derivatives $\partial^\alpha u$ are continuous up to $\Gamma$ from each side of the hypersurface.

More precisely, if $x_0 \in \Gamma$, then there should exist a neighborhood $U$ of $x_0$ in $\Omega$ such that $U = U^+ \cup \Gamma_U \cup U^-$, where $\Gamma_U = \Gamma \cap U$, $U^+$ and $U^-$ are open and the restrictions $u|_{U^+}$ and $u|_{U^-}$ can be extended to functions $u^+ \in C^\infty(U^+)$ and $u^- \in C^\infty(U^-)$.

In what follows we will often be interested in local questions only, where one should set from the beginning $U = \Omega$ and the above decomposition is of the form $\Omega = \Omega^+ \cup \Gamma \cup \Omega^-$, where $u^+ \in C^\infty(\Omega^+)$ and $u^- \in C^\infty(\Omega^-)$.

Our aim is to prove that all the potentials are smooth up to $\Gamma$ from each side (this in turn enables us to prove theorems about jumps of the potentials, and calculate the potentials). To this end we will first prove several auxiliary lemmas about functions which are smooth up to $\Gamma$ from each side.

If $u \in C^\infty(\Omega \setminus \Gamma)$ is smooth up to $\Gamma$ from each side, then it uniquely determines a distribution $[u] \in \mathcal{D}'(\Omega)$, such that $[u] \in L^1_{\text{loc}}(\Omega)$ and $[u]|_{\Omega \setminus \Gamma} = u$. Our immediate goal is to learn how to apply to $[u]$ differential operators
with smooth coefficients. Let us agree to write \([u]\) only when \(u\) is smooth up to \(\Gamma\) from each side.

In a neighborhood \(U\) of every point \(x_0 \in \Gamma\) we can construct a diffeomorphism of this neighborhood onto a domain in \(\mathbb{R}^n\), that transforms \(U \cap \Gamma\) into a part of the plane \(\{x : x_n = 0\}\). Here \(x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)\), where \(x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\). This diffeomorphism induces a change of variables in any differential operator transforming it into another differential operator of the same order (see Section 1.3). Thus, we will first consider a sufficiently small neighborhood \(U\) of \(0 \in \mathbb{R}^n\) and assume that the coordinates in \(U\) are selected so that

\[\Gamma = U \cap \{x : x_n = 0\},\]

\[U^\pm = \{x : x \in U, x = (x', x_n), \pm x_n > 0\},\]

and the normal \(\vec{n}\) to \(\Gamma\) is of the form \(\vec{n} = (0, \ldots, 0, 1)\), i.e., \(\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_n}\).

For simplicity of notations we will now temporarily assume that \(U = \Omega\), so \(U \cap \Gamma = \Gamma\).

Denote by \(\psi_k\) the jump of the \(k\)-th normal derivative of \(u\) on \(\Gamma\), i.e.

\[\psi_k = \frac{\partial^k u^+}{\partial x_n^k} \bigg|_{\Gamma} - \frac{\partial^k u^-}{\partial x_n^k} \bigg|_{\Gamma} \in C^\infty(\Gamma),\]

where \(k = 0, 1, 2, \ldots\). Note that the jumps of all the other derivatives are expressed via \(\psi_k\) by the formulas

\[(\partial^\alpha u^+)|_{\Gamma} - (\partial^\alpha u^-)|_{\Gamma} = \partial^\alpha' \psi_\alpha,\]

if \(\alpha = (\alpha', \alpha_n)\), where \(\alpha'\) is an \((n-1)\)-dimensional multi-index.

**Lemma 11.1.** The derivatives \(\frac{\partial [u]}{\partial x_n}\), \(\frac{\partial^2 [u]}{\partial x_n^2}\) and \(\frac{\partial^2 [u]}{\partial x_n \partial x_j}(j = 1, 2, \ldots, n-1)\) are expressed by the formulas

\[(11.5) \quad \frac{\partial [u]}{\partial x_n} = \left[ \frac{\partial u}{\partial x_n} \right] + \psi_0(x') \otimes \delta(x_n),\]

\[(11.6) \quad \frac{\partial [u]}{\partial x_j} = \left[ \frac{\partial u}{\partial x_j} \right], \quad j = 1, \ldots, n-1,\]

\[(11.7) \quad \frac{\partial^2 [u]}{\partial x_n^2} = \left[ \frac{\partial^2 u}{\partial x_n^2} \right] + \psi_1(x') \otimes \delta(x_n) + \psi_0(x') \otimes \delta'(x_n),\]

\[(11.8) \quad \frac{\partial^2 [u]}{\partial x_n \partial x_j} = \left[ \frac{\partial^2 u}{\partial x_n \partial x_j} \right] + \frac{\partial \psi_0}{\partial x_j} \otimes \delta(x_n).\]
11. Potentials and their calculations

Proof. For any \( \varphi \in \mathcal{D}(\Omega) \) we have

\[
\left\langle \frac{\partial u}{\partial x_j}, \varphi \right\rangle = -\left\langle [u], \frac{\partial \varphi}{\partial x_j} \right\rangle = -\int_{x_n > 0} [u] \frac{\partial \varphi}{\partial x_j} dx - \int_{x_n \leq 0} [u] \frac{\partial \varphi}{\partial x_j} dx
\]

which proves (11.5). Integrating by parts with respect to \( x_j \) easily leads to (11.6). The formula (11.7) is obtained if we twice apply (11.5), whereas (11.8) follows from (11.5) and (11.6). \( \square \)

Now, let \( a \in C^\infty(\Omega), a = a(x', x_n) \). We need to learn how to multiply distributions encountered in the right-hand sides of (11.5)--(11.8) by smooth functions. To this end, first, expand \( a(x', x_n) \) by the Taylor formula in \( x_n \) near \( x_n = 0 \). We have

\[
a(x', x_n) = \sum_{k=0}^{N-1} a_k(x') x_n^k + x_n^N a_N(x', x_n),
\]

where \( a_k \in C^\infty(\Gamma), a_N(x', x_n) \in C^\infty(V) \) (here \( V \) is a neighborhood of \( \Gamma \)), and

\[
a_k(x') = \frac{1}{k!} \left. \frac{\partial^k a}{\partial x_n^k} \right|_{x_n=0},
\]

\[
a_N(x', x_n) = \frac{1}{(N-1)!} \int_0^1 \frac{\partial^N a}{\partial x_n^N}(x', tx_n)(1-t)^{N-1} dt.
\]

Lemma 11.2. If \( \psi \in C^\infty(\Gamma) \), then

\[
a(x)(\psi(x') \otimes \delta(x_n)) = [a_0(x') \psi(x')] \otimes \delta(x_n),
\]

\[
a(x)(\psi(x') \otimes \delta'(x_n)) = [a_0(x') \psi(x')] \otimes \delta'(x_n) - [a_1(x') \psi(x')] \otimes \delta(x_n).
\]

Proof. Clearly,

\[
a_k(x') x_n^k(\psi(x') \otimes f(x_n)) = [a_k(x') \psi(x')] \otimes [x_n^k f(x_n)]
\]

for any distribution \( f \in \mathcal{D}'(\mathbb{R}) \). Further,

\[
a_N(x', x_n) x_n^N(\psi(x') \otimes f(x_n)) = a_N(x', x_n)(\psi(x') \otimes x_n^N f(x_n)).
\]

The direct calculation shows that

\[
x_n^p \delta^{(k)}(x_n) = 0 \text{ for } p > k;
\]
\[ x_n \delta'(x_n) = -\delta(x_n), \]

which with \(a(x',x_n)\) in the form (11.9) immediately yields (11.10) and (11.11). \(\square\)

**Lemma 11.3.** For any set of functions \(\psi_0, \psi_1, \ldots, \psi_N \in C^\infty(\Gamma)\), there exists a function \(u \in C^\infty(\Omega \setminus \Gamma)\) smooth up to \(\Gamma\) from each side, such that \(\psi_k\) is the jump of \(\partial_k u / \partial x_n\) on \(\Gamma\) for all \(k = 0, 1, \ldots, N\).

**Proof.** We may, for example, set
\[
u(x) = \sum_{k=0}^N \frac{1}{k!} \psi_k(x') x_n^k \theta(x_n),
\]
where \(\theta(t)\) is the Heaviside function. \(\square\)

**Lemma 11.4.** Let a distribution \(f \in \mathcal{D}'(\Omega)\) be of the form
\[
(11.12) \quad f(x) = [f_0(x)] + b_0(x') \otimes \delta(x_n) + b_1(x') \otimes \delta'(x_n),
\]
where \(b_0, b_1 \in C^\infty(\Gamma)\), and \(f_0\) is smooth up to \(\Gamma\) from each side. Let \(A\) be a 2nd order linear differential operator such that \(\Gamma\) is non-characteristic for \(A\) at every point. Then for any integer \(N \geq 0\) there exists a function \(u\) on a neighborhood \(\Omega'\) of hypersurface \(\Gamma\) in \(\Omega\) such that \(u\) is smooth up to \(\Gamma\) from each side, and
\[
(11.13) \quad A[u] - f \in C^N(\Omega').
\]

**Proof.** By Lemma 11.3 it is enough to select jumps \(\psi_0, \psi_1, \ldots, \psi_{N+2} \in C^\infty(\Gamma)\) of normal derivatives of \(u\). It is clear from Lemmas 11.1 and 11.2 that the distribution \(A[u] = \tilde{f}\) always has the form (11.12) (but perhaps with other \(f_0, b_0, b_1\)). We need to select the jumps \(\psi_0, \psi_1, \ldots, \psi_{N+2}\) in such a way that
\[
\tilde{f}(x) = [\tilde{f}_0(x)] + b_0(x') \otimes \delta(x_n) + b_1(x') \otimes \delta'(x_n)
\]
with the same functions \(b_0, b_1\) as in (11.12), and also the jumps of \(\partial_k f_0 / \partial x_n\) and \(\partial_k \tilde{f}_0 / \partial x_n\) on \(\Gamma\) coincide for \(k = 0, 1, \ldots, N\). (Then we will obviously have \(f - \tilde{f} \in C^N(\Omega),\) as required.)

First, note that the non-characteristicity of \(\Gamma\) for \(A\) means that \(A\) can be expressed in the form
\[
(11.14) \quad A = a(x) \frac{\partial^2}{\partial x_n^2} + A',
\]
where \( a \in C^\infty(\Omega), a(x', 0) \neq 0 \), \( A' \) does not contain \( \frac{\partial^2}{\partial x_n^2} \). Since \( a(x) \neq 0 \) in a neighborhood of \( \Gamma \) we can divide (11.13) by \( a(x) \) and reduce the problem to the case when \( a(x) = 1 \). Therefore, in what follows we assume that

\[
\text{(11.15)} \quad A = \frac{\partial^2}{\partial x_n^2} + A',
\]

where \( A' \) is a 2nd order differential operator that does not contain \( \frac{\partial^2}{\partial x_n^2} \).

Let us start with selecting a function \( v_0 \) such that

\[
A[v_0] = [f_0(x)] + \tilde{b}_0(x') \otimes \delta(x_n) + b_1(x') \otimes \delta'(x_n),
\]

where \( \tilde{f}_0 \) is any function (smooth up to \( \Gamma \) from each side), \( \tilde{b}_0 \in C^\infty(\Gamma) \). Lemmas 11.1 and 11.2 show that it suffices to take \( \psi_0^{(0)}(x') = b_1(x') \), where \( \psi_0^{(0)} \) is the jump of \( v_0 \) on \( \Gamma \). Now, (11.13) may be rewritten in the form

\[
\text{(11.16)} \quad A[u - v_0] = [f_1] + b_2(x') \otimes \delta(x_n)
\]

with some \( f_1 \) and \( b_2 \), where \( b_2 \in C^\infty(\Gamma) \).

Now, select a function \( v_1 \) such that

\[
A[v_1] = [\tilde{f}_1] + b_2(x') \otimes \delta(x_n),
\]

where \( b_2 \) is the same as in (11.16) and \( \tilde{f}_1 \) is arbitrary. If \( \psi_0^{(1)} \) and \( \psi_1^{(1)} \) are jumps of \( v_1 \) and \( \frac{\partial v_1}{\partial x_n} \) on \( \Gamma \), then it suffices to take

\[
\psi_0^{(1)} = 0, \quad \psi_1^{(1)}(x') = b_2(x').
\]

Setting \( u_2 = u - v_0 - v_1 \) we see that the condition (11.13) for \( u \) reduces to

\[
A[u_2] - [f_2] \in C^N(\Omega'),
\]

where \( f_2 \) is a function that is smooth up to \( \Gamma \) from each side. We will prove the possibility to select such \( u_2 \) by induction with respect to \( N \), starting with \( N = -1 \) and defining \( C^{-1}(\Omega') \) as the set of all functions in \( \Omega' \) that are smooth up to \( \Gamma \) from each side.

For \( N = -1 \) everything reduces to the simple condition \( u_2 \in C^1(\Omega') \), i.e., to the absence of jumps of \( u_2 \) and \( \frac{\partial u_2}{\partial x_n} \) on \( \Gamma \).

Now suppose that we have already selected a function \( u_{k+2} \) such that

\[
\text{(11.17)} \quad A[u_{k+2}] - [f_2] \in C^{k-1}(\Omega'),
\]

where \( k \) is a non-negative integer (as we have just seen this is possible for \( k = 0 \)). Then, as an induction step, we need to select a function \( u_{k+3} \) such that

\[
\text{(11.18)} \quad A[u_{k+3}] - [f_2] \in C^k(\Omega').
\]
11.2. Functions smooth up to the boundary

To this end set $u_{k+3} = u_{k+2} + v_{k+2}$, where $v_{k+2} \in C^{k+1}(\Omega')$ (we do this in order to preserve (11.17) when we replace $u_{k+2}$ by $u_{k+3}$). Clearly, $A'[u_{k+2}] \in C^k(\Omega')$ and, therefore, (11.18) reduces to

$$
\frac{\partial^2[v_{k+2}]}{\partial x_n^2} - [\tilde{f}_k] \in C^k(\Omega'),
$$

where $\tilde{f}_k = -(A[u_{k+2}] - [f_2]) \in C^{k-1}(\Omega')$. This can be obtained by setting, for example,

$$
v_{k+2}(x) = \frac{1}{(k+2)!} x_n^{k+2} \left[ \frac{\partial^k \tilde{f}_k^+}{\partial x_n^k} \bigg|_{\Gamma} - \frac{\partial^k \tilde{f}_k^-}{\partial x_n^k} \bigg|_{\Gamma} \right] \theta(x_n)
$$

in order to make the jumps of the $k$-th derivatives of the functions $\frac{\partial^2[v_{k+2}]}{\partial x_n^2}$ and $\tilde{f}_k$ on $\Gamma$ coincide. Thus, we have proved that the induction with respect to $N$ can be applied. This completes the proof. □

○ Global version. Fermi coordinates.

Now we will briefly describe a global version of the results above. Let us introduce appropriate notations.

Let $\Gamma$ be a closed hypersurface in $\mathbb{R}^n$, that is a $(n-1)$-dimensional compact closed submanifold of $\mathbb{R}^n$ without boundary. Let us take a function $s : \mathbb{R}^n \to \mathbb{R}$, defined as follows: $s(x) = \text{dist}(x, \Gamma)$ if $x$ is outside of the (closed, compact) body $B$ bounded by $\Gamma$ (in particular, $\Gamma = \partial B$), and $s(x) = -\text{dist}(x, \Gamma)$ if $x$ is inside this body, $s = 0$ on $\Gamma$. Clearly, $s < 0$ on the interior of $B$, $s > 0$ on $\mathbb{R}^n \setminus B$. It is easy to see that $s \in C^\infty$ in a sufficiently small neighborhood of $\Gamma$. Indeed, let $\bar{n} = \bar{n}(z)$ denote the outward unit normal vector to $\Gamma$ at the point $z \in \Gamma$. Consider the map

$$
(11.19) \quad \Gamma \times \mathbb{R} \to \mathbb{R}^n, \quad (z, s) \mapsto z + s\bar{n}.
$$

Its differential is obviously bijective at the points $(z, 0) \in \Gamma \times \mathbb{R}$. Therefore, by the inverse function theorem, the map (11.19) restricts to a diffeomorphism of $\Gamma \times (-\varepsilon, \varepsilon)$ onto an open subset $\Gamma_{\varepsilon}$ of $\mathbb{R}^n$, provided $\varepsilon > 0$ is sufficiently small.

In $\Gamma_{\varepsilon}$, it is convenient to use Fermi coordinates which have the form $(y', y_n)$, where $y' = (y_1, \ldots, y_{n-1})$ is a set of local coordinates on $\Gamma$, and $y_n = s \in (-\varepsilon, \varepsilon)$. If the $(n-1)$-tuple $y'$ represents a point $z \in \Gamma$, then, by definition, $y = (y', s)$ are coordinates of the point $z + s\bar{n} \in \Gamma_{\varepsilon}$. If we have a second order differential operator $A$ such that $\Gamma$ is non-characteristic for $A$ at every point, then, as in (11.14), we can write in any Fermi coordinate
system \((y', y_n)\):

\[
A = b(y) \frac{\partial^2}{\partial y_n^2} + B',
\]

where \(b \in C^\infty(\Gamma_\varepsilon)\), \(b(y', 0) \neq 0\), \(B'\) does not contain \(\frac{\partial^2}{\partial y_n^2}\). Now, if we pass from one Fermi coordinate system to another one \(\tilde{y} = (\tilde{y}', y_n)\), then the first term in the right hand side of (11.20) will not change its form, except change of variables in the scalar function \(b\). Therefore, the function \(b\) is a well defined scalar function on \(\Gamma_\varepsilon\). Then in any equation of the form \(Au = f\) we can divide by \(b\) on a small neighborhood of \(\Gamma = \Gamma \times \{0\}\). For example, we can replace \(\varepsilon\) by a smaller value and assume that \(b(y) \neq 0\) for all \(y \in \Gamma_\varepsilon\).

For the Laplacian \(\Delta\) in \(\mathbb{R}^n\) and Fermi coordinates \(y\) with respect to any hypersurface \(\Gamma\), the formula (11.20) simplifies: then \(b(y) = 1\) for all \(y\), so that we have

\[
\Delta = \frac{\partial^2}{\partial y_n^2} + B',
\]

and, moreover, \(B'\) does not contain any second derivatives of the form \(\frac{\partial^2}{\partial y_j \partial x_n}\) with \(j \leq n\). Both statements follow from a simple geometric fact (a generalization of a Gauss lemma), that in Fermi coordinates \(\frac{\partial}{\partial y_n}\) is the gradient of \(s = y_n\), or, equivalently, that the invariantly defined tangent vector field \(\frac{\partial}{\partial y_n}\) is orthogonal to the level sets of the signed distance function \(s = y_n\). (See the proof in a much more general context in Gray [9], Chapter 2; \(\mathbb{R}^n\) can be replaced by an arbitrary Riemannian manifold \(M\), and \(\Gamma\) by a closed submanifold of an arbitrary dimension of \(M\). As an exercise, the reader can find an elementary proof based on the fact that for small \(\varepsilon\) the straight line interval \(\{z + ts\bar{n} | 0 \leq t \leq 1\}\) is the shortest way between the level hypersurfaces \(y_n = 0\) and \(y_n = s\), where \(s \in [0, \varepsilon]\) is arbitrarily fixed.)

For more information and details about Fermi coordinates see Gray [9], Chapter 2.

To understand how the statement about the second derivatives follows from the above geometric fact, let us recall that in any local coordinates \(y = (y_1, \ldots, y_n)\) the Laplacian can be written in the form of the Laplace-Beltrami operator, which is given by

\[
\Delta u = \frac{1}{\sqrt{g}} \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial y_i} \left( g^{ij} \sqrt{g} \frac{\partial u}{\partial y_j} \right).
\]

Here \((g^{ij})\) is the inverse matrix for \((g_{ij})\), where \(g_{ij} = g_{ij}(y)\) is the symmetric positive definite matrix of the standard Riemannian metric on \(\mathbb{R}^n\) (i.e. \(ds^2 = \)
11.2. Functions smooth up to the boundary

\[
dx_1^2 + \cdots + dx_n^2 \in \text{coordinates } y, \text{ that is } \\
g_{ij}(x) = \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)_x,
\]

where the tangent vectors \( \frac{\partial}{\partial y_j} \) are considered as vectors in \( \mathbb{R}^n \), \( \langle \cdot, \cdot \rangle \) is the standard scalar product in \( \mathbb{R}^n \). (See, for example, Taylor \[30], Vol. 1, Sect. 2.4 for the details about the Laplace-Beltrami operator.)

Note that the vector \( \frac{\partial}{\partial y_n} \) has length one, hence \( g_{nn} = 1 \). The absence of the mixed second derivatives \( \frac{\partial^2}{\partial y_j \partial x_n} \), \( j < n \), is equivalent to \( g_{jn} = 0 \), which implies that for the inverse matrix \( (g^{ij}) \) we also have \( g^{nn} = 1 \) and \( g^{jn} = 0 \) if \( j < n \). This implies the desired absence of the mixed derivatives.

Now we can globalize the arguments given above in the proofs of Lemmas 11.1-11.4 (the local case), except we need to replace \( \psi(x') \otimes \delta(x_n) \) by \( \psi \delta_\Gamma \) where \( \psi \in C^\infty(\Gamma) \), and \( \psi(x') \otimes \delta'(x_n) \) by \( -\frac{\partial}{\partial n}(\psi \delta_\Gamma) \). This leads, in particular, to the following global version of Lemma 11.4:

**Lemma 11.5.** Let \( \Gamma \) be a closed compact hypersurface in \( \mathbb{R}^n \), and a distribution \( f \in \mathcal{D}'(\mathbb{R}^n) \) be of the form

\[
f(x) = [f_0(x)] + b_0(x) \delta_\Gamma + \frac{\partial}{\partial n}(b_1(x) \delta_\Gamma),
\]

where \( b_0, b_1 \in C^\infty(\Gamma) \), \( f_0 \) is locally integrable and smooth up to \( \Gamma \) from each side. Let \( A \) be a 2nd order linear differential operator in \( \mathbb{R}^n \), such that \( \Gamma \) is non-characteristic for \( A \) at every point. Then for any integer \( N \geq 0 \) there exists an integrable compactly supported function \( u \) in \( \mathbb{R}^n \) such that \( u \) is smooth up to \( \Gamma \) from each side, and

\[
A[u] - f \in C^N(\mathbb{R}^n).
\]

We leave the details of the proof as an exercise for the reader. Note only that if we found \( u \) satisfying (11.24) but without compact support, then we can replace \( u \) by \( \chi u \) where \( \chi \in C^\infty_0(\mathbb{R}^n) \), \( \chi = 1 \) in a neighborhood of \( \Gamma \). Then \( \chi u \) will satisfy all the requirements of the Lemma.

**Remark 11.6.** Lemma 11.5 is a simplified version of a more general result. It is not necessary to require compactness of \( \Gamma \). For example, the proof works if \( \Gamma \) is a hyperplane or an infinite cylinder with a compact base. In fact, it suffices that \( \Gamma \) is a closed submanifold in \( \mathbb{R}^n \), though the proof requires a minor modification: we need to glue local solutions into a global one by partition of unity, and this should be done separately on every induction step from the proof of Lemma 11.4. Moreover, the result holds in an even more
general setting, where $\mathbb{R}^n$ is replaced by any general Riemannian manifold (for example, any open set in $\mathbb{R}^n$), and $\Gamma$ by its closed submanifold.

11.3. Jumps of potentials

Now we are able to prove the main theorem on jumps of potentials.

**Theorem 11.7.** 1) Let $u$ be one of the potentials (11.2) – (11.4). Then $u$ is smooth up to $\Gamma$ from each side.

2) If $u$ is a volume potential, then $u$ and $\frac{\partial u}{\partial n}$ do not have jumps on $\Gamma$, so $u \in C^1(\mathbb{R}^n)$.

3) If $u$ is a single layer potential, then $u$ does not have any jump on $\Gamma$, so $u \in C(\mathbb{R}^n)$, whereas the jump of $\frac{\partial u}{\partial n}$ on $\Gamma$ is $(-\sigma)$, where $\sigma$ is the density of the charge on $\Gamma$ in the integral (11.3) defining $u$.

4) If $u$ is a double layer potential, then the jump of $u$ on $\Gamma$ is equal to $(-\alpha)$, where $\alpha$ is the density (of dipole moment) on $\Gamma$ in the integral (11.4) defining $u$.

Here the direction of the jumps is naturally determined by the normal vector field $\bar{n}$.

**Proof.** I will first explain the main idea of the proof. As we have seen in the previous section, the differentiation (of order 1 or 2) of a function $u$ which is smooth up to $\Gamma$ from each side, leads to another function of the same kind but with additional singular terms: terms containing the surface $\delta$-function $\delta_\Gamma$ and its normal derivative, with coefficients which are sums of jumps of $u$ and its normal derivative on $\Gamma$, taken with coefficients from $C^\infty(\Gamma)$. For our proof, we need, vice versa, to find some jumps by the information from the right-hand side of the equation $\Delta u = -f$, which is satisfied for all potentials with an explicitly given distribution $f$ which is supported on $\Gamma$ and has exactly the structure which we have described. Now, from the results of the previous section, we can produce a solution $\hat{u}$ of $\Delta \hat{u} = -f_1$, where $f_1$ differs from $f$ by an arbitrarily smooth function. Then $\Delta (u - \hat{u}) = f_1 - f$ can be done arbitrarily smooth, and so $u - \hat{u}$ can be done arbitrarily smooth too. It will follow that $u$ is smooth up to $\Gamma$ from each side, and the jumps can be recovered from $f$.

Now let us give more detail for each of the statements 1)-4) above.

1) By Lemma 11.5, for each of the potentials $u$ we may find a compactly supported integrable function $u_N$ smooth up to $\Gamma$ from each side, such that
11.3. Jumps of potentials

\[ \Delta[u_N] + f \in C^N(\mathbb{R}^n) \] (here \( f \) is the same as in (11.1)). Therefore, \( \Delta(u - u_N) = f_N \in C^N(\mathbb{R}^n) \).

Let us deduce from this that \( u - u_N \in C^N(\mathbb{R}^n) \). Without loss of generality we may assume that \( u_N \) and \( f_N \) have compact supports. But then \( \Delta(u - u_N - \mathcal{E}_n * f_N) = 0 \), which implies that \( u - u_N - \mathcal{E}_n * f_N \in C^\infty(\mathbb{R}^n) \).

At the same time, clearly, \( \mathcal{E}_n * f_N \in C^N(\mathbb{R}^n) \), because expressing this convolution as an integral

\[
\int \mathcal{E}_n(y) f_N(x - y) dy,
\]

we can differentiate, by dominated convergence theorem, under the integral \( N \) times with respect to \( x \). Therefore, \( u - u_N \in C^N(\mathbb{R}^n) \). This implies that the derivatives \( \partial^\alpha u \) are continuous for \( |\alpha| \leq N \) up to \( \Gamma \) from each side. Since \( N \) is arbitrary, this proves the first statement.

2) Let \( u \) be a volume potential. Clearly, \( u \in C^\infty(\mathbb{R}^n \setminus \Gamma) \). In a neighborhood of every point of \( \Gamma \) we can rectify \( \Gamma \) and use the equation \( Au = -f \), obtained by the change of variables from the equation \( \Delta u = -f \). Since \( \Delta \) is an elliptic operator, \( A \) is also elliptic. Therefore, \( \Gamma \) is non-characteristic. But then the inclusion \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and Lemmas 11.1 and 11.2 imply \( u \in C^1(\mathbb{R}^n) \). (If we assume opposite, then either \( u \) or \( \frac{\partial u}{\partial n} \) has a nontrivial jump on \( \Gamma \), and the formulas (11.5)–(11.8),(11.10), and (11.11) would clearly lead to the occurrence of a non-vanishing singular distribution of type \( \psi_0(x') \otimes \delta'(x_n) \) or \( \psi_0(x') \otimes \delta(x_n) \) in the right hand side of the equation \( Au = -f \), whereas \( f \) is locally integrable due to our assumptions in the case of the volume potential.)

3) In a neighborhood of a point \( x_0 \in \Gamma \) it is convenient to use Fermi coordinates \( y = (y', y_n) \). In particular, \( \frac{\partial u}{\partial n} = \frac{\partial u}{\partial y_n} \) at the boundary points. Let \( u \) be a simple layer potential, defined by (11.3), with the density \( \sigma \in C^\infty_0(\Gamma) \). Then the Poisson equation \( \Delta u = -\sigma \delta_\Gamma \) implies that \( u \) itself can not have a jump, because any non-vanishing jump \( \psi_0 \) of \( u \) would lead to the appearance of the term \( -\psi_0(y') \otimes \delta'(y_n) \) in \( f = -\Delta u \), which would contradict the Poisson equation. Therefore, \( u \) is continuous. On the other hand, it is clear from the form of the Laplacian in coordinates \( y \) given by (11.21) and from Lemma 11.1 that the jump of \( \frac{\partial u}{\partial n} \) should be \( -\sigma \) for \( u \) to satisfy the Poisson equation.

4) The same argument works for the double layer potential. Namely, a jump \( \psi_1 \) of \( u \) leads to the term \( \frac{\partial}{\partial n}(\psi_1) \delta_\Gamma \) in \( f = -\Delta u \), which implies the desired statement. □
Remark. In a similar way we may find the jumps of any derivative of the potentials.

11.4. Calculating potentials

We will give here simplest examples of calculation of potentials. The explicit calculation becomes possible in these examples with the help of:

a) symmetry considerations;
b) the Poisson equation satisfied by the potential;
c) theorems on jumps;
d) asymptotics at infinity.

As a rule, these considerations even give a surplus information which enables us to verify the result.

Example 11.1. The potential of a uniformly charged sphere. Let $\Gamma$ be the sphere of radius $R$ in $\mathbb{R}^3$ with the center at the origin. Suppose a charge is uniformly (with the surface density $\sigma_0$) distributed over the sphere. Denote the total charge of the sphere by $Q$, i.e., $Q = 4\pi R^2 \sigma_0$. Let us find the potential of the sphere and its field. Denote the desired potential by $u$.

First, symmetry considerations show that $u$ is invariant under rotations, hence $u(x) = f(r)$, where $r = |x|$, i.e., $u$ only depends on the distance to the center of the sphere. Now use the fact that $u$ is a harmonic function outside $\Gamma$. Since any spherically symmetric harmonic function is of the form $C_1 + C_2 r^{-1}$, we have

$$f(r) = \begin{cases} A + B r^{-1}, & r < R; \\ C + D r^{-1}, & r > R. \end{cases}$$

Here $A$, $B$, $C$, $D$ are real constants to be determined. First, $B = 0$ since $u$ is harmonic in a neighborhood of 0. This, in particular, implies that $u = \text{const}$ for $|x| < R$ and, therefore, $E = -\text{grad} u = 0$ for $|x| < R$, i.e., the field vanishes inside a uniformly charged sphere.

To find the three remaining constants, we need three more conditions for $u$. They can be obtained in different ways. Let us indicate the simplest of them.

1°. The continuity of $u(x)$ on $\Gamma$ implies $A = C + DR^{-1}$.

2°. Considering the representation of $u(x)$ as an integral, we see that $u(x) \to 0$ as $|x| \to \infty$, hence $C = 0$.
3°. Consider the integral that defines $u(x)$, in somewhat more detail. We have

$$u(x) = \frac{1}{4\pi} \int_{|y|=R} \sigma_0 dS_y = \frac{\sigma_0}{4\pi} \int_{|y|=R} dS_y \frac{1}{|x-y|} = \frac{\sigma_0}{4\pi|x|} \int_{|y|=R} dS_y \left(1 + O \left( \frac{1}{|x|} \right) \right)$$

$$= \frac{Q}{4\pi r} \left(1 + O \left( \frac{1}{r} \right) \right),$$

as $r \to \infty$. Hence $D = \frac{Q}{4\pi}$. From $1° - 3°$ we deduce that $B = C = 0$, $D = \frac{Q}{4\pi}$, $A = \frac{Q}{4\pi R}$. Thus,

$$u(x) = \begin{cases} \frac{Q}{4\pi r} & \text{for } |x| < R \\ \frac{Q}{4\pi} & \text{for } |x| > R \end{cases}.$$  

This means that outside the sphere the potential is the same as if the whole charge were concentrated at the center. Therefore the homogeneous sphere attracts the charges situated out of it as if the whole charge were supported in its center. This fact was first proved by Newton who considered gravitational forces. (He actually proved this theorem in a different way: by considering directly the integral that defines the attraction force.)

We will indicate other arguments that may be used to get equations on $A, C, D$. We can use them to verify the result.

4°. The jump of the normal derivative of $u$ on the sphere should be equal to $-\sigma_0$, i.e., $Dr^{-2}|_{|y|=R} = \sigma_0$ or $D = \sigma_0 R^2 = \frac{Q}{4\pi R^2} \cdot R^2 = \frac{Q}{4\pi}$ which agrees with the value of $D$ found above.

5°. The value of $u(x)$ at the center of the sphere is equal to

$$A = \frac{1}{4\pi} \int_{|y|=R} \frac{\sigma_0 dS_y}{R} = \frac{Q}{4\pi R} \cdot \sigma_0 4\pi R^2 = \frac{Q}{4\pi R},$$

which agrees with the value of $A$ found above.

Using the obtained result, we can immediately find the attracting force (and the potential) of the uniformly charged ball by summing the potentials (or attracting forces) of the spherical layers constituting the ball. In particular, the ball attracts a charge outside of it as if the total charge of the ball were concentrated at its center.

**Example 11.2.** *The potential of the double layer of the sphere.*

Consider the sphere of radius $R$ over which dipoles with the density of the dipole moment $\alpha_0$ are uniformly distributed. The potential of the double
11. Potentials and their calculations

Layer here is the integral (11.4), where

$$\Gamma = \{ x : x \in \mathbb{R}^3, |x| = R \}, \quad \alpha(y) = \alpha_0 = \text{const}.$$ 

Let us calculate this potential. This can be done in the same way as in the preceding example. We may immediately note, however, that $u(x) = 0$ for $|x| > R$ since $u(x)$ is the limit of the potentials of simple layers of a pair of concentric spheres uniformly charged by equal charges with opposite signs. In the limit, these spheres tend to each other, and charges grow in inverse proportion to the distance between the spheres. Even before the passage to the limit, the potential of this pair of spheres vanishes outside the largest of them. Therefore, $u(x) = 0$ for $|x| > R$. It is also clear that $u(x) = \text{const}$ for $|x| < R$, and the theorem on jumps implies (when the exterior normal is chosen) that $u(x) = \alpha_0$ for $|x| < R$.

**Example 11.3. The potential of a uniformly charged plane.**

Let us define and calculate the potential of a uniformly charged plane with the surface density of the charges $\sigma_0$. Let $x_3 = 0$ be the equation of the plane in $\mathbb{R}^3$. Observe that we can not use the definition of the potential by (11.3) because the integral diverges. Therefore, it is necessary to introduce first a new definition of the potential. This may be done in several ways. We will indicate two of them.

The first is to preserve the main equation

$$(11.25) \quad \Delta u = -\sigma_0 \delta_\Gamma = -\sigma_0 \delta(x_3)$$

and as many symmetry properties of the potential as possible. (Physicists often determine a value of an observable quantity from symmetry considerations without bothering much with the definitions.) If the integral (11.3) were convergent in our case, then (11.25) would be satisfied and the following symmetry conditions would hold:

a) $u$ is not changed by the translations along our plane, i.e., $u = u(x_3)$;

b) $u$ is invariant under the reflections with respect to our plane, i.e., $u(-x_3) = u(x_3)$.

Now, try to calculate $u(x_3)$ using (11.25) and the above symmetry properties. It turns out that $u(x_3)$ is uniquely defined up to a constant. Therefore, we can define the potential of the plane to be a distribution $u$ satisfying (11.25), a) and b).
11.5. Problems

From (11.25) it is clear that \( u(x_3) \) is linear in \( x_3 \) separately for \( x_3 > 0 \) and \( x_3 < 0 \), i.e.,

\[
\begin{align*}
  u(x_3) &= \begin{cases} 
    A + Bx_3, & x_3 > 0; \\
    C + Dx_3, & x_3 < 0.
  \end{cases}
\end{align*}
\]

Equation (11.25) means, besides, that the theorem on jumps holds, i.e., \( u \) is continuous on \( \Gamma \) (implying \( A = C \)) and the jump of \( \frac{\partial u}{\partial x_3} \) on \( \Gamma \) is equal to \( -\sigma_0 \) (implying \( B - D = -\sigma_0 \)). Finally, the fact that \( u(x_3) \) is even (condition b)) once again yields \( A = C \) and, besides, implies \( D = -B \). Hence, \( B = -D = -\frac{\sigma_0}{2} \) and

\[
u(x_3) = A - \frac{\sigma_0}{2} |x_3|.
\]

Thus we have found \( u \) up to an irrelevant additive constant.

Now, recall that our main goal is to find \( E = -\nabla u \). We see that \( E \) is everywhere perpendicular to the plane and if the direction from the plane is taken as the positive one, then the magnitude of the field is equal to \( \frac{\sigma_0}{2} \).

Here is another method to define \( u \). The field \( E \) can be found by the Coulomb law (\( E \) is given, as is easy to see, by a converging integral!) and then \( u \) is found from the equation \( E = -\nabla u \). Since the symmetry properties clearly hold and \( \text{div} \ E = \sigma_0 \delta(x_3) \) leads to (11.25), we can apply the same method to find \( u \). Therefore we have explicitly found the field strength \( E \) given by a relatively complicated integral.

11.5. Problems

11.1. Calculate the potential of a uniformly charged circle on the plane.

11.2. Calculate the potential of a uniformly charged disc on the plane.

11.3. Define and calculate the potential of the uniformly charged surface of a straight cylinder in \( \mathbb{R}^3 \).

11.4. Find the potential of the double layer of the circle in \( \mathbb{R}^2 \) and the surface of a straight cylinder in \( \mathbb{R}^3 \) with a constant density of the dipole moment.

11.5. Find the potential of a uniformly charged sphere in \( \mathbb{R}^n \) for \( n \geq 3 \).

11.6. A point charge \( Q \) is placed at the distance \( d \) from the center of a grounded conducting sphere of radius \( R \) (\( R < d \)). Find the total charge induced on the sphere.
11.7. Under the conditions of the preceding problem find the distribution of the induced charge over the surface of the sphere.

11.8. A thin elliptic wire is charged with the total charge $Q$. How will the charge distribute along the wire?
Wave fronts and short-wave asymptotics for hyperbolic equations

12.1. Characteristics as surfaces of jumps

The simplest situation in partial differential equations is when the equation or the boundary value problem can be explicitly solved. However, this case is exceptionally rare. You can more often meet the situation when only a part of the equation is known, or a part of the equation is known approximately. Then it is also very useful to represent the answer in an asymptotic form, when the solution is written in the form of an infinite expansion, and a finite part of the expansion is conjectured to be an approximate answer. This idea very often leads to a dramatic success. We will consider and explore several examples in this chapter.

We will start with a partial differential equation of the form $Au = f$, where $A$ is a linear differential operator with smooth coefficients in an open set $\Omega \subset \mathbb{R}^n$, $f$ is a known function (or distribution) in $\Omega$, $u$ is an unknown function or distribution in a possibly smaller open set.

Let $\Gamma$ be a closed smooth hypersurface in $\Omega$, i.e., a closed submanifold in $\Omega$ of codimension 1. Let $S$ be a closed hypersurface of codimension 1 in $\Gamma$. 
In this way, \( \Gamma \) will be closed in \( \Omega \), and
\[
\Omega \supset \Gamma \supset S,
\]
dim \( \Gamma = n - 1 \), dim \( S = n - 2 \).

Let a linear differential operator \( A \) of degree \( m \) (see Chapter 1),
\begin{equation}
A = a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,
\end{equation}
be given in \( \Omega \), with \( a_\alpha \in C^\infty(\Omega) \). \( a(x, \xi) \) be the total symbol of \( A \) (we usually consider (12.1) to be the definition of both \( A \) and \( a(x, D) \)). Recall that
\begin{equation}
a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha,
\end{equation}
so that \( a(x, D) \) is obtained from \( a(x, \xi) \) if all the \( \xi^\alpha \) are written on the right (as in (12.2)) and then replaced by \( D^\alpha \). The principal symbol \( a_m(x, \xi) \) of \( A \) is of the form
\[a_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha.\]

A hypersurface \( S \) is called a characteristic of \( A \) if for any point \( x \in S \) the unit normal \( \bar{n}_x \) to \( S \) at \( x \) is not 0 but satisfies
\begin{equation}
a_m(x, \bar{n}_x) = 0.
\end{equation}
For any characteristic it follows that the direction of the cotangent vector \( (x, \bar{n}_x) \in T^*_x \Omega \) vanishes. The vanishing of a corresponding vector direction is independent of the choice of local coordinates near \( x \). (See Section 1.3.)

Unfortunately, definitions of characteristic are ambiguous: there are many (not necessarily equivalent) versions. However, there are many important common features, which allow to create a common use of them.

Let us take now a real-valued function \( u \in C^\infty(\Omega \setminus S) \) which is infinitely differentiable up to \( S \) from each side (cf. definition in Section 11.2). Using the same notation as in Section 11.2, we locally have \( \Omega = \Omega^+ \cup S \cup \Omega^- \), and there exist \( u^+ \in C^\infty(\Omega^+) \) and \( u^- \in C^\infty(\Omega^-) \) such that \( u|_{\Omega^+} = u^+ \), \( u|_{\Omega^-} = u^- \).

We can consider \( u \) as a locally integrable function on \( \Omega \), and we will denote by \( u \) the corresponding distribution too. However, due to the implicit function theorem, we will actually have \( u \in C^\infty(\Gamma) \) under the assumptions above.
Note that it may happen that actually there is no discontinuity, i.e., $u$ can be extended to a function from $C^\infty(\Omega)$. This can be considered a regular case, whereas in the opposite (singular) case we will write that $S$ is the surface of jumps for $u$. This means that, first, $u$ is infinitely differentiable up to $S$ from each side and, secondly, for each point $x_0 \in S$ there exists a multi-index $\alpha$ such that $\partial^\alpha u$ is discontinuous at $x_0$, i.e., $\partial^\alpha u$ does not extend to a function on $\Omega$ continuous at $x_0$. In other words, in this case

$$(\partial^\alpha u^+)(x_0) \neq (\partial^\alpha u^-)(x_0).$$

Note that the same inequality also holds on $S$ for $x$ close to $x_0$ (with the same $\alpha$).

**Theorem 12.1.** If $S$ is a surface of jumps for $u \in L^1_{\text{loc}}(\Omega)$, and $u$ is a solution of the equation $Au = f$ in $\Omega$, with $f \in C^\infty(\Omega)$, then $S$ is a characteristic of $A$.

**Proof.** Since the theorem is local and the notion of a characteristic is invariant under diffeomorphisms, we may assume that $S$ is of the form

$$S = \{ x : x \in \Omega, x_n = 0 \},$$

where $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$. Since the normal to $S$ is of the form $(0, 0, \ldots, 1)$, the characteristic property of $S$ means that the coefficient of $\partial_m \partial x_n$ in $A$ vanishes on $S$.

We will prove the theorem arguing by contradiction. Let $S$ be non-characteristic at some point. This means that the coefficient of $\partial^m x_n$ is non-zero at this point; hence, near it. Then we may assume that it is non-zero everywhere in $\Omega$. Dividing the equation $Au = f$ by this coefficient, we may assume that the equation is of the form

$$(12.4) \quad \frac{\partial^m u}{\partial x_n^m} + \sum_{j=1}^m a_j(x, D') \frac{\partial^{m-j} u}{\partial x_n^{m-j}} = f,$$

where $a_j(x, D')$ is a differential operator in $\Omega$ without any derivatives with respect to $x_n$ (its order is $\leq j$ though this is of no importance now).

Let $u$ be a $C^\infty$-function up to $S$ from each side, $\Omega^\pm = \Omega \cap \{ x : \pm x_n > 0 \}$ and $u^\pm \in C^\infty(\Omega^\pm)$ be such that $u|_{\Omega^\pm} = u^\pm$. We want to prove that $u$ can be extended to a function from $C^\infty(\Omega)$, i.e., the jumps of all the derivatives actually vanish. Denote these jumps by $\varphi_\alpha$, i.e.,

$$\varphi_\alpha = (\partial^\alpha u^+)|_S - (\partial^\alpha u^-)|_S \in C^\infty(S).$$
As we have seen in Section 11.2, a special role is played by the jumps of the normal derivatives

\[(12.5) \quad \psi_k = \frac{\partial^k u^+}{\partial x_n^k} \bigg|_S - \frac{\partial^k u^-}{\partial x_n^k} \bigg|_S \in C^\infty(S).\]

All the jumps \(\varphi_\alpha\) are expressed in terms of the jumps of the normal derivatives \(\psi_k\) via the already mentioned formula

\[\varphi_\alpha = \partial_{x'}^\alpha \psi_{\alpha_n}, \quad \alpha = (\alpha', \alpha_n).\]

Therefore, it suffices to prove that \(\psi_k = 0\) for \(k = 0, 1, 2, \ldots\).

Let us find \(Au\) in \(\mathcal{D}'(\Omega)\). By Lemma 11.1 we have

\[\frac{\partial u}{\partial x_n} = \psi_0(x') \otimes \delta(x_n) + \left[ \frac{\partial u}{\partial x_n} \right],\]

\[\frac{\partial^2 u}{\partial x_n^2} = \psi_0(x') \otimes \delta'(x_n) + \psi_1(x') \otimes \delta(x_n) + \left[ \frac{\partial^2 u}{\partial x_n^2} \right].\]

Further calculations of the same type easily yield

\[(12.6) \quad \frac{\partial^k u}{\partial x_n^k} = \sum_{l=0}^{k-1} \psi_{k-l-1}(x') \otimes \delta^{(l)}(x_n) + \left[ \frac{\partial^k u}{\partial x_n^k} \right].\]

Now, if \(\alpha'\) is an \((n-1)\)-dimensional multi-index and \(\partial^{\alpha'} = \partial_{x'}^{\alpha'}\), then

\[(12.7) \quad \partial^{\alpha'} \frac{\partial^k u}{\partial x_n^k} = \sum_{l=0}^{k-1} [\partial^{\alpha'} \psi_{k-l-1}(x')] \otimes \delta^{(l)}(x_n) + \left[ \partial^{\alpha'} \frac{\partial^k u}{\partial x_n^k} \right].\]

Now, let \(a \in C^\infty(\Omega)\). Using the same arguments as in the proof of Lemma 11.2, we easily obtain

\[(12.8) \quad a(x', x_n)(\psi(x') \otimes \delta^{(l)}(x_n)) = \sum_{q=0}^{l} a_q(x') \otimes \delta^{(q)}(x_n),\]

where \(a_q \in C^\infty(S)\).

Formulas (12.6)–(12.8) show that if \(Au = f\) outside \(S\), then (12.4) may be rewritten in the form

\[(12.9) \quad \psi_0(x') \otimes \delta^{(m-1)}(x_n) + \sum_{l=1}^{m-1} a_l(x') \otimes \delta^{(m-1-l)}(x_n) = g(x),\]

where \(g \in L^1_{\text{loc}}(\Omega)\), \(a_l(x') \in C^\infty(S)\) and \(\psi_0\) is the jump of \(u\) on \(S\) (see formula (12.5) for \(k = 0\)). It is clear from (12.9) that \(g \equiv 0\) since the left-hand side of (12.9) is supported on \(S\). Further, after applying the distribution at the left hand side of (12.9) to a test function \(\varphi\) that is equal to \(\varphi_1(x')x_n^{m-1}\) in
12.1. Characteristics as surfaces of jumps

a neighborhood of $S$ (here $\varphi_1 \in C_0^\infty(S)$), we see that all the terms but the first vanish, and the first one is equal to $(-1)^{m-1}(m-1)! \int \psi_0(x') \varphi_1(x') dx'$. Since $\varphi_1$ is arbitrary, it is clear that $\psi_0(x') \equiv 0$, i.e., $u$ cannot have a jump on $S$.

One similarly proves that the derivatives up to order $m-1$ cannot have jumps on $S$. Namely, let $p$ be the smallest of all $k$ such that $\psi_k \not\equiv 0$. Therefore,

$$\psi_0 \equiv \psi_1 \equiv \ldots \equiv \psi_{p-1} \equiv 0, \quad \psi_p \not\equiv 0. \tag{12.10}$$

We wish to distinguish the most singular term in the left-hand side of (12.4). This is easy to do starting from the formulas (12.6)–(12.8). Namely, calculating the normal derivatives $\frac{\partial^k u}{\partial x_n^k}$ for $k \leq p$, we get functions from $L^1_{\text{loc}}(\Omega)$. For $k = p + 1$ we get

$$\frac{\partial^{p+1} u}{\partial x_n^{p+1}} = \psi_p(x') \otimes \delta(x_n) + \frac{\partial^{p+1} u}{\partial x_n^{p+1}}$$

and for $p \leq m-1$ equation (12.4) takes on the form

$$\psi_p(x') \otimes \delta^{(m-p-1)}(x_n) + \sum_{l=1}^{m-p-1} a_l(x') \otimes \delta^{(m-p-1-l)}(x') = g(x),$$

where $a_l(x') \in C_\infty(S), g \in L^1_{\text{loc}}(\Omega)$. As above, this implies $\psi_p \equiv 0$, contradicting the hypothesis. Thus, $\psi_p = 0$ for $p \leq m - 1$.

Let us prove that $\psi_m = 0$. This is immediately clear from (12.4) since all the derivatives $\partial^\alpha u$, where $\alpha = (\alpha', \alpha_n)$ and $\alpha_n \leq m - 1$, are continuous in $\Omega$. Therefore, the normal derivative of order $m$ has no jumps either.

Now, consider the case when $p > m$, where $p$ is the number of the first discontinuous normal derivative; see (12.10). This case can be easily reduced to the case $p = m$. Indeed, applying $\frac{\partial^{m-p} u}{\partial x_n^{m-p}}$ to (12.4), we see that $u$ satisfies an equation of the same form as (12.4) but of order $p$ instead of $m$. This implies that the case $p > m$ is impossible. This concludes the proof.

Theorem 12.1 implies that solutions of elliptic equations with smooth ($C^\infty$) coefficients and the right hand side cannot have jumps of the structure described above. It can be even proved that they are always $C^\infty$-functions. (The reader can find proofs in more advanced textbooks – see, for example, Taylor [30], vol. 2, Sect. 7.4, Hörmander [12], Ch. X, or [14], vol. I, Corollary 8.3.2.) The examples given below show that non-elliptic equations may have discontinuous solutions.
Example 12.1. The equation $u_{tt} = a^2 u_{xx}$ has a solution $\theta(x - at)$ with a jump along the characteristic line $x = at$. A jump of derivatives of arbitrarily large order along the characteristic $x = at$ can be arranged by considering the solution

$$u(t, x) = (x - at)^N \theta(x - at), \text{ where } N \in \mathbb{Z}_+.$$  

Similar discontinuities can be obviously constructed along any characteristic (all the characteristics here can be written in one of the forms $x - at = c_1$, $x + at = c_2$).

Example 12.2. Let us consider a more intricate case of the three-dimensional wave equation

$$u_{tt} = a^2 \Delta u, \quad u = u(t, x), \quad x \in \mathbb{R}^3.$$  

The simplest example of a discontinuous solution of this equation is the spherical wave $u(t, x) = \frac{\theta(|x| - at)}{|x|}$. This solution, clearly, has a jump on the upper sheet of the light cone $\{(t, x) : |x| = at, t > 0\}$. The sphere $\{x : |x| = at\} \subset \mathbb{R}^3$ is naturally called the wave front. The wave front moves at a speed $a$. In geometrical optics, a ray is a parametrically defined (with the parameter $t$) curve which is normal to the wave front at any moment of time. In our example the rays are straight lines $x = a\vec{e}t$, where $t > 0$, $\vec{e} \in \mathbb{R}^3$, $|\vec{e}| = 1$.

Sometimes any curve normal to all the wave fronts (without an explicitly given parameter) is also called a ray.

In what follows we will introduce more terminology of the geometric optics in connection with the Hamilton-Jacobi equation.

Example 12.3. Consider again the three-dimensional wave equation $u_{tt} = a^2 \Delta u$, but this time try to construct a solution which is discontinuous on a given smooth compact surface $\Gamma \subset \mathbb{R}^3$ at the initial moment. Let, for instance, $\Gamma = \partial \Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^3$, and the initial conditions are of the form

$$u|_{t=0} = \chi_{\Omega}, \quad u_t|_{t=0} = 0,$$

where $\chi_{\Omega}$ is the characteristic function of $\Omega$.

Let us try to find out how the jumps of $u$ behave as $t$ grows by looking at the Kirchhoff formula which we will use to construct $u$. In the Kirchhoff formula we are to integrate $\chi_{\Omega}$ over the sphere of radius $at$ with the center at $x$, divide the result by $4\pi a^2 t$ and then take the derivative with respect to $t$. It is easy to see that this $u(t, x)$ is a smooth function of $t, x$ for $t, x$ such
12.2. The Hamilton-Jacobi equation

that the sphere of radius at with the center at is not tangent to (i.e. this sphere either does not intersect or intersects it transversally.

Therefore, the jumps may occur only at the points where this sphere is tangent to . For small , the set of all such forms exactly the set of the points which are situated at the distance at from ("wave front"). For small , the wave front can be constructed as follows: draw all the normals to and mark on them points at the distance at on both sides. It is easy to see that on the wave front thus constructed, there is actually a jump of . For large normals may begin to intersect. An envelope of the family of rays appears and is called a caustic. The singularity of the solution on the caustic has a much more sophisticated structure and we will not deal with it.

Even this example shows how objects of geometric optics appear in the wave theory. In what follows we will consider this problem in somewhat more detail.

12.2. The Hamilton-Jacobi equation. Wave fronts, bicharacteristics and rays

The Hamilton-Jacobi equation is an equation of the form

\begin{equation}
H \left( x, \frac{\partial S}{\partial x} \right) = 0,
\end{equation}

where \( H = H(x, \xi) \) is a given real-valued function of variables \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \) (we will usually assume that \( H \in C^2 \)), and \( S = S(x) \) is an unknown (also real-valued) function with the gradient \( \frac{\partial S}{\partial x} \). The Hamilton-Jacobi equation plays an important role in mechanics and physics.

Recall briefly a method for integrating (12.11), more precisely, a connection of this equation with the Hamiltonian system

\begin{equation}
\begin{cases}
\dot{x} = H_\xi(x, \xi), \\
\dot{\xi} = -H_x(x, \xi),
\end{cases}
\end{equation}

where the Hamiltonian \( H = H(x, \xi) \) is the same function as in (12.11), \( H_\xi = \frac{\partial H}{\partial \xi} = (\frac{\partial H}{\partial \xi_1}, \ldots, \frac{\partial H}{\partial \xi_n}) \), \( H_x = \frac{\partial H}{\partial x} = (\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}) \). (We already discussed Hamiltonian systems in Sect. 10.3; here we use \( x, \xi \) instead of \( q, p \) in (10.13).)

Solutions of this system (curves \( (x(t), \xi(t)) \)) are called the Hamiltonian curves (or, more exactly, Hamiltonian curves of the Hamiltonian \( H(x, \xi) \)).
Vector field \((H_\xi, -H_x)\) in \(\mathbb{R}^{2n}_{x,\xi}\) defining the system (12.12) is called a Hamiltonian vector field with the Hamiltonian \(H\).

As was mentioned in Sect. 10.3, any Hamiltonian vector field is well defined on \(T^*\mathbb{R}^n_x\). This means that the system (12.12) retains the same form for any choice of curvilinear coordinates in \(\mathbb{R}^n_x\) assuming that \(\xi\) is a cotangent vector expressed in the coordinate system corresponding to a given coordinate system in \(\mathbb{R}^n_x\) (i.e., considering \(\xi_1, \ldots, \xi_n\) as coordinates of a cotangent vector with respect to the basis \(dx_1, \ldots, dx_n\), see Section 1.1). The reader can find more detail on this in Arnold [3].

**Proposition 12.2.** The Hamiltonian \(H(x, \xi)\) is a first integral of the system (12.12), i.e., \(H(x(t), \xi(t)) = \text{const}\) along any Hamiltonian curve \((x(t), \xi(t))\).

**Proof.** We have
\[
\frac{d}{dt} H(x(t), \xi(t)) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial \xi} \dot{\xi} = H_x \dot{x} + H_\xi (-H_x) = 0.
\]
(Here and below for the sake of simplicity we omit the dot standing for the inner product of two vectors.) This immediately implies the desired result.

□

In classical mechanics, Proposition 12.2 has the meaning of the energy conservation law. In fact, we already proved it for particular infinite-dimensional linear Hamiltonian system, which is equivalent to the wave equation (see Proposition 10.1).

The Hamiltonian curves with \(H(x(t), \xi(t)) \equiv 0\) are of particular importance for us. If \(H\) is the principal symbol of a differential operator, then they are called bicharacteristics (of the operator or of its symbol). By some abuse of terminology, we will call such a curve bicharacteristic for any \(H\) (even if \(H\) is not a principal symbol of any differential operator). As Proposition 12.2 says, if \((x(t), \xi(t))\) is a Hamiltonian curve defined on an interval of the \(t\)-axis and \(H(x(t_0), \xi(t_0)) = 0\) for some \(t_0\), then this curve is a bicharacteristic.

Now, consider the graph of the gradient of \(S(x)\), i.e., the \(n\)-dimensional submanifold in \(\mathbb{R}^{2n}_{x,\xi}\) of the form \(\Gamma_S = \{(x, S_x(x)), x \in \mathbb{R}^n\}\), where \(S_x = \frac{\partial S}{\partial x}\). Let us formulate the main statement necessary to integrate the Hamiltonian-Jacobi equation.

**Proposition 12.3.** If \(S\) is a \(C^2\) solution of the Hamilton-Jacobi equation (12.11), then the Hamiltonian field is tangent to \(\Gamma_S\) at its points.
12.2. The Hamilton-Jacobi equation

**Proof.** The statement means that if \((x(t), \xi(t))\) is a bicharacteristic and
\((x_0, \xi_0) = (x(t_0), \xi(t_0)) \in \Gamma_S\), then
\[
\frac{d}{dt}(x(t), S_x(x(t)))|_{t=t_0} = \frac{d}{dt}(x(t), \xi(t))|_{t=t_0}.
\]
Or, in a shorter form,
\[
\frac{d}{dt}S_x(x(t))|_{t=t_0} = \dot{\xi}(t_0).
\]
But
\[
\frac{d}{dt}S_x(x(t))|_{t=t_0} = S_{xx}(x(t_0))\dot{x}(t_0) = S_{xx}(x_0)H_\xi(x_0, \xi_0),
\]
where \(S_{xx} = \|\frac{\partial^2 S}{\partial x_i \partial x_j}\|_{i,j=1}^n\) is considered as a \(n \times n\) symmetric matrix. On
the other hand, \(\dot{\xi}(t_0) = -H_x(x_0, \xi_0)\), so we need to prove that
\[
(S_{xx}H_\xi + H_x)|_{\Gamma_S} = 0.
\]
But this immediately follows by differentiation of the Hamilton-Jacobi equation (12.11) with respect to \(x\). □

Proposition 12.3 directly implies

**Theorem 12.4.** \(\Gamma_S\) is invariant with respect to the Hamiltonian flow (i.e.,
with respect to the motion along the bicharacteristics).

This means that if \((x(t), \xi(t))\) is a bicharacteristic defined for \(t \in (a, b)\)
and \(\xi(t_0) = S_x(x(t_0))\) for some \(t_0 \in (a, b)\), then \(S_x(x(t)) = \xi(t)\) for all
\(t \in (a, b)\).

Using Theorem 12.4, we can construct a manifold \(\Gamma_S\) if it is known over
a submanifold \(M\) of codimension 1 in \(\mathbb{R}_x^n\). If we also know \(S\) over \(M\), then
\(S\) is recovered by simple integration over \(\Gamma_S\) (note, that if \(\Gamma_S\) is found, then
\(S\) is recovered from a value at a single point). It should be noted, however,
that all these problems are easy to study locally whereas we are rarely
successful in recovering \(S\) globally. We more often succeed in extending
\(\Gamma_S\) to a manifold which is not the graph of a gradient any longer but is
a so-called Lagrangian manifold; this often suffices to solve the majority of
problems of mathematical physics (the ambiguity of projecting a Lagrangian
manifold onto \(\mathbb{R}_x^n\) causes the appearance of caustics). The detailed analysis
of these questions, however, is beyond the scope of this textbook.

Let us indicate how to construct (locally) \(S\) if it itself and its gradient are
known over an initial manifold \(M\) of codimension 1 in \(\mathbb{R}_x^n\). If the bicharacter-

etistic \(L = \{(x(t), \xi(t))\}\) begins at \((x_0, \xi_0)\), where \(x_0 \in M\) and \(\xi_0 \in S_x(x_0)\),
and terminates at \((x_1, \xi_1)\), then, clearly,

\[(12.13) \quad S(x_1) - S(x_0) = \int_L S_x dx = \int_L \xi dx.
\]

The projections of bicharacteristics to \(\mathbb{R}^n_x\) are called rays. We will only consider bicharacteristics starting over \(M\) at the points of the form \((x, S_x(x))\), where \(x \in M\). Take two rays \(x_1(t)\) and \(x_2(t)\) corresponding to these bicharacteristics. They may intersect at some point \(x_3 \in \mathbb{R}^n_x\) and then (12.13) gives two (generally, distinct) values for \(S(x)\) at \(x_3\). For this reason the Cauchy problem for the Hamilton-Jacobi equation (finding \(S\) from the values of \(S|_M\) and \(S_x|_M\)) is rarely globally solvable.

The Cauchy problem is, however, often solvable locally. For its local solvability it suffices, for example, that rays of the described type that start at the points from \(M\), be transversal to \(M\) at the initial moment. Then we can consider the map \(f : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n_x\), which to a pair \(\{x_0 \in M, t \in (-\varepsilon, \varepsilon)\}\) assigns a point \(x(t)\) of the ray starting at \(x_0\) (i.e., \(x(0) = x_0\)) which is the projection of the bicharacteristic \((x(t), \xi(t))\) such that \(\xi(0) = S_x(x_0)\). (We assume that this map is well defined, i.e. the corresponding bicharacteristics are defined for \(t \in (-\varepsilon, \varepsilon)\).) By the implicit function theorem, \(f\) determines a diffeomorphism of \(U \times (-\varepsilon, \varepsilon)\) (where \(U\) is a small neighborhood of \(x_0\) in \(M\) and \(\varepsilon\) is sufficiently small) with an open set in \(\mathbb{R}^n_x\) if and only if \(\dot{x}(0)\) is not tangent to \(M\) (this is a necessary and sufficient condition for the derivative map of \(f\) to be an isomorphism \((T_{x_0}M \times \mathbb{R} \to T_{x_0} \mathbb{R}^n_x)\). In this case locally the rays do not intersect and the Cauchy problem is locally solvable.

Let us give an exact formulation of the Cauchy problem. Given a submanifold \(M \subset \mathbb{R}^n_x\) and \(S_0 \in C^\infty(M)\), let \(\Gamma_S|_M\) be a submanifold in \((T^* \mathbb{R}^n_x)|_M\) which projects to the graph of the gradient of \(S_0\) under the natural projection onto \(T^* M\) and belongs to the zero level surface of \(H(x, \xi)\). The Cauchy problem is then to find \(S\) such that \(S|_M = S_0\) and \(\Gamma_S|_M\) coincides with the submanifold thus denoted above and given over \(M\). If the above transversality condition holds at all points of \(M\), the Cauchy problem has a solution in a neighborhood of \(M\).

The level surfaces of \(S\) are usually called wave fronts. We will see below why the use of this term here agrees with that in Example 12.2.

**Example 12.4.** Consider a Hamilton-Jacobi equation of the form

\[(12.14) \quad |S_x(x)|^2 = \frac{1}{a^2}.
\]
The Hamiltonian in this case is
\[ H(x, \xi) = |\xi|^2 - \frac{1}{a^2} \]
and the Hamiltonian equations are
\[ \begin{cases} \dot{\xi} = 0, \\ \dot{x} = 2\xi, \end{cases} \]
which implies that the Hamiltonian curves are given by
\[ \begin{cases} \xi = \xi_0, \\ x = 2\xi_0 t + x_0. \end{cases} \]
In particular, the rays are straight lines in all directions. If such a ray is a projection of a bicharacteristic belonging to the graph \( \Gamma_S \) of the gradient of the solution \( S(x) \) (due to Theorem 12.4 this means that \( S_x(x_0) = \xi_0 \)), then
\[ S_x(x(t)) = \xi(t) = \xi_0 = \frac{\dot{x}}{2}, \]
along the whole ray, and the ray \( x(t) \) is orthogonal to all the wave fronts \( S(x) = \text{const} \) which it intersects. ▲

Note that in general rays are not necessarily orthogonal to wave fronts.

The function \( S(x) = a^{-1}|x| \) in \( \mathbb{R}^n \setminus 0 \) is one of the solutions of (12.14). Its level lines, i.e., wave fronts, are spheres \( \{ x : S(x) = t \} \), and they coincide with the wave fronts of Example 12.2. The rays in this case are straight lines which also coincide with the rays, which we mentioned in Example 12.2. ▲

Now consider the following particular case of the Hamilton-Jacobi equation:

\[ \frac{\partial S}{\partial t} + A \left( x, \frac{\partial S}{\partial x} \right) = 0, \tag{12.15} \]
where \( S = S(t, x) \); \( A = A(x, \xi) \); \( t \in \mathbb{R}^1 \); \( x, \xi \in \mathbb{R}^n \); \( A \in C^2 \) is assumed to be real-valued.

This is a Hamilton-Jacobi equation in \( \mathbb{R}_t^{n+1} \) with the Hamiltonian
\[ H(t, x, \tau, \xi) = \tau + A(x, \xi). \]
The Hamiltonian system that defines the Hamiltonian curves \((t(s), x(s), \tau(s), \xi(s))\) is

\[
\begin{aligned}
\dot{t} &= 1, \\
\dot{x} &= A\xi(x, \xi), \\
\dot{\tau} &= 0, \\
\dot{\xi} &= -A_x(x, \xi),
\end{aligned}
\]

(the “dot” stands for the derivative with respect to the parameter \(s\)). It follows that \(\tau(s) = \tau_0, t(s) = s + t_0\) and \((x(s), \xi(s))\) is a Hamiltonian curve of the Hamiltonian \(A(x, \xi)\). Since a shift of the parameter of a Hamiltonian curve does not affect anything, we may assume that \(t_0 = 0\), i.e., \(t = s\), and regard \(x\) and \(\xi\) as functions of \(t\). Further, when interested in bicharacteristics, we may arbitrarily set \(x(0) = x_0, \xi(0) = \xi_0\), and then \(\tau_0 = -A(x_0, \xi_0)\). Therefore arbitrary Hamiltonian curves \((x(t), \xi(t))\) of the Hamiltonian \(A(x, \xi)\) are in one-to-one correspondence with bicharacteristics for \(H(t, x, \tau, \xi)\) starting at \(t = 0\).

The rays corresponding to the described above bicharacteristics are of the form \((t, x(t))\). In particular, they are transversal to all the planes \(t = \text{const.}\)

We may set the Cauchy problem for the equation (12.15) imposing the initial condition

\[(12.16) \quad S|_{t=0} = S_0(x).\]

This implies \(S_x|_{t=0} = \frac{\partial S_0}{\partial x}\), and from (12.15) we find \(\frac{\partial S}{\partial t}|_{t=0}\). Therefore, the graph \(\Gamma_S\) of the gradient of \(S\) is known over the plane \(M = \{(t, x) : t = 0\}\). Since the rays are transversal to \(M\) at the initial moment, \(\Gamma_S\) may be extended to define first \(\Gamma_S\) and then \(S\) over a neighborhood of \(M\).

Namely, in accordance with (12.13), we obtain

\[S(t, x) = S_0(x_0) + \int_0^t [\tau_0 dt + \xi(t)\dot{x}(t)dt],\]

where \((x(t), \xi(t))\) is a Hamiltonian curve of the Hamiltonian \(A(x, \xi)\) such that \(x(0) = x_0\) and \(x(t) = x\), i.e., the points \(x_0\) and \(x\) are connected by a ray (clearly, \(x_0\) should be chosen so that the ray with \(x_0\) as the source hits \(x\) at the time \(t\)); besides,

\[\tau_0 = -A(x_0, \xi_0) = -A \left(x_0, \frac{\partial S_0}{\partial x}(x_0)\right) = -A(x(t), \xi(t))\]
12.2. The Hamilton-Jacobi equation

(this means that the corresponding bicharacteristic of \( H(t, x, \tau, \xi) \) belongs to \( \Gamma_S \)). Thus,

\[ S(t, x) = S_0(x_0) + \int_{x_0}^{x} (\xi dx - Adt), \tag{12.17} \]

where the integral is taken over a Hamiltonian curve of the Hamiltonian \( A \) such that its projection joins \( x_0 \) and \( x \). Instead of (12.17) we can also write

\[ S(t, x) = S_0(x_0) + \int_{x_0}^{x} Ldt, \tag{12.18} \]

where \( L = \xi \dot{x} - A \) is called the Lagrangian corresponding to the Hamiltonian \( A \) and the integral is taken again over the curve. Formulas (12.17) and (12.18) show that \( S \) is similar to the action of classical mechanics (\( \xi \in \mathbb{R}^n \) plays the role of momentum). Therefore, the wave fronts are level surfaces of the action.

We will not go any deeper into the Hamilton-Jacobi equation theory and its relations with mechanics and geometric optics. Details can be found in textbooks on mechanics (see e.g. Arnold \[3\]). Note, in particular, that we have proved here the uniqueness of a solution for the Cauchy problem and gave a method of its construction but did not verify its existence, i.e., we did not verify that this method actually produces a solution. This is really so in the domain where the above system of rays starting at \( t = 0 \) is diffeomorphic to the above system of parallel segments (in particular, this is true in a small neighborhood of the plane \( \{(t, x) : t = 0\} \)), but we skip the verification of this simple fact. ▲

▼ Let us give one further remark on the Hamilton-Jacobi equation of the form (12.15). Suppose we are interested not in the solution itself but only in the wave fronts \( \{(t, x) : S(t, x) = c\} \), where \( c \) is a constant. Fix one value of the constant and consider the system of surfaces \( \{x : S(t, x) = c\} \) belonging to \( \mathbb{R}^n \) and depending on \( t \) as on a parameter. If \( \frac{\partial S}{\partial t} \neq 0 \) somewhere, then we may (locally) solve the equation \( S(t, x) = c \) for \( t \) and express this system of surfaces in the form \( \{x : \tilde{S}(x) = t\} \), i.e., in the form of level surfaces of \( \tilde{S} \).

How to write an equation for \( \tilde{S} \)? Differentiating the identity \( S(\tilde{S}(x), x) \equiv c \) with respect to \( x \) we get

\[ \frac{\partial S}{\partial t} \tilde{S}_x + S_x = 0. \]
Substituting $S_x = -\tilde{S}_x \cdot \frac{\partial S}{\partial \tau}$ into (12.15), we get
\[
(12.19) \quad \frac{\partial S}{\partial t} + A \left( x, -\tilde{S}_x(x) \frac{\partial S}{\partial \tau} \right) = 0.
\]
Consider the case most often encountered and the most important for partial differential equations: the case when $A(x, \xi)$ is homogeneous of 1st order in $\xi$, i.e.,
\[
A(x, \lambda \xi) = \lambda A(x, \xi), \quad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n, \ \lambda > 0.
\]
We may assume that $\frac{\partial S}{\partial t} < 0$ (otherwise replace $S$ by $-S$ which does not affect the wave fronts). Dividing (12.19) by $(-\frac{\partial S}{\partial t})$ we get
\[
(12.20) \quad A(x, \tilde{S}_x(x)) = 1.
\]
We could get the same result just assuming that
\[
(12.21) \quad S(t, x) = \tilde{S}(x) - t + c.
\]
Then (12.20) is immediately obtained without requiring $A(x, \xi)$ to be homogeneous. The role of homogeneity is in the fact that under the assumption of homogeneity, (12.15) is actually just an equation for the direction of the normal vector to the wave fronts and does not depend in any way on parameterization of the wave fronts and the length of the normal vector.

Thus, assuming that $S(t, x)$ is of the form (12.21) or $A(x, \xi)$ is homogeneous of order $1$ in $\xi$, we see that the wave fronts are determined by the equation $\tilde{S}(x) = t$, where $\tilde{S}(x)$ is a solution of the time independent Hamilton-Jacobi equation (12.20).

\section*{12.3. The characteristics of hyperbolic equations}

We have not yet discussed the existence of characteristics. Clearly, the elliptic operator has no characteristics. We will see that a hyperbolic operator has plenty of characteristics.

Let
\[
A = a(t, x, D_t, D_x) = \sum_{|\alpha| \leq m} a_\alpha(t, x) D_t^{\alpha_0} D_x^{\alpha'},
\]
where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$; $\alpha = (\alpha_0, \alpha')$ is a $(n + 1)$-dimensional multi-index; $a_\alpha(t, x) \in C^\infty(\Omega)$, $\Omega$ is a domain in $\mathbb{R}^{n+1}$.

Let us recall what the hyperbolicity of $A$ with respect to a distinguished variable $t$ means (see Section 1.1). Consider the principal symbol
\[
(12.22) \quad a_m(t, x, \tau, \xi) = \sum_{|\alpha| = m} a_\alpha(t, x) \tau^{\alpha_0} \xi^{\alpha'}.
\]
The hyperbolicity of $A$ means that the equation

\begin{equation}
(12.23) \quad a_m(t, x, \tau, \xi) = 0,
\end{equation}

regarded as an equation with respect to $\tau$ has exactly $m$ real distinct roots for any $(t, x) \in \Omega, \xi \in \mathbb{R}^n \setminus 0$. Denote the roots by $\tau_1(t, x, \xi), \ldots, \tau_m(t, x, \xi)$. Clearly, $a_m$ is a polynomial of degree $m$ in $\tau$ and the hyperbolicity implies that its highest coefficient $a_{(m,0,\ldots,0)}$ (the coefficient by $\tau^m$) does not vanish for $(t, x) \in \Omega$ (this coefficient does not depend on $\xi$ due to (12.22). If we are only interested in the equation $Au = f$, then we can divide by this coefficient to make the highest coefficient identically 1. Then all coefficients of the principal symbol $a_m$ will become real. So we can (and will) assume that $a_{(m,0,\ldots,0)} \equiv 1$ from the very beginning.

Since the roots of $a_m$ with respect to $\tau$ are simple, we have

\[ \frac{\partial a_m}{\partial \tau} \bigg|_{\tau=\tau_j} \neq 0. \]

By the implicit function theorem, we can solve (12.23) for $\tau$ in a neighborhood of $\tau_j$ and obtain $\tau_j(t, x, \xi)$ as a locally smooth function of $(t, x, \xi) \in \Omega \times (\mathbb{R}^n \setminus 0)$, i.e., in a sufficiently small neighborhood of any fixed point $(t_0, x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$.

The last statement can be made global due to hyperbolicity. Indeed, we can enumerate the roots so that

\[ \tau_1(t, x, \xi) < \tau_2(t, x, \xi) < \cdots < \tau_m(t, x, \xi). \]

Then every $\tau_j$ will be a continuous (hence, smooth) function of $t, x, \xi$ on $\Omega \times (\mathbb{R}^n \setminus 0)$ because when we continuously move $t, x, \xi$ (keeping $\xi \neq 0$), the roots also move continuously and can not collide (a collision would contradict hyperbolicity), hence their order will be preserved for any such motion.

Further, by homogeneity of $a_m$, i.e., since

\[ a_m(t, x, s\tau, s\xi) = s^m a_m(t, x, \tau, \xi), \]

it is clear that the set of roots $\tau_1(t, x, s\xi), \ldots, \tau_m(t, x, s\xi)$ coincides with the set $s\tau_1(t, x, \xi), \ldots, s\tau_m(t, x, \xi)$. Since the order of the roots $\tau_j$ is preserved under the homothety, we see that the functions $\tau_j(t, x, \xi)$ are smooth, real-valued and homogeneous of order 1 in $\xi$:

\[ \tau_j(t, x, s\xi) = s\tau_j(t, x, \xi), \quad s > 0. \]
In particular, if we know them for $|\xi| = 1$, then they can be extended on all values of $\xi \neq 0$.

Suppose the surface $\{(t, x) : S(t, x) = 0\}$ is a characteristic. Then the vector $(\tau, \xi) = (\frac{\partial S}{\partial t}, S_x)$ on this surface satisfies (12.23). But then it is clear that for some $j$ we locally have

$$\frac{\partial S}{\partial t} = \tau_j \left( t, x, \frac{\partial S}{\partial x} \right)$$

(again on the surface $S = 0$). In particular, for $S$ we may take any solution of the Hamilton-Jacobi equation (12.24). We only need for the surface $S = 0$ to be non-singular, i.e., $S = 0$ should imply $\text{grad} S \neq 0$. Such a function $S$ may be locally obtained, for example, by solving the Cauchy problem for the equation (12.24) with the initial condition

$$S|_{t=t_0} = S_0(x),$$

where $\text{grad}_x S_0(x) \neq 0$. For instance, we may construct a characteristic passing through $(t_0, x_0)$ taking $S_0(x) = \xi \cdot (x - x_0)$, where $\xi \in \mathbb{R}^n \setminus 0$.

In fact, taking $S_0(x)$ to be arbitrary, so that $\text{grad} S_0(x) \neq 0$ on $\{x : S_0(x) = 0\}$, and choosing turn by turn all $\tau_j, j = 1, \ldots, m$, we will find $m$ different characteristics, whose intersection with the plane $t = t_0$ is the same non-singular hypersurface in this plane. This means that the wave front, preassigned for $t = t_0$, may propagate in $m$ different directions.

We may verify that the obtained characteristics, whose intersection with the plane $t = t_0$ is the given non-singular hypersurface, do not actually depend on the arbitrariness in the choice of the initial function $S_0$ if the initial surface $\{x : S_0(x) = 0\}$ is fixed. We skip a simple verification of this fact; it follows, for example, from the study of the above method of constructing solutions for the Hamilton-Jacobi equation. For $n = 1$ such a verification is performed in Chapter 1, where we proved that in $\mathbb{R}^2$ exactly two characteristics of any second-order hyperbolic equation pass through each point.

### 12.4. Rapidly oscillating solutions. The eikonal equation and the transport equations

Now we wish to understand how the wave optics turns into the geometric optics for very short waves (or, which is the same, for high frequencies).
Recall that the plane wave with frequency $\omega$ and wave vector $k$ is of the form
\[
u(t, x) = e^{i(\omega t - k \cdot x)} = e^{i\omega(t - \frac{k}{a} x)}.
\]
The length of the vector $\frac{k}{a}$ is equal to $\frac{1}{a}$, where $a$ is the speed of the wave propagation. For the three-dimensional wave equation, $a$ is a constant, i.e., it does not depend on $\omega$, whereas $\omega$ may be viewed as arbitrary, in particular, arbitrarily large. Therefore, for any vector $k_0$ of length $\frac{1}{a}$ and any $\omega$ there exists a plane wave of the form
\[
u(t, x) = e^{i\omega(t - k_0 \cdot x)}.
\]
We may assume now $k_0$ to be fixed, and let $\omega$ tend to $+\infty$. This is the limit transition to geometric optics for the case of a plane wave.

The argument of the exponent, $\varphi = \omega(t - k_0 \cdot x)$, is called the phase of the plane wave (12.25). Let us fix $\omega$ and $k_0$ and consider constant phase surfaces $\varphi = \text{const}$, which are called wave fronts. For each $t$ a wave front is the plane
\[
\{ x : k_0 \cdot x = t + c \} \subset \mathbb{R}^3.
\]A wave front moves with the speed $a$ in the direction of the vector $k_0$. Therefore, the straight lines in the direction $k_0$ are naturally called rays.

Now consider a general differential operator
\[
A = a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad x \in \Omega \subset \mathbb{R}^n
\]and try to find a solution of the equation $Au = 0$, by analogy with the plane wave (12.25), in the form
\[
u(x) = e^{i\lambda S(x)},
\]
where $\lambda$ is a large parameter, $S(x)$ a real-valued function called phase.

Calculating $Au$, we see that
\[
Au = \lambda^m a_m(x, S_x(x)) e^{i\lambda S(x)} + O(\lambda^{m-1}),
\]where $a_m(x, \xi)$ is the principal symbol of $A$ (see Section 1.2). For the moment we will ignore the terms denoted in (12.27) by $O(\lambda^{m-1})$ (they are of the form of $e^{i\lambda S(x)}$ times a polynomial of $\lambda$ of degree not higher than $m - 1$). But it is clear from (12.27) that if we want $Au = 0$ to be satisfied for arbitrarily large values of $\lambda$, then $S(x)$ should satisfy the equation
\[
a_m \left( x, \frac{\partial S}{\partial x} \right) = 0
\]
which is called the *eikonal equation* in geometric optics. It is a Hamilton-Jacobi equation if the principal symbol $a_m(x, \xi)$ is real-valued (only this case will be considered in what follows). The equation (12.28) means the fulfillment of $Au = 0$ in the first approximation. More precisely, if $u(x)$ is of the form (12.26), then (12.28) is equivalent to

$$Au = O(\lambda^{m-1}) \quad \text{as} \quad \lambda \to \infty.$$ 

We see that the search for wave fronts (constant phase surfaces) reduces in this case to solving the Hamilton-Jacobi equation.

Suppose that the solvability conditions for the Cauchy problem hold for the Hamilton-Jacobi equation (12.28) (this is true, for example, if $A$ is hyperbolic with respect to a variable denoted by $t$). Then we may construct wave fronts by methods of geometric optics proceeding from their disposition on the initial manifold (at $t = 0$ in the hyperbolic case).

Now let us try to satisfy $Au = 0$ with a better precision. It turns out that it is a good idea to seek $u$ in the form

$$u = e^{i\lambda S(x)}b(x, \lambda), \quad (12.29)$$

where

$$b(x, \lambda) = \sum_{j=0}^{\infty} b_j(x)\lambda^{-j}$$

is a formal power series (in $\lambda$) called the *amplitude*. Therefore $u = u(x, \lambda)$ is also a formal series.

Let us explain the meaning of $Au$. Clearly,

$$A(f(x)e^{i\lambda S(x)}) = e^{i\lambda S(x)} \sum_{l=0}^{m} \lambda^{m-l}(A_l f),$$

where $A_l$ are linear differential operators. Applying $A$ to the $k$-th term of (12.29) we get

$$A(e^{i\lambda S(x)}b_k(x)\lambda^{-k}) = e^{i\lambda S(x)} \sum_{l=0}^{m} \lambda^{m-l-k}(A_l b_k).$$

Set

$$Au = e^{i\lambda S(x)} \sum_{j=0}^{\infty} v_j(x)\lambda^{m-j} \quad (12.32)$$

which is the formal series obtained by applying $A$ termwise to (12.29) and grouping together all the terms with the same power of $\lambda$ (as is clear from (12.31), there is only finite number of terms with any given power of $\lambda$). We
call \( u \) an asymptotic or rapidly oscillating solution if \( Au = 0 \), i.e., if all the coefficients \( v_j \) in (12.32) vanish.

Let \( u \) be an asymptotic solution of the form (12.29). Consider a finite segment of the series (12.29):

\[
u_N(x, \lambda) = e^{i\lambda S(x)} \sum_{j=0}^{N} b_j(x) \lambda^{-j}.
\]

Then \( u_N \) is a (regular) function of \( x \) depending on \( \lambda \) as on a parameter. The series \( u - u_N \) begins with \( \lambda^{-N-1} \). Therefore, the series \( A(u - u_N) \) begins with \( \lambda^{-N-1+m} \). But \( Au = 0 \), therefore, \( A(u - u_N) = -Au_N \), implying

\[
Au_N = O(\lambda^{-N-1+m}).
\]

Thus, if we know an asymptotic solution \( u(x, \lambda) \), then its finite segments \( u_N(x, \lambda) \) for large \( \lambda \) are “near-solutions” which become ever more accurate as \( N \) increases.

Similar arguments are applicable to a non-homogeneous equation \( Au = f \) and to boundary-value problems for this equation. Let a boundary-value problem for \( Au = f \) has a unique solution continuously depending on \( f \) (this is often possible to deduce, for example, from energy estimates). Then, having found \( u_N \) such that \( Au_N = f_N \) differs but little from \( f \) (for large \( \lambda \)), we see that since \( A(u - u_N) = f - f_N \), the approximate solution differs only a bit from the exact solution \( u \) (we assume that \( u_N \) satisfies the same boundary conditions as \( u \)). The “exact solution”, however, does not have greater physical meaning than the approximate one since in physics the equations themselves are usually approximate. Therefore, in the sequel we will speak about approximate solutions and not bother with “exact” one.

Note, however, that an approximate solution satisfies the equation with good accuracy only for large \( \lambda \) (for “high frequencies” of “in short-wave range”).

How to find asymptotic solutions? Note first that we may assume \( b_0(x) \neq 0 \) (if \( b_0(x) \equiv 0 \), then we could factor out \( \lambda \) and reenumerate the coefficients \( b_j \)). It may happen that \( b_0(x) \neq 0 \) but \( b_0|_U \equiv 0 \) for some open subset \( U \subset \mathbb{R}^n \). In this case we should consider \( U \) and its complement separately. With this remark in mind we will assume that \( b_0|_U \neq 0 \) for any \( U \subset \Omega \). But then the highest term in the series \( Au \) is

\[
\lambda^m a_m(x, S_x(x)) b_0(x) e^{i\lambda S(x)}.
\]
If we wish it to be 0, then the phase $S(x)$ should satisfy the Hamilton-Jacobi equation (12.28) and if this equation is satisfied, then (12.33) vanishes whatever $b_0(x)$. Thus, let $S(x)$ be a solution of the Hamilton-Jacobi equation.

Now, let us try to find $b_j(x)$ so that the succeeding terms would vanish. For this, let us write the coefficients $v_j$ in the series $Au$ (formula (12.32)) more explicitly. In the above notation, we clearly have

$$v_j(x) = \sum_{l+k=j, 0 \leq l \leq m, k = 0, 1, 2, \ldots} (A_l b_k), \quad j = 0, 1, 2, \ldots,$$

where $A_l$ are linear differential operators depending on $S$ and defined by (12.30). The sum in (12.34) runs over $l \in \{0, 1, \ldots, m\}$ and integers $k \geq 0$ such that $l+k = j$; therefore, the sum is finite. For $j = 0$ the sum contains one summand: $A_0 b_0 = a_m(x, S_x(x))b_0(x)$. This makes it clear that $A_0 = 0$ due to the choice of $S(x)$. Therefore, the sum over $l$ in (12.34) runs actually from 1 to $m$. We have

$$v_1 = A_1 b_0,$$
$$v_2 = A_1 b_1 + A_2 b_0,$$
$$v_3 = A_1 b_2 + A_2 b_1 + A_3 b_0,$$
$$\cdots \cdots \cdots$$

Therefore, the conditions $v_j = 0, j = 0, 1, 2, \ldots$ yield, in addition to the already written Hamilton-Jacobi equation, also linear differential equations for functions $b_j$. These equations may be written in the form

$$A_1 b_0 = 0,$$
$$A_1 b_1 = -A_2 b_0,$$
$$A_1 b_2 = -A_2 b_1 - A_3 b_0,$$
$$\cdots \cdots \cdots$$

i.e., all of them are of the form

(12.35) \[ A_1 b_j = f_j, \]

where $f_j$ is expressed in terms of $b_0, \ldots, b_{j-1}$.

Equations (12.35) are called transport equations. They show the importance of the role played by $A_1$.

Let us find $A_1$. We assume that $S(x)$ satisfies the eikonal equation (12.28). Then $A_1$ is determined from the formula

$$A(f(x)e^{i\lambda S(x)}) = e^{i\lambda S(x)}\lambda^{m-1}(A_1 f) + O(\lambda^{m-2}).$$
The direct computation, using the Leibniz product differentiation rule, shows that for $|\alpha| = m$ we have

$$D^\alpha(f(x)e^{i\lambda S(x)}) = \lambda^m(S_x(x))^{\alpha}f(x)e^{i\lambda S(x)} + \lambda^{m-1}\sum_{j=1}^n \alpha_j(S_x(x))^{\alpha-e_j}[D_j f(x)]e^{i\lambda S(x)} +$$

$$\lambda^{m-1}c_\alpha(x)f(x)e^{i\lambda S(x)} + O(\lambda^{m-2}),$$

where $D_j = i^{-1}\frac{\partial}{\partial x_j}$, $e_j$ is the multi-index with only the $j$th non-zero entry which is equal to 1, $c_\alpha$ is a $C^\infty$ function which depends upon the first and second derivatives of $S$. Since

$$\alpha_j \xi^{\alpha-e_j} = \frac{\partial}{\partial \xi_j} \xi^\alpha,$$

it follows that

$$A(f(x)e^{i\lambda S(x)}) = \lambda^{m-1} \left[ \sum_{j=1}^n \frac{\partial a_m}{\partial \xi_j} \bigg|_{\xi=S_x(x)} (D_j f(x)) + c(x)f(x) \right] e^{i\lambda S(x)} + O(\lambda^{m-2}),$$

implying

$$A_1 = \sum_{j=1}^n \frac{\partial a_m}{\partial \xi_j} \bigg|_{\xi=S_x(x)} D_j + c(x) = i^{-1}L_1 + c(x),$$

where $L_1$ is the derivative along the vector field

$$V(x) = \frac{\partial a_m}{\partial \xi} \bigg|_{\xi=S_x(x)} \cdot \frac{\partial}{\partial x}.$$

What is the geometric meaning of $V(x)$? Let us write the Hamiltonian system with the Hamiltonian $a_m(x, \xi)$:

$$\begin{cases}
\dot{x} = \frac{\partial a_m}{\partial \xi} \\
\dot{\xi} = -\frac{\partial a_m}{\partial x}
\end{cases}$$

and take a bicharacteristic belonging to the graph of the gradient $S$, i.e., a bicharacteristic $(x(t), \xi(t))$ such that $\xi(t) = S_x(x(t))$. Then $V(x(t))$ is the tangent vector to the ray $x(t)$ corresponding to such a bicharacteristic. Therefore, the transport equations (12.35) are of the form

$$\frac{1}{i} \frac{d}{dt} b_j(x(t)) + c(x(t))b_j(x(t)) = f_j(x(t)),$$

i.e., are ordinary linear differential equations along rays. Therefore, the value $b_j(x(t))$ along the ray $x(t)$ is uniquely determined by $b_j(x(0))$. 

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12.4. The eikonal and transport equations
Example 12.5. Consider the wave equation in a non-homogeneous medium
\[ \frac{\partial^2 u}{\partial t^2} - c^2(x) \Delta u = 0, \quad x \in \mathbb{R}^3, \]
where \( c(x) \) is a smooth positive function of \( x \). Due to the general recipe, we seek a rapidly oscillating solution in the form
\[ u(t, x, \lambda) = e^{i \lambda S(x)} \sum_{j=0}^{\infty} b_j(t, x) \lambda^{-j}. \]
The principal symbol of the corresponding wave operator is
\[ H(t, x, \tau, \xi) = -\tau^2 + c^2(x) |\xi|^2, \]
so the phase \( S(x, t) \) should satisfy the following eikonal equation:
\[ \left( \frac{\partial S}{\partial t} \right)^2 - c^2(x) \left( \frac{\partial S}{\partial x} \right)^2 = 0. \]
The equations of Hamiltonian curves are
\[
\begin{align*}
\dot{t} &= -2 \tau \\
\dot{x} &= 2 c^2(x) \xi \\
\dot{\tau} &= 0 \\
\dot{\xi} &= -2 c(x) |\xi|^2 \cdot \frac{\partial c}{\partial x} 
\end{align*}
\]
implying \( \tau = \tau_0 \), \( t = -2 \tau_0 \sigma + t_0 \), where \( \sigma \) is the parameter along the curve. Since we are only interested in bicharacteristics, we should also assume that \( \tau^2 = c^2(x) |\xi|^2 \) or \( \tau = \mp c(x) |\xi| \). From this we find for \( |\xi| \neq 0 \):
\[ \frac{dx}{dt} = \frac{\dot{x}}{\dot{t}} = \frac{2 c^2(x) \xi}{-2 \tau_0} = \pm c(x) \frac{\xi}{|\xi|}. \]
In particular, the absolute value of the ray’s speed is \( c(x) \). This easily implies that \( c(x) \) is the speed of the wave front propagation at each point. ▲

The first and the last equations of bicharacteristics also imply
\[ \frac{d\xi}{dt} = \frac{-2 c(x) |\xi|^2 c_x(x)}{-2 \tau_0} = \mp |\xi| \cdot c_x(x). \]
Thus, \( x(t) \) and \( \xi(t) \) satisfy
\[ \begin{align*}
\frac{dx}{dt} &= \pm c(x) \frac{\xi}{|\xi|}, \\
\frac{d\xi}{dt} &= \mp |\xi| \cdot c_x(x).
\end{align*} \]
This is a Hamiltonian system with the Hamiltonian \( H(x, \xi) = \pm c(x) |\xi| \). In particular, \( c(x) |\xi| = H_0 = \text{const} \) along the trajectories. The choice of the
constant \( H_0 \) (determined by the initial conditions) does not affect the rays \( x(t) \) because \( (x(t), \lambda \xi(t)) \) is a solution of (12.38) for any \( \lambda > 0 \) as soon as \( (x(t), \xi(t)) \) is. Using this fact, let us try to exclude \( \xi(t) \). Replacing \( |\xi| \) by \( \frac{H_0}{c(x)} \) we derive from (12.38) that

\[
\begin{align*}
\frac{dx}{dt} &= \pm H_0^{-1} c^2(x) \xi \\
\frac{d\xi}{dt} &= \mp H_0 \frac{c_x(x)}{c(x)}
\end{align*}
\] (12.39)

Now, substituting the expression

\[
\xi = \pm \frac{H_0}{c^2(x)} \frac{dx}{dt}
\] (12.40)

found from the first equation into the second one, we get

\[
\frac{d}{dt} \left( \frac{1}{c^2(x)} \frac{dx}{dt} \right) + \frac{c_x(x)}{c(x)} = 0.
\] (12.41)

Thus, we have shown that every ray \( x(t) \) satisfies the 2nd order differential equation (12.41). This is a non-linear differential equation (more precisely, a system of \( n \) such equations). It can be resolved with respect to \( \frac{d^2x}{dt^2} \) (its coefficient is equal to \( \frac{1}{c^2(x)} \) and we have assumed that \( c(x) > 0 \)). Therefore, the initial conditions \( x(0) = x_0, x'(0) = v_0 \) (the “prime” stands for the derivative with respect to \( t \)) uniquely define \( x(t) \).

Not every solution \( x(t) \) of this equation, however, determines a ray, since the passage from (12.38) to (12.41) is not exactly an equivalence. Let us consider this in detail.

Let \( x(t) \) be a solution of (12.41). Determine \( \xi(t) \) via (12.40) choosing (so far, arbitrarily) a positive constant \( H_0 \). Then (12.41) is equivalent to the second of equations (12.39), whereas the first of equations (12.39) means simply that \( \xi(t) \) is obtained via (12.40). Therefore, if \( (x(t), \xi(t)) \) is a solution of (12.39) for some \( H_0 > 0 \), then \( x(t) \) is a solution of (12.41) and, conversely, if \( x(t) \) is a solution of (12.41) and \( H_0 \) is an arbitrary positive constant, then there exists a unique function \( \xi(t) \), such that \( (x(t), \xi(t)) \) is a solution of (12.39).

How to pass from (12.39) to (12.38)? If \( (x(t), \xi(t)) \) is a solution of (12.39) and \( \xi(t) \neq 0 \), then clearly (12.38) holds if and only if \( c(x(t))|\xi(t)| = H_0 \). Note that for this it suffices that \( c(x(0))|\xi(0)| = H_0 \). Indeed, it suffices to verify that the vector field

\[
\left( \pm H_0^{-1} c^2(x) \xi, \mp H_0 \frac{c_x(x)}{c(x)} \right)
\]
is tangent to the manifold
\[ M = \{(x, \xi) : c(x)\xi = H_0\} \subset \mathbb{R}^{2n} \, x, \xi. \]

Take \((x_0, \xi_0) \in M\) and a curve \((x(t), \xi(t))\) which is a solution of (12.39) such that \((x(0), \xi(0)) = (x_0, \xi_0)\). In particular, the velocity vector \((\dot{x}(0), \dot{\xi}(0))\) coincides with the vector field at \((x_0, \xi_0)\). We are to verify that \((\dot{x}(0), \dot{\xi}(0))\) is tangent to \(M\) at \((x_0, \xi_0)\), i.e., that
\[ \frac{d}{dt}(c^2(x(t))|\xi(t)|^2)|_{t=0} = 0. \]

But
\[
\frac{d}{dt}(c^2(x(t))|\xi(t)|^2)|_{t=0} = 2c(x_0)[c_x(x_0) \cdot x'(0)]|\xi_0|^2 + 2c^2(x_0)\xi_0 \cdot \xi'(0)
\]
\[= 2c(x_0)|\xi_0|^2c_x(x_0) \cdot [\pm H_0^{-1}c^2(x_0)\xi_0] + 2c^2(x_0)\xi_0 \cdot [\mp H_0c^{-1}(x_0)c_x(x_0)]
\]
\[= 2c(x_0)[\xi_0 \cdot c_x(x_0)] [\pm H_0^{-1}c^2(x_0)|\xi_0|^2 \mp H_0] = 0,
\]
because \(c(x_0)|\xi_0| = H_0\) due to our assumption that \((x_0, \xi_0) \in M\).

Thus, if \(c(x(0))|\xi(0)| = H_0\), then a solution \((x(t), \xi(t))\) of (12.39) is also a solution of (12.38); hence, \(x(t)\) is a ray.

Now note that (12.40) implies that for each fixed \(t\) the condition \(c(x(t))|\xi(t)| = H_0\) is equivalent to \(|x'(t)| = c(x(t))\). In particular, if \(|x'(0)| = c(x(0))\) and \(x(t)\) is a solution of (12.41), then \(x(t)\) is a ray and \(|x'(t)| = c(x(t))\) for all \(t\). Thus, the rays \(x(t)\) are distinguished among the solutions of (12.41) by the fact that their initial values \(x(0) = x_0, x'(0) = v_0\) satisfy \(v_0 = c(x_0)\).

Let us verify “Fermat’s principle”, which means that that the rays are extrema of the functional \(T = T(\gamma)\) defining time for our wave (e.g. light) to pass along the curve \(\gamma\), i.e.,
\[ T(\gamma) = \int_\gamma \frac{ds}{c}, \]
where \(ds\) is the arc-length element of the curve \(\gamma\) and \(c = c(x)\) is considered as a function on \(\gamma\). The extremum is understood with respect to all variations preserving the endpoints, i.e., in the class of the paths \(\gamma\) that connect two fixed points. Denote these points \(x_0\) and \(x_1\). Since the extremality of a curve does not depend on a choice of a parameter, it is convenient to assume that the parameter runs over \([0, 1]\). Thus, let \(\gamma = \{x(\tau) : \tau \in [0, 1]\}\). Take variations of \(\gamma\) preserving the conditions \(x(0) = x_0, x(1) = x_1\). A curve is an extremum if it satisfies the Euler–Lagrange equations (see Sect. 2.5) for
the functional

\[ T(\{x(\tau)\}) = \int_0^1 \frac{|\dot{x}(\tau)|}{c(x(\tau))} d\tau, \]

where the “dot” denotes the derivative with respect to \( \tau \). The Euler-Lagrange equations

\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]

for the Lagrangian \( L = \frac{|\dot{x}|}{c(x)} \) can be rewritten as

\[ \frac{d}{d\tau} \left( \frac{1}{c(x(\tau))} \frac{1}{|\dot{x}(\tau)|} \frac{dx(\tau)}{d\tau} \right) + \frac{|\dot{x}(\tau)|}{c^2(x(\tau))} c_x(x(\tau)) = 0. \]

Clearly, the extremality of a curve is independent upon a choice of the parameter \( \tau \) and its range, and so the Euler-Lagrange equations hold for every such choice, or for none. So the parameter \( \tau \) may run over any fixed segment \([a,b]\). Let now \( \{x(t)\} \) be a ray with the parameter \( t \) (the same as in (12.38)). We know that \( |\dot{x}(t)| = c(x(t)) \) along the ray. But then setting \( \tau = t \), we see that the Lagrange equation (12.42) turns into (12.41) derived above for rays. Thus, the rays satisfy the Fermat principle. This agrees, in particular, with the fact that in a homogeneous medium (i.e., for \( c(x) = c = \text{const} \)) the rays are straight lines.

Let us write down the transport equations in the described situation. For this purpose, substitute a series \( u(t, x, \lambda) \) of the form (12.37) into equation (12.36). We will assume that \( S(t, x) \) satisfies the eikonal equation. Then the series \( \frac{\partial^2 u}{\partial \tau^2} - c^2(x) \Delta u \) begins with \( \lambda^1 \). We have
12. Wave fronts and short-wave asymptotics

\[ u_t(t, x, \lambda) = i\lambda S_t e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + e^{i\lambda S} \sum_{j=0}^{\infty} \frac{\partial b_j}{\partial t} \lambda^{-j}; \]

\[ u_{tt}(t, x, \lambda) = -\lambda^2 S_t^2 e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + i\lambda S_{tt} e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + \]

\[ 2i\lambda S_t e^{i\lambda S} \sum_{j=0}^{\infty} \frac{\partial b_j}{\partial t} \lambda^{-j} + e^{i\lambda S} \sum_{j=0}^{\infty} \frac{\partial^2 b_j}{\partial t^2} \lambda^{-j}; \]

\[ u_x(t, x, \lambda) = i\lambda S_x e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + e^{i\lambda S} \sum_{j=0}^{\infty} \frac{\partial b_j}{\partial x} \lambda^{-j}; \]

\[ \Delta u(t, x, \lambda) = -\lambda^2 S_x^2 e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + i\lambda S \Delta e^{i\lambda S} \sum_{j=0}^{\infty} b_j \lambda^{-j} + \]

\[ 2i\lambda e^{i\lambda S} \sum_{j=0}^{\infty} S_x \cdot \frac{\partial b_j}{\partial x} \lambda^{-j} + e^{i\lambda S} \sum_{j=0}^{\infty} (\Delta b_j) \lambda^{-j}. \]

(Here \( S_x^2 \) means \( S_x \cdot S_x = |S_x|^2 \).)

Now, compute \( u_{tt} - c^2(x) \Delta u \). The principal terms (with \( \lambda^2 \)) in the expressions for \( u_{tt} \) and \( c^2(x) \Delta u \) cancel due to the eikonal equation. Equating the coefficients of all the powers of \( \lambda \) to zero and dividing by \( 2i \), we get

\[
\begin{align*}
S_t \frac{\partial b_0}{\partial t} - c^2(x) S_x \cdot \frac{\partial b_0}{\partial x} + \frac{1}{2} [S_{tt} - c^2(x) \Delta S] b_0 &= 0; \\
S_t \frac{\partial b_0}{\partial t} - c^2(x) S_x \cdot \frac{\partial b_0}{\partial x} + \frac{1}{2} [S_{tt} - c^2(x) \Delta S] b_1 + (2i)^{-1} \left[ \frac{\partial^2 b_0}{\partial t^2} - c^2(x) \Delta b_0 \right] &= 0; \\
S_t \frac{\partial b_1}{\partial t} - c^2(x) S_x \cdot \frac{\partial b_1}{\partial x} + \frac{1}{2} [S_{tt} - c^2(x) \Delta S] b_1 + (2i)^{-1} \left[ \frac{\partial^2 b_1}{\partial t^2} - c^2(x) \Delta b_1 \right] &= 0; \\
\end{align*}
\]

These are the transport equations. The Hamiltonian equations imply that the vector field

\[ (S_t, -c^2(x) S_x) = (\tau, -c^2(x) \xi) \]

is tangent to the rays, i.e., the equations (12.43) are of the form

\[ \frac{d}{d\sigma} b_j(t(\sigma), x(\sigma)) + a(\sigma) b_j(t(\sigma), x(\sigma)) + f(\sigma) = 0, \]
where \( a(\sigma) \) only depends on \( S \) and \( f(\sigma) \) depends on \( S, b_0, \ldots, b_{j-1} \) while \((t(\sigma), x(\sigma))\) is a ray with the parameter \( \sigma \) considered at the beginning of the discussion of this example.

\[\nabla\]

Let us indicate another method of analyzing this example and examples of a similar structure. We may seek solutions of (12.36) in the form
\[
(12.44) \quad u(t, x, \omega) = e^{i\omega t}v(x, \omega),
\]
where \( \omega \) is a large parameter and \( v \) is a formal series of the form
\[
v(x, \omega) = e^{i\omega S(x)} \sum_{k=0}^{\infty} v_k(x)\omega^{-k}.
\]
Equation (12.36) for a function \( u(t, x, \omega) \) of the form (12.44) turns into an equation of the Helmholtz type for \( v(x, \omega) \)
\[
[\Delta + \omega^2 c^{-2}(x)]v(x, \omega) = 0.
\]
For \( S(x) \) we get the eikonal equation
\[
S^2_x(x) = \frac{1}{c^2(x)}.
\]
We could get the same result assuming initially \( S(t, x) = t - S(x) \). We will not write now the transport equations for functions \( v_k(x) \). The structure of these equations is close to that of non-stationary transport equations (12.43).

**Remark.** Transport equations describe transport of energy, polarization and other important physical phenomena in physical problems. We will not dwell, however, on these questions which rather belong to physics.

12.5. The Cauchy problem with rapidly oscillating initial data

Consider the equation \( Au = 0 \), where \( A = a(t, x, D_t, D_x) \), \( t \in \mathbb{R} \), \( x \in \mathbb{R}^n \), and \( A \) is hyperbolic with respect to \( t \). Let us consider a formal series
\[
(12.45) \quad u(t, x, \lambda) = e^{i\lambda S(t, x)} \sum_{j=0}^{\infty} b_j(t, x)\lambda^{-j}
\]
and try to find out which initial data at \( t = 0 \) enable us to recover this series if it is an asymptotic solution. We will only consider the phase functions \( S(t, x) \) such that \( \text{grad}_x S(t, x) \neq 0 \). The function \( S(t, x) \) should satisfy the eikonal equation
\[
(12.46) \quad a_m(t, x, S_t, S_x) = 0,
\]
where \( a_m(t, x, \tau, \xi) \) is the principal symbol of \( A \) (which can be assumed real without loss of generality). But as we have seen above in Section 12.3, this is equivalent to the fulfillment of one of \( m \) equations

\[
\frac{\partial S}{\partial t} = \tau_l(t, x, S_x)
\]

where \( \tau_l(t, x, \xi) \), for \( l = 1, 2, \ldots, m \), is the complete set of roots of the equation \( a_m(t, x, \tau, \xi) = 0 \), chosen, say, in the increasing order. Setting the initial condition

\[
S|_{t=0} = S_0(x),
\]

where \( S_0(x) \) is such that \( \text{grad}_x S_0(x) \neq 0 \), we may construct exactly \( m \) different solutions of the eikonal equation (12.46) satisfying this initial condition, namely, solutions of each of the equation (12.47) with the same initial condition (12.48) (such a solution is unique as soon as we have chosen the root \( \tau_l \)). Solutions of equations (12.47) exist in a small neighborhood of the plane \( t = 0 \) determined as indicated in Section 12.2. In the same neighborhood, the functions \( b_j(t, x) \) are uniquely defined from the transport equations provided the initial values are given:

\[
b_j|_{t=0} = b_{j,0}(x), \quad j = 0, 1, 2, \ldots
\]

Let us now formulate the Cauchy problem for the equation \( Au = 0 \) by imposing the initial conditions of the form

\[
\begin{align*}
    u|_{t=0} &= e^{i\lambda S_0(x)} \sum_{j=0}^{\infty} c_j^{(0)}(x) \lambda^{-j}, \\
    u_t|_{t=0} &= \lambda e^{i\lambda S_0(x)} \sum_{j=0}^{\infty} c_j^{(1)}(x) \lambda^{-j}, \\
    \frac{\partial^{m-1}u}{\partial t^{m-1}}|_{t=0} &= \lambda^{m-1} e^{i\lambda S_0(x)} \sum_{j=0}^{\infty} c_j^{(m-1)}(x) \lambda^{-j}.
\end{align*}
\]

Solving this problem would allow to find with arbitrarily good accuracy (for large \( \lambda \)) solutions of the equation \( Au = 0 \) with rapidly oscillating initial values that are finite segments of the series on the right-hand sides of (12.49).

A solution of this problem is not just a series of the form (12.45): there may be no such a series. We should take a finite sum of these series with different phases \( S(t, x) \). Thus, consider a sum

\[
u(t, x, \lambda) = \sum_{r=0}^{m} u^r(t, x, \lambda),
\]

where

\[
u^r(t, x, \lambda) = e^{i\lambda S^r(t, x)} \sum_{j=0}^{\infty} b_j^{(r)}(t, x) \lambda^{-j}.
\]
Sum (12.50) is called a solution of the equation \( Au = 0 \) if each \( u^r(t, x, \lambda) \) is a solution of this equation. Since all the series \( \frac{\partial^k u}{\partial t^k} \bigg|_{t=0} \) are of the form
\[
\lambda^k e^{i\lambda S_0(x)} \sum_{j=0}^{\infty} c_j(x) \lambda^{-j},
\]
the initial values \( u|_{t=0}, u_t|_{t=0}, \ldots, \frac{\partial^{m-1} u}{\partial t^{m-1}}|_{t=0} \) also are of the same form. Therefore, it makes sense to say that \( u(t, x, \lambda) \) satisfies the initial conditions (12.49).

For instance, we may consider a particular case when the series in (12.49) contain one term each:
\[
\begin{align*}
u|_{t=0} &= e^{i\lambda S_0(x)} \varphi_0(x), \\
\frac{\partial^{m-1} u}{\partial t^{m-1}}|_{t=0} &= \lambda^{m-1} e^{i\lambda S_0(x)} \varphi_{m-1}(x).
\end{align*}
\]
Cutting off the series for \( u \) at a sufficiently remote term, we see that any asymptotic solution provides (regular) functions \( u(t, x, \lambda) \), such that
\[
Au = O(\lambda^{-N}),
\]
\[
u|_{t=0} = e^{i\lambda S_0(x)} \varphi_0(x) + O(\lambda^{-N}),
\]
\[
\frac{\partial^{m-1} u}{\partial t^{m-1}}|_{t=0} = \lambda^{m-1} e^{i\lambda S_0(x)} \varphi_{m-1}(x) + O(\lambda^{-N}),
\]
where \( N \) may be taken arbitrarily large.

**Theorem 12.5.** The above Cauchy problem for the hyperbolic equation \( Au = 0 \) with initial conditions (12.49) is uniquely solvable in a small neighborhood of the initial plane \( t = 0 \).

**Proof** Each of the phases \( S^r(t, x) \) should satisfy the eikonal equation and the initial condition \( S|_{t=0} = S_0(x) \). There exist, as we have seen, exactly \( m \) such phases \( S^1, S^2, \ldots, S^m \). Now the transport equations for the amplitudes \( b^{(r)}_j(t, x) \) show that it suffices to set \( b^{(r)}_j|_{t=0} = b_{j,r}(x) \) to uniquely define all functions \( b^{(r)}_j(t, x) \). Thus we need to show that the initial conditions (12.49) uniquely determine \( b_{j,r}(x) \). Substitute \( u = \sum_{r=1}^{m} u^r \) into the initial conditions (12.49) and write the obtained equations for \( b^{(r)}_j(t, x) \) equating the coefficients of all the powers of \( \lambda \). The first of equations (12.49) gives
\[
\sum_{r=1}^{m} b^{(r)}_j|_{t=0} = c^{(0)}_j, \quad j = 0, 1, \ldots
\]
The second of equations (12.49) becomes

$$
\sum_{r=1}^{m} \left( \frac{i}{\partial t} S_r b_j^{(r)} \right)_{t=0} + \sum_{r=1}^{m} \left( \frac{\partial b_{r-1}^{(r)}}{\partial t} \right)_{t=0} = c_j^{(1)}, \quad j = 0, 1, \ldots
$$

In general, the equation which determines $\frac{\partial u}{\partial x} \bigg|_{t=0}$ becomes

(12.51)

$$
\sum_{r=1}^{m} \left( \frac{i}{\partial t} S_r b_j^{(r)} \right)_{t=0} = f_j^{(k)}(x), \quad k = 0, 1, \ldots, m - 1; \quad j = 0, 1, \ldots,
$$

where $f_j^{(k)}(x)$ only depends on $b_0^{(r)}, \ldots, b_{j-1}^{(r)}$. Now, note that

$$
\frac{\partial S_r}{\partial t} \bigg|_{t=0} = \tau_r \left( 0, x, \frac{\partial S_0}{\partial x} \right),
$$

where $\tau_r(0, x, \xi) \neq \tau_l(0, x, \xi)$ if $r \neq l$ and $\xi \neq 0$. Therefore, if $\frac{\partial S_0}{\partial x} \neq 0$ (which we assume to be the case in the considered problem), then the system (12.51) for a fixed $j$ is a system of linear equations with respect to $b_j^{(r)} \big|_{t=0} = b_{j,r}(x)$ with the determinant equal to the Vandermonde determinant

$$
\begin{vmatrix}
1 & \ldots & 1 \\
i \tau_1 & \ldots & i \tau_m \\
\vdots & \vdots & \vdots \\
(i \tau_1)^{m-1} & \ldots & (i \tau_m)^{m-1}
\end{vmatrix},
$$

where $\tau_r = \tau_r(0, x, \frac{\partial S_0}{\partial x})$. Since, due to hyperbolicity, $\tau_j \neq \tau_k$ for $j \neq k$, the determinant does not vanish anywhere (recall that we assumed $\frac{\partial S_0}{\partial x} \neq 0$). The right-hand sides of (12.51) depend only on $b_0^{(r)}, \ldots, b_{j-1}^{(r)}$. Therefore, the sequence of the systems of linear equations (12.51) enables us to uniquely determine all functions $b_j^{(r)}$ by induction. Namely, having found $b_{0,r}(x)$ for $r = 1, \ldots, m$ from (12.51) with $j = 0$, we may determine $b_0^{(r)}(t, x)$ from the transport equations with the initial conditions $b_0^{(r)}|_{t=0} = b_{0,r}(x)$. If $b_0, \ldots, b_{j-1}$ are determined, then the functions $b_j^{(r)}|_{t=0}$, $r = 1, \ldots, m$, are recovered from (12.51), and then the transport equations enable us to determine $b_j^{(r)}(t, x)$. □

Let us briefly indicate the connection of Theorem 12.5 with the behavior of the singularities for solutions of the Cauchy problem. It is well known that the singularities of $f(x)$ are connected with the behavior of its Fourier transform $\hat{f}(\xi)$ at infinity. For instance, the inversion formula

(12.52)

$$
f(x) = (2\pi)^{-n} \int \hat{f}(\xi)e^{ix\xi} \, d\xi,
$$
implies that if $|f(\hat{\xi})| \leq C(1 + |\xi|)^{-N}$, then $f \in C^{N-n-1}(\mathbb{R}^n)$. Therefore, to find the singularities of a solution $u(t,x)$ it suffices to solve the Cauchy problem up to functions whose Fourier transform with respect to $x$ decreases sufficiently rapidly as $|\xi| \to \infty$.

Formula (12.52) can be regarded as a representation of $f(x)$ in the form of a linear combination of exponents $e^{ik \cdot x}$. Since the problem is a linear one, to find solutions of the problem with one of the initial condition equal to $f(x)$, it suffices to consider a solution with the initial data $f(x)$ replaced by $e^{ix \cdot \xi}$ and then take the same linear combination of solutions.

If $f \in \mathcal{E}'(\mathbb{R}^n)$ and supp$f \subset K$, where $K$ is a compact in $\mathbb{R}^n$, it is more convenient to somewhat modify the procedure. Namely, let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ in a neighborhood of $K$. Then

\begin{equation}
(12.53) \quad f(x) = \varphi(x)f(x) = (2\pi)^{-n} \int \hat{f}(\xi)\varphi(x)e^{ix \cdot \xi}d\xi
\end{equation}

and it suffices to solve the initial problem by replacing $f(x)$ by $\varphi(x)e^{ix \cdot \xi}$. For instance, if $f(x) = \delta(x)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi(0) = 1$, then (12.53) holds with $\hat{f}(\xi) \equiv 1$; therefore, having solved the Cauchy problem

\begin{equation}
(12.54) \quad \begin{cases}
Au = 0, \\
u|_{t=0} = 0, \\
t_{t=0} = 0, \\
\frac{\partial^{m-1}u}{\partial t^{m-1}}|_{t=0} = \varphi(x)e^{ix \cdot \xi},
\end{cases}
\end{equation}

and having denoted the solution by $u_\xi(t,x)$, we can write

\[ \mathcal{E}(t,x) = (2\pi)^{-n} \int u_\xi(t,x)d\xi \]

(if the integral exists in some sense, e.g. converges for any fixed $t$ in the topology of $\mathcal{D}'(\mathbb{R}^n_z)$), and then $\mathcal{E}(t,x)$ is a solution of the Cauchy problem

\begin{equation}
(12.55) \quad \begin{cases}
A\mathcal{E} = 0, \\
\mathcal{E}|_{t=0} = 0, \\
\mathcal{E}_t|_{t=0} = 0, \\
\frac{\partial^{m-1}\mathcal{E}}{\partial t^{m-1}}|_{t=0} = \delta(x).
\end{cases}
\end{equation}

If we only wish to trace the singularities of $\mathcal{E}(x)$, we should investigate the behavior of $u_\xi(x)$ as $|\xi| \to +\infty$. We may fix the direction of $\xi$ introducing
a large parameter $\lambda = |\xi|$ and setting $\eta = \xi / |\xi|$. Then the Cauchy problem (12.54) takes on the form of the Cauchy problem with rapidly oscillating initial data. Namely, in (12.49) we should set

\[
S_0(x) = \eta \cdot x, \\
c_j^{(0)}(x) = c_j^{(1)}(x) = \ldots = c_j^{(m-2)}(x) = 0, \quad \text{for } j = 0, 1, \ldots; \\
c_j^{(m-1)}(x) = 0, \quad \text{for } j \neq m - 1, \\
c_{m-1}(x) = \varphi(x).
\]

We may find a solution $u_\xi(t, x) = u_\eta(t, x, \lambda)$ which satisfies (12.54) with accuracy up to $O(\lambda^{-N})$ for arbitrarily large $N$. Then (12.55) are satisfied up to functions of class $C^{N-n-1}(\mathbb{R}^n)$. It can be proved that singularities of the genuine solution of (12.55) are the same as for the obtained approximate solution, i.e., the genuine solution differs from the approximate one by an arbitrarily smooth function (for large values of $N$).

Where are the singularities of $E(t, x)$? To find out, we should consider the integral

\[
E_N(t, x) = (2\pi)^{-n} \int u_{\xi, N}(t, x) d\xi,
\]

where $u_{\xi, N}$ is the sum of the first $N + 1$ terms of the series which represents the asymptotic solution $u_\xi$. All these terms are of the form

\[
\lambda^k \int e^{i\lambda S^r(t, x, \eta)} \varphi(t, x, \eta) d\xi,
\]

where $\lambda = |\xi|$, $\eta = \xi / |\xi|$, $S^r(t, x, \eta)$ is one of the phase functions satisfying the eikonal equation with the initial condition $S^r|_{t=0} = S_0(x) = \eta \cdot x$; $\varphi(t, x, \eta)$ is infinitely differentiable with respect to all variables; $\varphi$ is defined for small $t$ and has the $(t, x)$-support belonging to an arbitrarily small neighborhood of the set of rays $(t, x(t))$ with the source in $(0, 0)$. The last statement is deduced from the form of the transport equations. Therefore we see that the support of $E_N(t, x)$ belongs to an arbitrarily small neighborhood of the set of rays passing through $(0, 0)$. Taking into account the fact that the singularities of $E$ and $E_N$ coincide (with any accuracy), we see that the distribution $E(t, x)$ is infinitely differentiable outside of the set of rays with $(0, 0)$ as the source. Here we should take all the rays corresponding to all the functions $S^r(t, x, \eta)$ for each $\eta$ and each $r = 1, \ldots, m$. We get the set of $m$ curved cones formed by all rays starting from $(0, 0)$ (for each $\eta \in \mathbb{R}^n$, $|\eta| = 1$, and each $r = 1, \ldots, m$ we get a ray smoothly depending on $\eta$).
A more detailed description is left to the reader as a useful exercise which should be worked out for 2nd order equations, for example, for the equation (12.36) describing light propagation in a non-homogeneous medium.

Problems

12.1. For the equation \( u_{tt} = a^2 \Delta u \), where \( u = u(t, x) \), \( x \in \mathbb{R}^2 \), find the characteristics whose intersection with the plane \( t = 0 \) is

a) the line \( \eta \cdot x = 0 \), where \( \eta \in \mathbb{R}^2 \setminus \{0\} \); 
b) the circle \( \{ x : |x| = R \} \).

12.2. Describe the relation between the rays and characteristics of the equation \( u_{tt} = c^2(x)u_{xx} \).

12.3. For the Cauchy problem

\[
\begin{cases}
    u_{tt} = x^2 u_{xx} \\
    u|_{t=0} = 0 \\
    u_t|_{t=0} = e^{i\lambda x}
\end{cases}
\]

write and solve the eikonal equation and the first transport equation. Use this to find the function \( u(t, x, \lambda) \), such that for \( x \neq 0 \), we have as \( \lambda \to \infty \):

\[
\begin{cases}
    u_{tt} - x^2 u_{xx} = O(\lambda^{-1}) \\
    u|_{t=0} = O(\lambda^{-2}) \\
    u_t|_{t=0} = e^{i\lambda x} + O(\lambda^{-1}).
\end{cases}
\]
Answers and Hints.

Solutions

1.1. a) \( u_{pp} + u_{qq} + u_{rr} = 0 \); the change of variables is:
\[
\begin{align*}
    p &= x, \\
    q &= -x + y, \\
    r &= x - \frac{1}{2}y + \frac{1}{2}z.
\end{align*}
\]
(The reader should keep in mind that a change of variables leading to the canonical form is not unique; the above one is one of them, and may very well happen that the reader will find another one).

b) \( u_{pp} - u_{qq} + 2u_p = 0 \); the change of variables is:
\[
\begin{align*}
    p &= x + y, \\
    q &= x - y, \\
    r &= y + z.
\end{align*}
\]

1.2. a) \( u_{zw} - \frac{1}{w}u_z - \frac{1}{w}u_w - \frac{1}{z^2}u = 0 \); the change of variables is:
\[
\begin{align*}
    z &= yx^{-3}, \\
    w &= xy.
\end{align*}
\]

b) \( u_{zz} + u_{ww} + \frac{1}{z+w}u_z + \frac{1}{2w}u_w = 0 \); the change of variables is:
\[
\begin{align*}
    z &= y^2 - x^2, \\
    w &= x^2.
\end{align*}
\]
c) \( u_{ww} + 2u_z + u_w = 0 \); the change of variables is:
\[
\begin{aligned}
z &= x + y, \\
w &= x.
\end{aligned}
\]

1.3. a) \( u(x, y) = f\left(\frac{x}{y}\right) + \sqrt{\frac{x}{y}} g(xy) \), where \( f, g \) are arbitrary functions of one independent variable.

\( \text{Hint.} \) The change of variables
\[
\begin{aligned}
z &= \ln |xy|, \\
w &= \ln |y/x|,
\end{aligned}
\]
reduces the equation to the form \( 2u_w + u_z = 0 \), i.e., \( \frac{\partial}{\partial z}(2u_w + u) = 0 \) which implies \( 2u_w + u = F(w) \) and \( u(z, w) = a(w) + \exp(-w/2)b(z) \).

b) \( u(x, y) = \ln |y/x| f(xy) + g(xy) \), where \( f, g \) are arbitrary functions of one independent variable.

\( \text{Hint.} \) The change of variables:
\[
\begin{aligned}
z &= \ln |xy|, \\
w &= \ln |y/x|,
\end{aligned}
\]
reduces the equation to the form \( u_{ww} = 0 \), hence \( u(z, w) = w\varphi(z) + \psi(z) \).

2.1. a) \( u_x|_{x=0} = 0 \)

\( \text{Hint.} \) Write the dynamical Newton equation of the motion of the ring.

b) \( (mu_{tt} - Tu_x)|_{x=0} = 0 \).

\( \text{Hint.} \) See Hint to a).

c) \( (mu_{tt} + \alpha u_t - Tu_x)|_{x=0} = 0 \).

2.2. a) \( E(t) = \frac{1}{2} \int_0^l [\rho u_t^2(t, x) + Tu_x^2(t, x)] dx = E_0 = \text{const} \).

b), c) Let
\[
E(t) = \frac{1}{2} mu_t^2|_{x=0} + \int_0^l [\rho u_t^2(t, x) + Tu_x^2(t, x)] dx.
\]
Then \( E(t) \equiv E_0 = \text{const in case b)} \) and \( E(t) - E(0) = A_{fr} = -\int_0^t \alpha u_t^2|_{x=0} dt \) in case c) (here \( A_{fr} \) is the work of the friction force).

2.3. \( \rho u_{tt} = Eu_{xx} \) or \( u_{tt} = a^2 u_{xx} \) with \( a = \sqrt{E/\rho} \). Here \( u = u(t, x) \) is a longitudinal displacement of a point with position \( x \) in equilibrium, \( \rho \) the volume density of the material of the rod, \( E \) the Young modulus.
(called also the modilus of elongation) which describes the force arising in the deformed material by the formula $F = ES\frac{\Delta l}{l}$ (Hook’s law), $S$ the cross-section where the force is measured, $\Delta l/l$ the deformation of the small piece of material around the point of measurement ($\Delta l/l$ is often called the relative lengthening, here $l$ is the length of the piece in equilibrium position and $\Delta l$ is the increment of this length caused by the force).

**Hint.** Prove that the relative lengthening of the rod at the point with position $x$ in equilibrium is $u_x = u_x(t, x)$ so the force $F = ESu_x(t, x)$ is acting on the left part of the rod in the corresponding cross-section at the point $x$. Consider the motion of the part $[x, x + \Delta x]$ of the rod, write Newton’s dynamical equation for this part and then take the limit as $\Delta x \to 0$.

2.4. $u_x|_{x=0} = 0$.

2.5. $(ESu_x - ku)|_{x=0} = 0$. Here $k$ is the rigidity coefficient of the spring, i.e. the force per unit lengthening of the spring.

2.6. a), b)

$$E(t) = \frac{1}{2} \int_0^L \left[ \rho u_t^2(t, x) + E u_x^2(t, x) \right] dx = E_0 = \text{const}.$$  

$c) E(t) \equiv \frac{1}{2} ku^2|_{x=0} + \frac{1}{2} \int_0^L \left[ \rho u_t^2(t, x) + ESu_x^2(t, x) \right] dx = E_0 = \text{const}.$

2.7. $\rho S(x) u_{tt} = \frac{\partial}{\partial x} \left( ES(x) \frac{\partial u}{\partial x} \right)$, where $S(x)$ is the area of the cross-section of the rod at the point with position $x$ in equilibrium (the $x$-axis should be directed along the axis of the rod).

2.9. $E(t) \equiv \frac{1}{2} \int_{-\infty}^\infty \left[ \rho u_t^2(t, x) + Tu_x^2(t, x) \right] dx = E_0 = \text{const}$ (perhaps $+\infty$ for all $t$)

**Hint.** Using Problem 2.8 prove that $E(t') \leq E(t)$ for $t' \geq t$, and then reverse the direction of the time-variable.

2.10. See Fig. 1.

2.11. $u \equiv 0$

**Hint.** The Cauchy initial conditions are $u|_{x=x_0} \equiv 0$, $\frac{\partial u}{\partial x}|_{x=x_0} \equiv 0$.

2.12. $u = \begin{cases} f(x - at) & \text{for } x > x_0 + \varepsilon, \text{ where } f(x) = 0 \text{ for } x < x_0 + \varepsilon, \\ g(x + at) & \text{for } x < x_0 - \varepsilon, \text{ where } g(x) = 0 \text{ for } x > x_0 - \varepsilon. \end{cases}$

2.13. See Fig. 2.

2.14. See Fig. 3.
13. Answers and Hints. Solutions

Figure 1

Hint. Use the formula
$$\frac{1}{2a} \int_{x-at}^{x+at} \psi(s)ds = \Psi(x+at) - \Psi(x-at),$$
where $\Psi(z) = \frac{1}{2a} \int_0^z \psi(s)ds$.

The graphs of $\Psi(x+at)$ and $\Psi(x-at)$ are drawn on Fig. 3 by dashed lines.

2.15. $E(t) \equiv \frac{1}{2} \int_0^\infty (\rho u_t^2(t,x) + Tu_x^2(t,x))dx = E_0 = \text{const.}$

Hint. See Hint to Problem 2.9.

2.16. The reflected wave is
$$\left(\frac{k^2}{E^2S^2} + \frac{\omega^2}{a^2}\right)^{-1} \left\{ \frac{2\omega k}{aES} \left[ \cos \omega \left( t - \frac{x}{a} \right) - \exp \frac{k}{ES} (x - at) \right] + \left( \frac{\omega^2}{a^2} - \frac{k^2}{E^2S^2} \right) \sin \omega \left( t - \frac{x}{a} \right) \right\},$$
with notations as in the answer to Problem 2.3.

2.17. On the left $u = \sin \omega \frac{x+at}{a+v}$ with the frequency $\omega_l = \frac{\omega a}{a+v} < \omega$.

On the right $u = \sin \omega \frac{x-at}{v-a}$ with the frequency $\omega_r = \frac{\omega a}{a-v} > \omega$.

2.18. The standing waves are $X_0(x)(at+b)$ and $X_n(x)(a_n \cos \omega_n t + b_n \sin \omega_n t)$ for $n = 1, 2, \ldots$ with $X_n(x) = \cos \frac{n\pi x}{l}$ and $\omega_n = \frac{n\pi a}{l}$.

The graphs of the first $X_n(x)$, $n = 1, 2, 3$, see on Fig. 4.

2.19. The boundary conditions are $u|_{x=0} = 0$ and $(ESu_x + ku)|_{x=l} = 0$. The standing waves are $X_p(x)(a_p \cos \omega_p t + b_p \sin \omega_p t)$, where $X_p(x) = \sin \frac{\omega_p x}{l}$, $\omega_p = \gamma_p a/l$, and the $\gamma_p$ are the solutions of the equation
$$\cot \gamma = - \frac{kl}{ES \gamma}.$$
such that $\gamma_p \in (p\pi, (p + 1)\pi)$ for $p = 0, 1, 2, \ldots$. The system $\{X_p : p = 0, 1, \ldots\}$ is orthogonal in $L^2([0, l])$, and every function $X_p$ is an eigenfunction of the operator $L = -d^2/dx^2$ with the boundary conditions $X(0) = 0, X'(0) + \alpha X(0) = 0$ for $\alpha = k/(ES)$. The corresponding eigenvalue is $\lambda_p = (\gamma_p/l)^2$ and its shortwave asymptotic is

$$
\lambda_p = \left[ \frac{\pi(2p + 1)}{2l} \right]^2 \left( 1 + O\left( \frac{1}{p^2} \right) \right).
$$

Figure 2
2.20. a) The resonance frequencies are \( \omega_k = \frac{(2k+1)\pi a}{2l}, k = 0, 1, 2, \ldots \); the condition for a resonance is \( \omega = \omega_k \) for some \( k \). If \( \omega \neq \omega_k \) then

\[
\begin{align*}
  u(t, x) &= \frac{F_0 a}{E S \omega_0 \cos \frac{\pi x}{a}} \sin \frac{\omega x}{a} \cdot \sin \omega_0 t + \\
                 &\sum_{k=0}^{\infty} (-1)^k \frac{2F_0 \omega^2}{\omega_k IS(\omega^2 - \omega_k^2)} \sin \omega_k t \cdot \sin \frac{(2k+1)\pi x}{2l}.
\end{align*}
\]
13. Answers and Hints. Solutions

If $\omega = \omega_k$ for some $k$, then

$$u(t, x) = (-1)^k \frac{F_0 a^2}{ESl} \left[ \frac{3}{2} \sin \omega t \cdot \sin \frac{\omega}{a} x - \frac{\omega}{a} \sin \omega t \cdot \cos \frac{\omega}{a} x - \omega t \cos \omega t \cdot \sin \frac{\omega}{a} x \right] +$$

$$\sum_{p \geq 0, p \neq k} (-1)^p \frac{2F_0 \omega a^2}{\omega_p l E S (\omega^2 - \omega_p^2)} \sin \frac{(2p+1)\pi x}{l}.$$

**Hint.** Seek a particular solution in the form $u_0 = X(x) \sin \omega t$ satisfying the boundary conditions (but not the initial conditions), then find $v = u - u_0$ by the Fourier method in case $\omega \neq \omega_k$. In case of a resonance $\omega = \omega_k$, take the limit of the non-resonance formula for $u$ as $\omega \to \omega_k$.

b) Resonance frequencies is $\omega_k = \frac{k\pi a}{l}$, $k = 1, 2, \ldots$, the condition for a resonance is $\omega = \omega_k$ for some $k$. If $\omega \neq \omega_k$, then

$$u(t, x) = - \frac{F_0 a}{ES \omega \sin \frac{\omega}{a} x} \sin \omega t \cdot \cos \frac{\omega}{a} x + \frac{F_0 a^2 t}{ESl \omega} +$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{2F_0 \omega a^2}{\omega_k l E S (\omega^2 - \omega_k^2)} \sin \omega_k t \cdot \cos \frac{k\pi x}{l}.$$

If $\omega = \omega_k$ for some $k$, then

$$u(t, x) = (-1)^k \frac{F_0 a^2}{ESl} \left[ \frac{3}{2} \sin \omega t \cdot \sin \frac{\omega}{a} x - \frac{1}{\omega} \sin \omega t \cdot \cos \frac{\omega}{a} x + \omega t \cos \omega t \cdot \sin \frac{\omega}{a} x \right] +$$

$$\sum_{p \geq 1, p \neq k} (-1)^p \frac{2F_0 \omega a^2}{\omega_p l E S (\omega^2 - \omega_p^2)} \sin \omega_p t \cdot \cos \frac{p\pi x}{l}.$$

**Hint.** See Hint to a).
2.21.  

a) The resonance frequencies are \( \omega_k = \frac{k\pi a}{l} \) for \( k = 1, 2, \ldots \); the condition for a resonance is \( \omega = \omega_k \) for some \( k \). If \( \omega \neq \omega_k \), then

\[
    u(t, x) = A \sin \frac{\omega x}{a} \sin \omega t + \sum_{k=1}^{\infty} \frac{2\omega A a}{l(\omega^2 - \omega_k^2)} \sin \omega_k t \cdot \sin \frac{k\pi x}{l}.
\]

If \( \omega = \omega_k \) for some \( k \), then

\[
    u(t, x) = (-1)^k \frac{A a}{l} \left[ \frac{1}{2\omega} \sin \omega t \cdot \sin \frac{\omega x}{a} + \frac{\omega}{a} \sin \omega t \cdot \cos \frac{\omega x}{a} + \frac{1}{2}\sin \omega t \cdot \sin \frac{\omega x}{a} \right] + \sum_{p \geq 1, p \neq k} (-1)^p \frac{2\omega A a}{l(\omega^2 - \omega_p^2)} \sin \omega_p t \cdot \sin \frac{p\pi x}{l}.
\]

**Hint.** See Hint to 2.20 a).

b) The resonance frequencies are \( \omega_k = \frac{(2k+1)\pi a}{2l} \), \( k = 0, 1, 2, \ldots \), the condition for a resonance is \( \omega = \omega_k \) for some \( k \). If \( \omega \neq \omega_k \), then

\[
    u(t, x) = A \cos \frac{\omega x}{a} \sin \omega t + \sum_{k=0}^{\infty} (-1)^k \frac{2\omega A a}{l(\omega^2 - \omega_k^2)} \sin \omega_k t \cdot \cos \left( \frac{(2k+1)\pi a}{2l} \right).
\]

If \( \omega = \omega_k \) for some \( k \), then

\[
    u(t, x) = (-1)^k \frac{A a}{l} \left[ \frac{1}{2\omega} \sin \omega t \cdot \cos \frac{\omega x}{a} + \frac{\omega}{a} \sin \omega t \cdot \sin \frac{\omega x}{a} - t \cos \omega t \cdot \cos \frac{\omega x}{a} \right] + \sum_{p \geq 0, p \neq k} (-1)^p \frac{2\omega A a}{l(\omega^2 - \omega_p^2)} \sin \omega_p t \cdot \cos \left( \frac{(2p+1)\pi a}{2l} \right).
\]

**Hint.** See Hint to 2.20 a).

c) The resonance frequencies are \( \omega_p = \gamma_p a/l \), where \( \gamma_p \) are as in answer to 2.19, \( p = 1, 2, \ldots \). The condition for a resonance is \( \omega = \omega_p \) for some \( p \). If \( \omega \neq \omega_p \), then with \( \gamma = \omega l/a \):

\[
    u(t, x) = A(\cos \frac{\omega x}{a} + \frac{ka}{ES\gamma} \sin \frac{\omega x}{a})^{-1} \left[ \cos \frac{\omega t}{a} + \frac{ka}{ES\gamma} \sin \frac{\omega t}{a} \right] \sin \omega t + \sum_{p=0}^{\infty} \frac{4A\gamma_p \sin \gamma_p}{(2\gamma_p - \sin(2\gamma_p)) \sin \frac{\gamma_p x}{a}} \sin \omega_p t \cdot \left[ \cos \frac{\gamma_p t}{l} + \frac{kl}{ES\gamma_p} \sin \frac{\gamma_p x}{a} \right].
\]

If \( \omega = \omega_p \) for some \( p \), then

\[
    u(t, x) = \frac{2\gamma A}{2\gamma - \sin 2\gamma} \left[ \frac{\omega}{a} \sin \omega t \cdot \cos(\gamma(\frac{\omega}{a} - 1)) + \frac{at}{l} \cos \omega t \cdot \sin(\gamma(\frac{\omega}{a} - 1)) \right] + \frac{A\gamma(1-\cos(2\gamma)) \sin \omega t \sin(\gamma(\frac{\omega}{a} - 1))}{(2\gamma - \sin 2\gamma) \gamma_p} \sin \omega_p t \cdot \left[ \cos \frac{\gamma_p t}{l} + \frac{kl}{ES\gamma_p} \sin \frac{\gamma_p x}{a} \right].
\]

**Hint.** See Hint to 2.20 a).
3.1. \( \Phi(x) = \cos kx + \frac{1}{k} \int_0^x \sin k(x-t)q(t)\Phi(t)dt; \)
\[
\Phi(x) = \cos kx + O(\frac{1}{k}) = \\
\cos kx + \frac{1}{k} \int_0^x \sin k(x-t) \cos kt q(t)dt + O(\frac{1}{k^2}); \\
\Phi'(x) = -k \sin kx + O(1) = \\
-k \sin kx + \int_0^x \cos k(x-t) \cos kt q(t)dt + O(\frac{1}{k}).
\]

3.2. **Hint.** The values of \( k \) satisfying \( \Phi(l) = \Phi(l,k) = 0 \) can be found by implicit function theorem near \( k \) such that \( \cos kl = 0 \). Differentiation with respect to \( k \) of the integral equation of Problem 3.1 gives an integral equation for \( \partial \Phi/\partial k \) providing information needed to apply the implicit function theorem.

3.3. \( G(x,\xi) = \frac{1}{l}[\theta(\xi-x)x(l-\xi) + \theta(x-\xi)(l-x)\xi]. \)

A physical interpretation: \( G(x,\xi) \) is the form of a string which is pulled at point \( \xi \) by the vertical point force \( F = T \), where \( T \) is the tension force of the string.

3.4. \( G(x,\xi) = \frac{1}{\sinh l}[\theta(\xi-x)\cosh x \cdot \cosh(l-\xi) + \theta(x-\xi) \cosh(l-x) \cdot \cosh \xi]. \)

A physical interpretation: \( G(x,\xi) \) is the form of a string which is pulled at a point \( \xi \) by the vertical point force \( F = T \), has free ends (ends which are allowed to move freely in the vertical direction) and is subject to the action of an elastically returning uniformly distributed force; see Fig. 5, where the returning force is realized by the little springs attached to the points of the string and to a rigid massive body.

3.6. **Hint.** A solution \( y = y(x) \not= 0 \) of the equation \(-y'' + q(x)y = 0\) with \( q \geq 0 \) has at most one root.

3.7. **Hint.** Use the inequality \((G\varphi,\varphi) \geq 0\) for \( G = L^{-1} \); take
\[
\varphi(x) = \sum_{j=1}^n c_j \varphi_\varepsilon(x-x_j)
\]
with a $\delta$-like family $\varphi$.

3.8. a) $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \int_0^1 G(x,x)dx.$

**Hint.** Use the expansion

$$G(x,\xi) = \sum_{j=1}^{\infty} \frac{X_j(x)X_j(\xi)}{\lambda_j}$$

which can be obtained, e.g. as the orthogonal expansion of $G(\cdot,\xi)$ with respect to $\{X_j(x), j = 1,2,\ldots\}$.

b) $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \int_0^1 \int_0^1 |G(x,\xi)|^2 dxd\xi.$

**Hint.** This is the Parseval identity for the orthogonal expansion of $G = G(x,\xi)$ with respect to the system

$$\{X_j(x)X_k(\xi) : j,k = 1,2,\ldots\}.$$

In the particular case $L = -d^2/dx^2$ on $[0,\pi]$ with boundary conditions $X(0) = X(\pi) = 0$ we obtain $\lambda_n = n^2$, and taking into account the answer to Problem 3.3, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

4.1. $0$.

4.2. $2\delta(x)$.

4.3. $\pi\delta(x)$.

4.4. **Hint.** $f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)dt = x \int_0^1 f'(tx)dt$.

4.5. **Hint.** Write explicit formulas for the map $\mathcal{D}'(S^1) \to \mathcal{D}'_c(\mathbb{R})$ and its inverse.

4.6. **Hint.** Use the continuity of a distribution from $\mathcal{D}'(S^1)$ with respect to a seminorm in $C^\infty(S^1)$.

4.7. **Hint.** Use duality considerations.

4.8. $\sum_{k \in \mathbb{Z}} \delta(x+2k\pi) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{imx}$.

4.9. **Hint.** Multiply the above formula in Answer to 4.9 by $f(x)$ and integrate.

4.10. a) $u(x) = v.p. \frac{1}{x} + C \delta(x) = \frac{1}{x+i0} + C_1 \delta(x) = \frac{1}{x-i0} + C_2 \delta(x)$.

b) $u(x) = \ln |x| + C \theta(x) + C_1$. 

13. Answers and Hints. Solutions

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c) \( u(x) = -\delta(x) + C\theta(x) + C_1 \).

5.1. a) \(-2\pi i\theta(\xi)\).

b) \( 4\pi r_0 \frac{\sin r_0|\xi|}{|\xi|} \).

**Hint.** Use the spherical symmetry and take
\[
\xi = |\xi|(1,0,0) = (\xi_1,0,0).
\]

c) \( \frac{2\pi^2}{\rho^2} \delta(|\xi| - \rho_0) \).

**Hint.** Use the Answer to b).

d) \( \frac{(-1)^k+1}{(\xi-\xi_0)^{k+1}} \).

5.2. a) \( \frac{1}{(n-2)\sigma_{n-1}|x|^{n-2}} \).

**Hint.** It should be a spherically symmetric fundamental solution for 
\( (-\Delta) \).

b) \( \frac{1}{4\pi|x|} e^{-k|x|} \).

**Hint.** Using the spherical symmetry take \( \xi = (\xi_1,0,0) \). Use the regularization with multiplication by \( \exp(-\varepsilon|\xi|) \). Calculate the arising 1-dimensional integral using residues.

c) \( \frac{1}{4\pi|x|} e^{-ik|x|} \).

**Hint.** Calculate \( F^{-1}\left(\frac{1}{|\xi|^2-k^2+i\varepsilon}\right) \) as in b) and pass to the limit as \( \varepsilon \to +0 \).

5.3. \( \frac{1}{4\pi|x|} e^{-k|x|} \).

5.6. 0 as \( t \to +\infty \) and \(-2\pi i\delta(x)\) as \( t \to -\infty \).

6.1. Prove the statements inverse to the ones of Theorems 6.1, 6.2: if a
continuous function \( u \) in an open set \( \Omega \subset \mathbb{R}^n \) has mean value property for
all balls \( B \) with \( \bar{B} \subset \Omega \) or for all spheres which are boundaries of such
balls (that is, either (6.2) holds for such spheres or (6.4) holds for all such
balls), then \( u \) is harmonic in \( \Omega \). Moreover, it suffices to take balls or spheres,
centered at every point \( x \) with sufficiently small radii \( r \in (0,\varepsilon(x)) \), where
\( \varepsilon(x) > 0 \) continuously depends upon \( x \in \Omega \).

6.2. Let \( u \) be a real-valued \( C^2 \) function on an open ball \( B = B_r(x) \subset \mathbb{R}^n \),
which extends to a continuous function on the closure \( \bar{B} \), and \( \Delta u \geq 0 \)
everywhere in \( B \). Prove that
\[
(13.1) \quad u(x) \leq \frac{1}{\text{vol}_{n-1}(S)} \int_S u(y) dS_y = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{S_r(x)} u(y) dS_y,
\]
and
\begin{equation}
(13.2) \quad u(x) \leq \frac{1}{\text{vol}(B)} \int_B u(y) dy = \frac{n}{\sigma_{n-1} r^n} \int_{B_r(x)} u(y) dy,
\end{equation}
where the notations are the same as in Section 6.1.

6.3. Prove the inverse statement to the one in the previous problem: if \( u \in C^2(\Omega) \) is real-valued and for all balls \( B \) with \( \bar{B} \subset \Omega \) (or for all spheres which are boundaries of such balls), (13.2) (respectively, (13.1)) holds, then \( \Delta u \geq 0 \) in \( \Omega \). Moreover, it suffices to take balls or spheres, centered at every point \( x \) with sufficiently small radii \( r \in (0, \varepsilon(x)) \), where \( \varepsilon(x) > 0 \) continuously depends upon \( x \in \Omega \).

Remark 13.1. Let a real-valued function \( u \in C^2(\Omega) \) be given. It is called \textit{subharmonic}, subharmonic function if \( \Delta u \geq 0 \) in \( \Omega \). Note, however, that the general notion of subharmonicity does not require that \( u \in C^2 \) and allows more general functions \( u \) (or even distributions, in which case \( \Delta u \geq 0 \) means that \( \Delta u \) is a positive Radon measure).

6.4. Let \( u \) be a real-valued subharmonic function in a connected open set \( \Omega \subset \mathbb{R}^n \). Prove that \( u \) cannot have local maxima in \( \Omega \), unless it is constant. In other words, if \( x_0 \in \Omega \) and \( u(x_0) \geq u(x) \) for all \( x \) in a neighborhood of \( x_0 \), then \( u(x) = u(x_0) \) for all \( x \in \Omega \).

6.5. Consider the initial value problem (6.11) with \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( \psi \equiv 0 \). Assume that it has a solution \( u \). Prove that in this case \( \varphi \equiv 0 \) on \( \mathbb{R} \) and \( u \equiv 0 \) on \( \Pi_T \).

6.6. Let \( U, V, W \) be open sets in \( \mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p \) respectively, and \( g : U \to V, f : V \to W \) be maps which are locally \( C^{k,\gamma} \) (that is, they belong to \( C^{k,\gamma} \) on any relatively compact subdomain of the corresponding domain of definition).

(a) Assume that \( k \geq 1 \). Then prove that the composition \( f \circ g : U \to W \) is also locally \( C^{k,\gamma} \).

(b) Prove that the statement (a) above does not hold for \( k = 0 \).

(c) Prove that if \( k \geq 1, \Omega \) is a domain in \( \mathbb{R}^n \) with \( \partial \Omega \in C^{k,\gamma} \), \( U \) is a neighborhood of \( \Omega \) in \( \mathbb{R}^n \), and \( \psi : U \to \mathbb{R}^n \) is a \( C^{k,\gamma} \) diffeomorphism of \( U \) onto a domain \( V \subset \mathbb{R}^n \), then the image \( \Omega' = \psi(\Omega) \) is a domain with a \( C^{k,\gamma} \) boundary, \( \overline{\Omega'} \subset V \).
6.7. (a) Prove that if $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $\partial \Omega \in C^{k, \gamma}$ with $k \geq 1$, then $\partial \Omega$ is a $C^{k, \gamma}$ manifold, $\dim \partial \Omega = n - 1$.

(b) Prove that for any non-negative integers $k, k'$ and numbers $\gamma, \gamma' \in (0, 1)$, such that $k + \gamma \geq k' + \gamma'$, $k \geq 1$, the class $C^{k', \gamma'}(\partial \Omega)$ is locally invariant under $C^{k, \gamma}$ diffeomorphisms. In particular, function spaces $C^{k', \gamma'}(\partial \Omega)$ are well defined.

(c) Under the same assumptions as in (a) and (b), for a real-valued function $\varphi : \partial \Omega \to \mathbb{R}$, the inclusion $\varphi \in C^{k', \gamma'}(\partial \Omega)$ is equivalent to the existence of an extension $\hat{\varphi} \in C^{k', \gamma'}(\Omega)$, such that $\hat{\varphi}|_{\partial \Omega} = \varphi$.

6.8. Assume that $k \geq 1$ is an integer, $\Omega$ is a bounded domain in $\mathbb{R}^n$, such that $\partial \Omega \in C^{k+2}$, $f \in C^k(\bar{\Omega})$ and $\varphi \in C^{k+2}(\partial \Omega)$. Using Schauder’s Theorem 6.20, prove that the problem (6.10) has a solution $u \in C^{k+1}(\bar{\Omega})$.

6.9. Assume that $\partial \Omega \in C^2$, and there exists Green’s function $G = G(x, y)$ of $\Omega$, such that $G \in C^2$ for $x \neq y$, $x, y \in \bar{\Omega}$. Consider an integral representation of a function $u \in C^2(\bar{\Omega})$ given by the formula (6.28), with $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial \Omega)$. Prove that the pair $\{f, \varphi\}$, representing a fixed function $u \in C^2(\bar{\Omega})$ by (6.28), is unique.

**Hint.** The function $f$ can be recovered from $u$ by the formula $f = \Delta u$, whereas at any regular point of the boundary $x \in \partial \Omega$, $\varphi$ is the jump of the outward normal derivative of $u$ on $\partial \Omega$:

$$\varphi(x) = \lim_{\varepsilon \to 0^+} \left( \frac{\partial u}{\partial n}(x + \varepsilon n) - \frac{\partial u}{\partial n}(x - \varepsilon n) \right).$$

6.10. Provide detailed proofs of identities (6.44) and (6.45).

6.11. Provide detailed proofs of identities (6.46), (6.47) and (6.48).

7.1. $u(t, x) = \frac{u_1 + u_2}{2} + \sum_{p=0}^{\infty} \frac{2(u_1 - u_2)(-1)^p}{(2p+1)\pi} \exp\left[-\left(\frac{(2p+1)\pi a}{l}\right)^2 t\right] \cos\left(\frac{(2p+1)\pi x}{l}\right)$.

Relaxation time is $t_r \simeq \frac{l^2}{4\pi^2\rho a} \simeq \frac{l^2}{40a^2} \simeq \frac{l^2 c p}{40 k}$, where $k$ is the thermal conductivity, $c$ the specific heat (per units of mass and temperature), $\rho$ the density (mass per volume), $l$ the length of the rod.

**Hint.** The first term in the sum above is much greater than the other terms at times comparable with the relaxation time.
7.2. Let $I$ be the current, $R$ the electric resistance of the rod, $V$ the volume of the rod, $k$ the thermal conductivity, and $c_p = \frac{2}{l} \int_0^l \left[ \phi(x) - \frac{I^2 R}{2V k} x(l-x) \right] \sin \frac{p\pi x}{l} \, dx$, where $\phi = u|_{t=0}$.

Then

$$u(t, x) = \frac{I^2 R}{2V k} x(l-x) + \sum_{p=1}^{\infty} c_p \exp \left[ - \left( \frac{p\pi a}{l} \right)^2 t \right] \sin \frac{p\pi x}{l}$$

and

$$\lim_{t \to +\infty} u(t, x) = \frac{I^2 R}{2V k} x(l-x).$$

**Hint.** The equation describing the process is

$$u_t = a^2 u_{xx} + \frac{I^2 R}{c_p V}$$

7.3. \( \lim_{t \to +\infty} u(t, x) = \frac{b+c}{x} \).

**Hint.** Consider Poisson’s integral giving an explicit formula for $u(t, x)$.

8.1. $u(x, y) = a^2 (x^2 - y^2)$.

**Hint.** The Fourier method gives in the polar coordinates

$$u(r, \varphi) = a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos k\varphi + b_k \sin k\varphi).$$

8.4. $u(t, x) = \frac{1}{12} (x^4 - y^4) - \frac{a^2}{12} (x^2 - y^2)$.

**Hint.** Find a particular solution $u_p$ of $\Delta u = x^2 - y^2$ and then seek $v = u - u_p$.

8.5. In the polar coordinates

$$u(r, \varphi) = 1 + \frac{b}{2} \ln \frac{r}{a} + \frac{b^3}{4(a^4 + b^4)} (r^2 - a^4 r^{-2}) \cos 2\varphi.$$

**Hint.** The Fourier method gives in the polar coordinates

$$u(r, \varphi) = A_0 + B_0 \ln r + \sum_{k=1}^{\infty} [(A_k r^k + B_k r^{-k}) \cos k\varphi + (C_k r^k + D_k r^{-k}) \sin k\varphi].$$

8.6. The answer to the last question is

$$u(x, y) = B \frac{\sinh \frac{\pi b y}{a}}{\sinh \frac{\pi b}{a}} \sin \frac{\pi x}{a} +$$
Hint. The general scheme is as follows.

Step 1. Reduce the problem to the case

\[
\varphi_0(0) = \varphi_0(a) = \varphi_1(0) = \psi_0(0) = \psi_0(b) = \psi_1(0) = \psi_1(b) = 0
\]

by subtracting a function of the form \( A_0 + A_1 x + A_2 y + A_3 xy \).

Step 2. Seek \( u \) in the form \( u = u_1 + u_2 \) with \( u_1, u_2 \) such that \( \Delta u_1 = \Delta u_2 = 0; \) \( u_1 \) satisfies the Dirichlet boundary conditions with \( \psi_0 = \psi_1 = 0 \) and the same \( \varphi_0, \varphi_1 \) as for \( u \); \( u_2 \) satisfies the Dirichlet boundary conditions with \( \varphi_0 = \varphi_1 = 0 \) and the same \( \psi_0, \psi_1 \) as for \( u \).

Then

\[
\begin{align*}
  u_1(x, y) &= \sum_{k=1}^{\infty} (C_k e^{-k\pi x} + D_k e^{-k\pi x}) \sin \frac{k\pi y}{b} = \\
  u_2(x, y) &= \sum_{k=1}^{\infty} A_k \sin \frac{k\pi e^{-k\pi x}}{b} + B_k \sin \frac{k\pi (a-x)}{b} \sin \frac{k\pi y}{b},
\end{align*}
\]

where

\[
\begin{align*}
  A_k &= \frac{2}{a} \int_a^b \varphi_1(x) \sin \frac{k\pi x}{a} \, dx, \\
  B_k &= \frac{2}{a} \int_a^b \psi_0(x) \sin \frac{k\pi x}{a} \, dx,
\end{align*}
\]

8.7. Hint. If \( \hat{u}(\xi, y) = \int e^{-i\xi x} u(x, y) \, dx \), then \( -\xi^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2} = 0 \) and \( \hat{u}(\xi, y) = C_1(\xi) \exp(-|\xi|y) + C_2(\xi) \exp(|\xi|y) \). The second summand should vanish almost everywhere if we want \( u \) to be bounded. Hence, \( \hat{u}(\xi, y) = \hat{\varphi}(\xi) \exp(-|\xi|y) \) and \( u(x, y) = \frac{1}{2\pi} \int e^{-|\xi|y+i(\xi-\xi_0)\xi} \varphi(z) \, dz \, d\xi \). Now, change the order of integration.

8.8. \( \alpha \in \mathbb{Z}_+ \) or \( \alpha - s > -n/2 \).

8.9. No.

Hint. Use Friedrichs’ inequality.

8.10. Hint. Work with the Fourier transform. Show that the question can be reduced to the case of a multiplication operator in \( L^2(\mathbb{R}^n) \).

9.1. Physical interpretation: \( G(x, y) \) is the potential at \( x \) of a unit point charge located at the point \( y \in \Omega \) inside a conducting grounded surface \( \partial \Omega \).

9.2. The same as for \( E_n(x-y) \), where \( E_n \) is the fundamental solution of \( \Delta \).
9.3. **Hint.** Use the fact that \( L^{-1} \) is self-adjoint.

9.4. **Hint.** Apply Green's formula (4.32) with \( v(x) = G(x, y) \) (consider \( y \) as a parameter) and \( \Omega \) replaced by \( \Omega \setminus B(y, \varepsilon) \), where \( B(y, \varepsilon) = \{ x : |x - y| \leq \varepsilon \} \). Then pass to the limit as \( \varepsilon \to +0 \).

9.5. \( G(x, y) = \begin{cases} 
\frac{1}{4\pi} \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{(x_1 - y_1)^2 + (x_2 + y_2)^2} & \text{for } n = 2; \\
\frac{1}{(2-n)\sigma_{n-1}} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|x-\bar{y}|^{n-2}} \right) & \text{for } n \geq 3,
\end{cases} \)

where \( \bar{y} = (y_1, \ldots, y_{n-1}, -y_n) \) for \( y = (y_1, \ldots, y_{n-1}, y_n) \).

**Hint.** Seek the Green function in the form

\[
G(x, y) = \mathcal{E}_n(x - y) + c\mathcal{E}_n(x - \bar{y}),
\]

where \( \bar{y} \notin \Omega \) and \( c \) is a constant.

9.6.

\[
u(x) = \int_{\mathbb{R}^{n-1}} \frac{2x_n \varphi(y')}{|x-(y,0)|^n} = \frac{2}{\sigma_{n-1}} \int_{\mathbb{R}^{n-1}} \frac{x_n \varphi(y_1, \ldots, y_{n-1}) dy_1 \ldots y_{n-1}}{|(x_1 - y_1)^2 + \ldots + (x_{n-1} - y_{n-1})^2 + y_n^2|^n}.
\]

**Hint.** Use the formula in Problem 9.4.

9.7. Let the disc or the ball be \( \{ x : |x| < R \} \). Then

\[
G(x, y) = \begin{cases} 
\frac{1}{2\pi} \ln \frac{R|x-y|}{|y|} & \text{for } n = 2; \\
(2-n)\sigma_{n-1}|x-y|^{n-2} - \frac{R^{n-2}}{|y|^{n-2}(2-n)\sigma_{n-1}|x-\bar{y}|^{n-2}} & \text{for } n \geq 3,
\end{cases} \]

where \( \bar{y} = \frac{R^2 y}{|y|^2} \) is the point which is obtained from \( y \) by the inversion with respect to the circle (sphere) \( \{ x : |x| = R \} \).

**Hint.** If \( n = 2 \), seek \( G(x, y) \) in the form

\[
\mathcal{E}_2(x - y) - \mathcal{E}_2(x - \bar{y}) - c(y).
\]

If \( n \geq 3 \), seek \( G(x, y) \) in the form \( \mathcal{E}_n(x - y) - c(y)\mathcal{E}_n(x - \bar{y}) \), where (in both cases) \( \bar{y} = \frac{R^2 y}{|y|^2} \). Show geometrically that if \( |x| = R \), then \( \frac{|x - y|}{|x - \bar{y}|} = \frac{|y|}{R} \) does not depend on \( x \).

9.8.

\[
u(x) = \frac{1}{\sigma_{n-1} R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x - y|^n} f(y) dS_y.
\]

Here the disc (or the ball) is \( \Omega = \{ x : |x| < R \} \), \( f = u|_{\partial \Omega} \).
**Hint.** Use the formula from Problem 8.4. When calculating $\frac{\partial}{\partial y} G(x, y)$, use that $|y| = R$ implies
\[
\frac{\partial}{\partial y} |x-y| = -\frac{\partial}{\partial y} |x-y| = -(\nabla_y |x-y|, \frac{y}{|y|}) = -(\frac{y-x}{|y-x|}, \frac{y}{|y|}) = \frac{(x,y)-R^2}{R|x-y|}.
\]

9.9. For the half-ball $\{x : |x| \leq R, x_n \geq 0\}$ the answer is
\[
G(x, y) = G_0(x, y) - G_0(x, \overline{y}),
\]
where $G_0$ is the Green function of the ball $\{x : |x| < R\}$ and $\overline{y} = (y_1, \ldots, y_{n-1}, -y_n)$ for $y = (y_1, \ldots, y_{n-1}, y_n)$.

9.10.
\[
G(x, y) = \frac{1}{4\pi \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}} + \frac{1}{4\pi \sqrt{(x_1-y_1)^2+(x_2+y_2)^2+(x_3-y_3)^2}} - \frac{1}{4\pi \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}} - \frac{1}{4\pi \sqrt{(x_1-y_1)^2+(x_2+y_2)^2+(x_3+y_3)^2}}.
\]

**Hint.** $G(x, y) = \mathcal{E}_3(x-y) - \mathcal{E}_3(x-T_2y) - \mathcal{E}_3(x-T_3y) + \mathcal{E}_3(x-T_2T_3y)$, where $T_2(y_1, y_2, y_3) = (y_1, -y_2, y_3), T_3(y_1, y_2, y_3) = (y_1, y_2, -y_3)$.

9.11. **Hint.** Use the fact that $\Gamma(z)$ has simple poles for $z = 0, -1, -2, \ldots$

9.12. **Hint.** Use the Liouville formula for the Wronskian of two linearly independent solutions of the Bessel equation.

9.13. **Hint.** Expand the left-hand side in power series in $t$ and $x$, and use the expansion of $J_n(x)$ given in Problem 8.11.

9.14. **Hint.** Take $t = e^{i\varphi}$ in the formula of Problem 8.13 and use the result of Problem 8.11.

9.15. For the domain $\{x = (x_1, \ldots, x_n) : 0 < x_j < a_j, j = 1, \ldots, n\}$ the eigenfunctions are $\prod_{j=1}^n \sin k_j \pi x_j$ for $k_j = 1, 2, \ldots$ and the eigenvalues are $\lambda_{k_1 \ldots k_n} = \left(\frac{k_1 \pi}{a_1}\right)^2 + \cdots + \left(\frac{k_n \pi}{a_n}\right)^2$.

9.16. **Hint.** Use the fact that $J_k(\alpha, x)$ for all $n = 1, 2, \ldots$, are eigenfunctions of the same operator $L = \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{k^2}{x^2}$ (on the space of functions vanishing at $x = 1$). This operator is symmetric in the space $L^2((0, 1), xdx)$ consisting of the functions on $(0, 1)$ which are square integrable with respect to the measure $xdx$.

9.17. For the disc $\{x : |x| < R\}$, in the polar coordinates the desired system is
\[
J_k \left(\frac{\alpha_k \pi}{R} r\right) e^{ik\varphi}, \text{ for } k \in \mathbb{Z}_+, \ n = 1, 2, \ldots,
\]
where $\alpha_{k,n}$ are defined in Problem 8.16.

**Hint.** Write $\Delta$ in the polar coordinates and expand the eigenfunction into Fourier series $u(r, \varphi) = \sum_k X_k(r)e^{ik\varphi}$. Then each term $X_k(r)e^{ik\varphi}$ will be an eigenfunction and $X_k(r)$ satisfies the Bessel equation. Using regularity at 0, show that $X_k(r) = cJ_k(\alpha r)$.

9.18. **Hint.** For the cylinder $\Omega = \{(x,y,z) : x^2 + y^2 < R^2, 0 < z < H\}$ reduce the equation to the case $\Delta u = f$, $u|_{\partial\Omega} = 0$, then expand $u$ and $f$ in the eigenfunctions of $\Delta$ in $\Omega$ which are of the form $\left\{e^{ik\varphi}J_k(\alpha_{k,n}r/R) \sin \frac{m\pi z}{H} : k \in \mathbb{Z}_+, n = 1,2,\ldots, m = 1,2,\ldots\right\}$.

10.1. $u(t,x) = e^{i\omega t}J_0(\frac{\omega}{a}\rho), \rho = \sqrt{x_1^2 + x_2^2}$. (The axis of the cylinder is $\{x : x_1 = x_2 = 0\}$.)

10.2. See Fig. 6.

$$u(t,r) = \begin{cases} 1 & \text{if } r \leq R - at; \\ \frac{1}{2} - \frac{at}{2r} & \text{if } r + at > R \text{ and } -R < r - at < R; \\ 0 & \text{if } r - at \geq R. \end{cases}$$

**Hint.** Introduce a new function $v = ru$ instead of $u$. (Here $r = |x|$.) Then $v(t,r) = \frac{1}{2}[\hat{\varphi}(r + at) + \hat{\varphi}(r - at)]$, where $\hat{\varphi}$ is the extension of $r\varphi$ to $\mathbb{R}$ as an odd function.

10.3. See Fig. 7.

$$u(t,r) = \begin{cases} \frac{t}{4ar}[R^2 - (r - at)^2] & \text{if } r + at \leq R, \\ 0 & \text{if } r - at \geq R. \end{cases}$$

**Hint.** Introduce a new function $v = ru$ with $r = |x|$. Then $v(t,r) = \frac{1}{2a} \int_{r-at}^{r+at} \hat{\psi}(s)ds = \Psi(r + at) - \Psi(r - at)$, where $\Psi(r) = \frac{1}{2a} \int_0^r \hat{\psi}(s)ds$ and $\hat{\psi}$ is the extension of $r\psi$ to $\mathbb{R}$ as an odd function.
10.5. **Hint.** Consider the Cauchy problem with initial conditions $u|_{t=0} = 0$ and $u|_{t=0} = \psi(x)$. Then

$$u(t, x) = (2\pi)^{-n} \int \frac{1}{2i|\xi|} e^{i(x-y)\cdot\xi + at|\xi|} \psi(y)dyd\xi - (2\pi)^{-n} \int \frac{1}{2i|\xi|} e^{i(x-y)\cdot\xi - at|\xi|} \psi(y)dyd\xi.$$
Consider, e.g. the first summand. Multiplying the integrand by a $C^\infty$-function $\chi(\xi)$ that is equal to 1 for $|\xi| > 1$ and vanishes for $|\xi| < 1/2$ (this does not affect the singularities of $u$), try to find a differential operator $L = \sum_{j=1}^{n} c_j(t, x, y, \xi) \frac{\partial}{\partial \xi_j}$ such that $Le^{i[(x-y)\cdot \xi + at|\xi|]} = e^{i[(x-y)\cdot \xi + at|\xi|]}$.

Prove that such an operator $L$ exists if and only if $|x-y|^2 \neq a^2 t^2$ (here $x, y, t$ are considered as parameters) and integrate by parts $N$ times thus moving $L$ from the exponential to other terms.

10.6. $\{(t, x, y) : t \geq 0, \ t^2 = x^2 + y^2/4\}$.

**Hint.** Reduce the operator to the canonical form.

10.7. a) $\{(t, x, y) : 0 \leq t \leq \min\left(\frac{a}{2}, \frac{b}{2}\right), \ x \in [t, a-t], \ y \in [t, b-t]\}$
10.8. a) Let \( \{ (x,y,z) : x \in [0,a], y \in [0,b], z \in [0,c] \} \) be the given parallelepiped.

Then the answer is
\[
\left\{ (t,x,y,z) : 0 \leq t \leq \min \left( \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \right), \ t \leq x \leq a - t, \ t \leq y \leq b - t, \ t \leq z \leq c - t \right\}
\]

The section by the hyperplane \( t = 1 \) is a smaller parallelepiped, or a rectangle, or an interval, or the empty set (Fig. 10).

b) \( \{ (t,x,y,z) : \text{dist}((x,y,z),\Pi) \leq t \} \), where \( \Pi \) is the given parallelepiped. The section by the hyperplane \( t = 1 \) is plotted on Fig. 11.

11.1.

\[
u(x) = \begin{cases} 
-\frac{Q}{2\pi} \ln R & \text{if } |x| \leq R, \\
-\frac{Q}{2\pi} \ln r & \text{if } |x| \geq R,
\end{cases}
\]
where $r = |x|$, the circle is $\{x : |x| = R\}$, $Q$ is the total charge of the circle, i.e.,

$Q = \int_{|y|=R} \sigma(y) dy$, where $\sigma$ is the density that defines the potential.

11.2.

$u(x) = \begin{cases} 
-\frac{Qr^2}{4\pi R^2} + \frac{Q}{4\pi} - \frac{Q}{2\pi} \ln R & \text{if } r \leq R; \\
-\frac{Q}{2\pi} \ln r & \text{if } r \geq R.
\end{cases}$

Here $Q$ is the total charge of the disc $\{x : |x| \leq R\}$, $r = |x|$.

11.3. The same answer as in Problem 10.1 with $r = \sqrt{x^2 + y^2}$ in the $(x, y, z)$-space, where the $z$-axis is the axis of the cylinder, $R$ is the radius of its cross-section by the $xy$-plane.

11.4.

$u(x) = \begin{cases} 
\alpha_0, & r < R, \\
0, & r > R,
\end{cases}$

where $\alpha_0$ is the density of the dipole moment, $r$ is the distance to the center of the circle or to the axis of the cylinder, $R$ is the radius of the circle or of the cross-section of the cylinder.

11.5.

$u(x) = \begin{cases} 
\frac{Q}{(n-2)|\sigma_{n-1}|R^{n-2}} & \text{if } |x| \leq R, \\
\frac{Q}{(n-2)|\sigma_{n-1}| |x|^{n-2}} & \text{if } |x| > R,
\end{cases}$

where the sphere of radius $R$ is taken with the center at the origin, $Q$ is the total charge of the sphere.

11.6. The induced charge is $q = -\frac{QR}{d}$.

**Hint.** The potential vanishes on the surface of the sphere, hence inside of the sphere. Calculate the potential in the center assuming that we know the distribution of the charge on the surface.
11.7. $\sigma(x) = \frac{Q(R^2 - d^2)}{4\pi R |x - y|^3}$, where $y$ is the point at which the initial charge $Q$ is situated.

**Hint.** Let $\overline{y}$ be the point obtained by reflecting $y$ with respect to the sphere (if the sphere is $\{x : |x| = R\}$, then $\overline{y} = \frac{R^2 y}{|y|^2}$). Place the charge $q$ at the point $\overline{y}$; then the potential $u(x) = \frac{q}{4\pi |x - y|} + \frac{Q}{4\pi |x - y|}$ coincides with the potential of all real charges ($Q$ and induced ones) outside the sphere because of the uniqueness of the solution of the Dirichlet problem in the exterior. Then

$$\sigma(x) = -\frac{\partial u(x)}{\partial r}, \text{ where } \frac{\partial u}{\partial r} = \left(\nabla u, \frac{x}{|x|}\right).$$

11.8. The charge will be distributed uniformly over the whole wire.

12.1. a) at $\pm x \cdot \frac{y}{|y|} = 0$;
   b) at $\pm (|x| - R) = 0$.

12.2. Rays and characteristics are the same curves in the $(t, x)$-plane.

**Hint.** Use the description of rays as solutions of (12.39) satisfying $|\dot{x}(t)| = c(x(t))$.

12.3. $u(t, x, \lambda) = \frac{i\lambda^{-1}}{2x} [\exp(i\lambda xe^{-t} + \frac{3}{2} t) - \exp(i\lambda xe^{-t} - \frac{3}{2} t)]$. 

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13. Answers and Hints. Solutions

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