The value of a liability cash flow in discrete time subject to capital requirements

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Abstract

The aim of this paper is to define the market-consistent value of a liability cash flow in discrete time subject to repeated capital requirements, and explore its properties. Our multi-period market-consistent valuation approach is based on defining a criterion for selecting a static replicating portfolio and defining the value of the residual liability, whose cash flow is the difference between the original liability cash flow and that of the replicating portfolio. The value of the residual cash flow is obtained as a solution to a backward recursion that is implied by the procedure for financing the repeated capital requirements, and no-arbitrage arguments. We show that the liability value resulting from no-arbitrage pricing of the dividends to capital providers may be expressed as a multi-period cost-of-capital valuation. Explicit valuation formulas are obtained under Gaussian model assumptions.

1 Introduction

The aim of this paper is to define the market-consistent value of a liability cash flow in discrete time subject to repeated capital requirements, and explore its properties. The liability should be interpreted as the aggregate liability of a company, i.e. at the level on which capital requirements are imposed. Our multi-period valuation approach is based on defining a criterion for selecting a static replicating portfolio and defining the value of the residual liability whose cash flow is the difference between the original liability cash flow and that of the replicating portfolio. For defining the value of the residual cash flow we do not impose a particular valuation functional. Instead we derive the value as a solution to a backward recursion that is implied by the procedure for financing the repeated capital requirements, and no-arbitrage arguments.

The approach to market-consistent liability valuation presented in [13] has been the main source of inspiration for the current paper. Similarly to what is advocated in [13], and as is explicitly stated in current insurance market regulation, we consider a hypothetical transfer of the liability to a so-called reference undertaking whose only purpose is to manage the runoff of the liability. The repeated capital requirements are financed by capital providers with limited liability. In [13], a valuation framework based on dynamic replication and cost-of-capital arguments was presented. In [9] a valuation framework, inspired by [13], based on dynamic monetary risk measures and dynamic monetary utility functions was presented and explicit valuation formulas were derived under Gaussian model assumptions. An essential difference between [13] and [9] is that initial static replication, instead of dynamic replication, of the liability cash flow is considered in [9]. The static replicating portfolio is transferred to the reference undertaking together with the liability. Static replication is a reasonable assumption since sophisticated dynamic hedging may be unrealistic for an entity only designed to manage a liability in runoff.

In [9], the static replicating portfolio was assumed to be given and the analysis only focused on the multi-period valuation of the residual liability cash flow. Criteria for selection of a replicating portfolio were not analyzed. A large part of the current paper focuses on presenting properties of a particular criterion for selection of the replicating portfolio that forms the basis for defining the value of the liability. This criterion says that a good replicating portfolio is one that makes the need for capital injections small. Moreover, in the current paper the value of the residual liability is implied by no-arbitrage pricing of a derivative security with optionality written on the cumulative cash flow to capital provider. We demonstrate that there is a correspondence between the choice of pricing measure used for pricing the derivative security and an adapted process of cost-of-capital rates that defines the capital providers' acceptability criteria for providing solvency capital throughout the runoff of the liability.

Replicating portfolio theory for capital requirement calculation has attracted much interest in recent years. There, the value of a liability cash flow at a future time is modeled as a conditional expected value with respect to the market's pricing measure of the sum of discounted future liability cash flows. Since computation of this liability value is typically not feasible, one seeks an accurate approximation by replacing the liability cash flow (or its value) by that of a portfolio of traded replication instruments. Then, a risk measure is applied to the approximation of the liability value yielding an approximation of the capital requirement. In [3], [14], [15] and [16] various aspects of this replicating portfolio approach to capital calculations are clarified. A fact that somewhat complicates the analysis is that risk measures defining capital requirements are defined with respect to the real-world probability measure $\mathbb{P}$, whereas the replication criteria are usually expressed in terms of the market's pricing measure $\mathbb{Q}$. Comparisons of properties and effects of different replication criteria are presented in [14], [15] and [16]. In [3], it is shown how replicating portfolio theory can be formulated in order to allow for efficient replication of liability values exhibiting path-dependence.

Common to the works [3], [14], [15] and [16] is that the liability value is defined as a conditional expected value of the sum of discounted liability cash flows. This is very different from the approach presented here. As explained above, we do not define the value of the liability from the outset. Rather we consider the dividends to the capital provider that finances the capital requirements of the residual liability cash flow that remains after imperfect initial replication.

Dynamic risk measures and dynamic risk-adjusted values have been analyzed in great detail during the last decade, see e.g. [1], [2], [4], [5], [6], [8] and the references therein for important contributions. Much of the research in this area has been aimed at establishing properties and representation results for dynamic risk measures in general functional analytic settings, particularly for bounded stochastic processes and under convexity requirements for the risk measures. We want to allow for models for unbounded liability cash flows. Moreover, limited liability for the capital providers in our setting implies that the dynamic valuation mappings appearing here will not be concave even when the conditional risk measures are convex. We will only assume very basic properties of the conditional risk measures defining capital requirements, namely, so-called translation invariance, monotonicity and normalization. In particular, we want to allow for conditional versions of the risk measure Value-at-Risk that is extensively used in practice.

Another approach to market-consistent liability valuation is presented in [17], combining no-arbitrage valuation and actuarial valuation into a general framework. Both the current paper and [17] advocates a two-step valuation. However, this has a quite different meaning in [17] compared to the approach presented here. In [17], an actuarial pricing principle is used to price the residual risk. However, the residual risk in [17] does not correspond to our residual liability cash flow. Moreover, as described above, we do not price the residual liability cash flow by a given pricing operator; in our setting the value of the residual liability cash flow is implied by no-arbitrage valuation of the cumulative dividends to capital providers.

In [11] an approach to valuation of an insurance liability cash flow is presented that defines the value as a sum of a best-estimate reserve and a risk margin. The best-estimate reserve is obtained by sequential local risk minimization, from [12], and can, under additional assumptions on the incomplete financial market, be expressed as the expected sum of discounted cash flows with respect to an equivalent pricing measure expressed in terms of a certain state-price deflator. The risk margin is obtained as the difference between the expected sum of discounted cash flows using another state-price deflator and the best-estimate reserve. In our setting the best-estimate reserve is not an object that appears naturally in the valuation of the liability cash flow: only under special circumstances may the value of the liability be interpreted in terms of a best-estimate reserve.

The paper is organized as follows: The liability valuation framework is presented in Section 2 which is divided into three subsections. Section 2.1 presents the procedure for financing the repeated capital requirements, imposed on the reference undertaking, by capital injections from capital providers. In particular, it is shown that no-arbitrage pricing of the derivative security written on the cumulative cash flows to the capital providers leads to a backward recursion for the value of the residual liability cash flow. Section 2.2 presents the mathematical framework for valuation of the residual liability cash flow when capital requirements are expressed in terms of a dynamic monetary risk measure. Section 2.3 focuses on criteria for selecting the static replicating portfolio, proposes a particular criterion and explores its properties. Based on this criterion and the framework presented in Section 2.2 the value of the liability is defined. Explicit valuation formulas are obtained in Section 3 under Gaussian model assumptions. The proofs are found in Section 4.

2 The valuation framework

We consider time periods $1, \dots, T$, corresponding time points $0, 1, \dots, T$, and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ with $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_T = \mathcal{F}$, and $\mathbb{P}$ denotes the real-world measure. We write $L^p(\mathcal{F}_t, \mathbb{P})$ for the normed linear space of $\mathcal{F}_t$-measurable random variables X with norm $\mathbb{E}^{\mathbb{P}}[|X|^p]^{1/p}$. Equalities and inequalities between random variables should be interpreted in the $\mathbb{P}$-almost sure sense. We assume a given numéraire process $(N_t)_{t=0}^T$ and that all financial values are discounted by this numéraire. Although the choice of numéraire is irrelevant for the analysis, we take the numéraire to be the bank account numéraire: $N_0 = 1$ and N_t is the amount at time t from rolling forward an initial unit investment in one-period risk-free bonds. A value $N_t Y$ at time t has discounted value Y at time t.

We assume that there exist a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale $(D_t)_{t=0}^T$ with $\mathbb{E}^{\mathbb{P}}[D_T] = 1$ defining the equivalent pricing measure $\mathbb{Q}$ of an arbitrage-free incomplete financial market via $D_t = d\mathbb{Q}/d\mathbb{P} \mid \mathcal{F}_t$, i.e. for $u > t$ and a sufficiently integrable $\mathcal{F}_u$-measurable Z,

$$\mathbb{E}_t^{\mathbb{Q}}[Z] = \frac{1}{D_t}\mathbb{E}_t^{\mathbb{P}}[D_u Z],$$

where subscript t in $\mathbb{E}_t^{\mathbb{Q}}$ and $\mathbb{E}_t^{\mathbb{P}}$ means conditioning on $\mathcal{F}_t$. The choice of $\mathbb{Q}$ should reflect market participants' demands for compensation for providing capital financing capital requirements for non-hedgeable (insurance) risks. Whereas the pricing measure $\mathbb{Q}$ is neutral to financial trading risk, it may reflect risk aversion towards non-hedgeable risks.

2.1 The liability derivative instrument

A discounted liability cash flow corresponds to an $\mathbb{F}$-adapted stochastic process $X^o = (X_t^o)_{t=1}^T$ interpreted as a discounted cash flow from an aggregate insurance liability in runoff. Our aim is to give a precise meaning to the market-consistent value of the liability, explore its properties, and provide results that allow this value to be computed.

As is done in e.g. [13] and prescribed by EIOPA, see [7, Article 38], we take the point of view that an aggregate liability cash flow should be valued by considering a hypothetical transfer of the liability and its associated replicating portfolio to a separate entity, a so-called reference undertaking, whose assets have the purpose of matching the value or cash flow of the liability as well as possible.

We will give a meaning to the liability value by a particular valuation procedure. At time 0, a replicating portfolio is purchased with the aim of replicating the liability value or its cash flow at all times. Let $X^r = (X_t^r)_{t=1}^T$ denote the discounted cash flow of the replicating portfolio and note that, from standard assumptions of no arbitrage, its initial price is $\sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X_t^r]$.

Externally imposed capital requirements imply that the reference undertaking typically needs capital injections throughout the liability runoff. A capital provider is the owner of the reference undertaking for as long as the necessary capital injections are provided. The capital provider may at any time choose to stop providing capital and, in that case, has no further obligations towards the reference undertaking. From the capital provider's perspective, ownership of the reference undertaking is equivalent to holding a derivative security with optionality, described in detail below, written on the residual liability cash flow $X := X^o - X^r$.

We will define the value at time 0 of the liability cash flow X^o as

$$\sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X_t^r] + V_0(X) = \mathbb{E}_0^{\mathbb{Q}}\Big[\sum_{t=1}^T X_t^r + V_0(X)\Big],$$

where X^r is the discounted cash flow of a particular replicating portfolio and $V_0 := V_0(X)$ is a position in the numéraire asset at time 0 that corresponds to the discounted value at time 0 of the residual liability cash flow. V_0 will be determined from X by solving a non-linear backward recursion that appears as the consequence of capital requirements and the procedure for handling the residual liability cash flow X. The $\mathbb{F}$-adapted sequence $(V_t)_{t=0}^T$, whose terms are the discounted values of the residual liability cash flows $(X_s)_{s=t+1}^T$, $t = 0, \dots, T-1$, will be determined from no-arbitrage pricing of a particular derivative security with optionality written on the discounted residual cash flow X. The basis for defining the value of the liability cash flow is the following:

- At time 0, the liability, replicating portfolio and a position V_0 in the numéraire asset are transferred to a reference undertaking.

- At time t, provided that the holder of the derivative has not exercised the right to stop, denote the discounted available capital, all invested in the numéraire asset, by V_t and the discounted solvency capital requirement by R_t, required for the residual liability cash flow in runoff. The holder of the derivative is required to offset the difference by paying an amount with discounted value $R_t - V_t$. The position R_t in the numéraire asset is held until time $t + 1$.

- At time $t + 1$ the discounted value of the payoff to the holder of the derivative is $(R_t - X_{t+1} - V_{t+1})_+ := \max(R_t - X_{t+1} - V_{t+1}, 0)$ upon stopping, and $R_t - X_{t+1} - V_{t+1}$ upon not stopping.

- The random sequence $(V_t)_{t=0}^T$ is determined from the requirement that the market price of the derivative is zero at all times. Let $H_{t,t'}$, $t' \geq t+1$, be the discounted gain for the holder of the derivative from time t to time t' upon stopping at time t'. Notice that

$$\begin{aligned} H_{t,t+1} &= -(R_t - V_t) + (R_t - X_{t+1} - V_{t+1})_+, \\ H_{t,t'} &= -(R_t - V_t) + (R_t - X_{t+1} - V_{t+1}) + H_{t+1,t'}, \quad t' > t+1. \end{aligned}$$

The holder of the derivative is the owner of the reference undertaking. An essential feature is that the owner of the reference undertaking neither pays nor is paid anything for the ownership, i.e. for the transfer of the liability, the replicating portfolio and the position in the numéraire asset. Moreover, the position V_t in the numéraire asset at any time t is such that the value of continued ownership is zero. This requirement, and its consequences, are given in the following result. Here ess sup refers to the essential supremum with respect to $\mathbb{P}$, see Appendix A.5 in [10]. Details on arbitrage-free pricing of American contingent claims can be found in Section 6.3 in [10]. Let $\mathcal{S}_{t+1}$ denote the set of stopping times $\tau : \Omega \to \{t+1, \dots, T\} \cup \{+\infty\}$ and set $\mathcal{S}_{t+1,T} := \{\tau \wedge T : \tau \in \mathcal{S}_{t+1}\}$. We use the convention $\inf \emptyset := +\infty$.

Theorem 1. *For all $t \in \{0, \dots, T-1\}$, assume that $R_t, V_t \in L^1(\mathcal{F}_t, \mathbb{Q})$ and set $R_T, V_T := 0$. For all $t \in \{1, \dots, T\}$, assume that $X_t \in L^1(\mathcal{F}_t, \mathbb{Q})$. Fix $s \in \{0, \dots, T-1\}$ and consider the following statements:*
(i) For all $t \in \{s, \dots, T-1\}$,

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau}] = 0.$$

(ii) For all $t \in \{s, \dots, T-1\}$,

$$V_t = R_t - \mathbb{E}_t^{\mathbb{Q}}\big[(R_t - X_{t+1} - V_{t+1})_+\big], \quad V_T := 0. \tag{1}$$

The statements (i) and (ii) are equivalent and either one implies that

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau}] = \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau^*_{t+1}}],$$

where $\tau^*_{t+1} := \inf\{u \in \{t+1, \dots, T\} : R_{u-1} - X_u - V_u < 0\} \wedge T$.

Remark 1. *The backward recursion* (1) *for the values of the residual liability cash flow involves the pricing measure* $\mathbb{Q}$*. It may equivalently be expressed in terms of the real-world probability measure* $\mathbb{P}$ *and an* $\mathbb{F}$*-adapted sequence* $(\eta_t)_{t=0}^{T-1}$ *of cost-of-capital rates. Notice that*

$$\begin{aligned} V_t &= R_t - \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+] \\ &= R_t - \frac{1}{1+\eta_t} \mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+] \end{aligned}$$

if we define

$$\frac{1}{1+\eta_t} := \frac{\mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+]}{\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+]}.$$

In particular, $\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+] = (1+\eta_t)(R_t - V_t)$ *which may be interpreted as the capital provider's acceptability condition at time t for injecting capital in the numéraire asset at time t until time* $t+1$ *to ensure that the capital requirement is met. Since the residual liability cash flow is a replication error, it is reasonable to expect nonnegative cost-of-capital rates* η_t*, which provides guidance on how to choose the pricing measure* $\mathbb{Q}$*.*

2.2 The value of the residual liability cash flow

In order to ensure that the value of the liability, to be defined, is a sensible object we require three basic properties of the risk measures quantifying solvency capital requirements.

By conditional and dynamic monetary risk measures quantifying one-period capital requirements we mean the following:

Definition 1. *For* $p \in [0, \infty]$ *and* $t \in \{0, \dots, T-1\}$*, a conditional monetary risk measure is a mapping* $R_t : L^p(\mathcal{F}_{t+1}, \mathbb{P}) \to L^p(\mathcal{F}_t, \mathbb{P})$ *satisfying*

$$\text{if } \lambda \in L^p(\mathcal{F}_t, \mathbb{P}) \text{ and } Y \in L^p(\mathcal{F}_{t+1}, \mathbb{P}), \text{ then } R_t(Y + \lambda) = R_t(Y) - \lambda, \quad (2)$$

$$\text{if } Y, \widetilde{Y} \in L^p(\mathcal{F}_{t+1}, \mathbb{P}) \text{ and } Y \le \widetilde{Y}, \text{ then } R_t(Y) \ge R_t(\widetilde{Y}), \quad (3)$$

$$R_t(0) = 0. \quad (4)$$

A sequence $(R_t)_{t=0}^{T-1}$ *of conditional monetary risk measures is called a dynamic monetary risk measure.*

For $t \geq 0$, $x \in \mathbb{R}$, $u \in (0,1)$ and an $\mathcal{F}_{t+1}$-measurable Z, let

$$\begin{aligned} F_{t,-Z}(x) &:= \mathbb{P}(-Z \leq x \mid \mathcal{F}_t), \\ F_{t,-Z}^{-1}(1-u) &:= \min\{m \in \mathbb{R} : F_{t,-Z}(m) \geq 1-u\}, \end{aligned}$$

and define conditional versions of Value-at-Risk and Expected Shortfall as

$$\begin{aligned} \mathrm{VaR}_{t,u}(Z) &:= F_{t,-Z}^{-1}(1-u), \\ \mathrm{ES}_{t,u}(Z) &:= \frac{1}{u}\int_0^u \mathrm{VaR}_{t,v}(Z)dv. \end{aligned}$$

$\mathrm{VaR}_{t,u}$ and $\mathrm{ES}_{t,u}$ are special cases of the following more general type of conditional monetary risk measure.

Definition 2. *Let $t \in \{0,\dots,T-1\}$ and let M be a probability distribution on the Borel subsets of $(0,1)$ such that either M has a bounded density with respect to the Lebesgue measure or the support of M is bounded away from 0 and 1. Define*

$$R_t(Z) := \int_0^1 F_{t,-Z}^{-1}(u)dM(u).$$

Theorem 2. *For $p \in [1,\infty]$, R_t in Definition 2 is a conditional risk measure in the sense of Definition 1. In particular, for $p \in [1,\infty]$, $\mathrm{VaR}_{t,u}$ and $\mathrm{ES}_{t,u}$ are conditional monetary risk measures in the sense of Definition 1.*

The statement of Theorem 2 follows from combining Proposition 4 (i) and Remark 5 in [9]; the proof is therefore omitted. Notice that $\mathrm{VaR}_{t,v}$ is obtained by choosing M such that $M(\{1-v\}) = 1$, and $\mathrm{ES}_{t,v}$ is obtained by choosing M with density $u \mapsto v^{-1}1_{(1-v,1)}(u)$.

From (1) follows that V_t is determined recursively from X_{t+1} and V_{t+1} as follows:

$$V_t := W_t(X_{t+1} + V_{t+1}), \quad V_T := 0, \tag{5}$$

$$W_t(Y) := R_t(-Y) - \mathbb{E}_t^{\mathbb{Q}}\big[\big(R_t(-Y) - Y\big)_+\big]. \tag{6}$$

It remains to define the mappings W_t properly. This can be done in various ways and we will focus on the ones that fit our purposes. From the fact that

$$\mathbb{E}_t^{\mathbb{Q}}\big[\big(R_t(-Y) - Y\big)_+\big] = \frac{1}{D_t}\mathbb{E}_t^{\mathbb{P}}\big[D_{t+1}\big(R_t(-Y) - Y\big)_+\big]$$

applications of Hölder's and Minkowski's inequalities allow us to define W_t.

Theorem 3. *(i) Fix $t \in \{0,\dots,T-1\}$ and $p \in [1,\infty]$. Suppose $D_{t+1}/D_t \in L^\infty(\mathcal{F}_{t+1},\mathbb{P})$ and that R_t is a conditional monetary risk measure according* *to Definition 1. Then W_t in (6) is a mapping from $L^p(\mathcal{F}_{t+1},\mathbb{P})$ to $L^p(\mathcal{F}_t,\mathbb{P})$ having the properties*

$$\text{if } \lambda \in L^p(\mathcal{F}_t,\mathbb{P}) \text{ and } Y \in L^p(\mathcal{F}_{t+1},\mathbb{P}), \text{ then } W_t(Y+\lambda) = W_t(Y) + \lambda, \quad (7)$$

$$\text{if } Y, \widetilde{Y} \in L^p(\mathcal{F}_{t+1},\mathbb{P}) \text{ and } Y \le \widetilde{Y}, \text{ then } W_t(Y) \le W_t(\widetilde{Y}), \quad (8)$$

$$W_t(0) = 0. \quad (9)$$

(ii) Fix $t \in \{0,\dots,T-1\}$ and $1 \le p_1 < p_2$. Suppose $D_{t+1}/D_t \in L^r(\mathcal{F}_{t+1},\mathbb{P})$ for every $r \ge 1$. Suppose further that for any $p \in [p_1,p_2]$, R_t is a conditional monetary risk measure according to Definition 1. Then, for any $\epsilon > 0$ such that $p-\epsilon \ge p_1$, W_t in (6) can be defined as a mapping from $L^p(\mathcal{F}_{t+1},\mathbb{P})$ to $L^{p-\epsilon}(\mathcal{F}_t,\mathbb{P})$ having the properties (7)-(9).

The requirement $D_{t+1}/D_t \in L^\infty(\mathcal{F}_{t+1},\mathbb{P})$ in statement (i) of Theorem 3 leads to a cleaner definition of the mapping W_t. However, the boundedness of D_{t+1}/D_t may be a too restrictive requirement. Finiteness of all moments of D_{t+1}/D_t, as in statement (ii), will be an appropriate requirement for the subsequent analysis here.

Under the assumptions of Theorem 3 (i) or (ii), it follows from (5) and (7) that

$$V_t = W_t \circ \cdots \circ W_{T-1}(X_{t+1} + \cdots + X_T), \quad (10)$$

where $W_t \circ \cdots \circ W_{T-1}$ denotes the composition of mappings $W_t,\dots,W_{T-1}$, and that $V_t \in L^p(\mathcal{F}_t,\mathbb{P})$ in case of Theorem 3 (i) applies or, for any $\epsilon > 0$ such that $p-\epsilon > 0$, $V_t \in L^{p-\epsilon}(\mathcal{F}_t,\mathbb{P})$ in case Theorem 3 (ii) applies.

In statements involving V_t we will in what follows, in order to avoid irrelevant lengthy technical statements, assume that suitable conditions are satisfied ensuring that $V_t = V_t(X)$ is well-defined from (10) as a mapping from $L^p((\mathcal{F}_t)_{t=1}^T,\mathbb{P})$ to $L^{p-\epsilon}(\mathcal{F}_t,\mathbb{P})$ for relevant values of p and ϵ such that (7)-(9) hold.

The following result is an immediate consequence of the representation (10) combined with Theorem 3. The proof is therefore omitted. The property (ii) below, known as time consistency, is well known, see [6] for more general results of this kind.

Theorem 4. *(i) Let $b = (b_s)_{s=1}^T$ with $b_s \in L^p(\mathcal{F}_t,\mathbb{P})$ for each s, and let $X_s \le \widetilde{X}_s$ for each s. Then, for every $t < T$,*

$$V_t(0) = 0, \quad V_t(X+b) = V_t(X) + \sum_{s=t+1}^{T} b_s, \quad V_t(X) \le V_t(\widetilde{X}).$$

(ii) For every pair of times (s,t) with $s \le t$, the two conditions $(X_u)_{u=1}^t = (\widetilde{X}_u)_{u=1}^t$ and $V_t(X) \le V_t(\widetilde{X})$ together imply $V_s(X) \le V_s(\widetilde{X})$.

2.3 The value of the liability cash flow

In order to define the value of the liability we need to specify the available replication instruments and their cash flows. Consider m discounted cash flows $X^{f,k} = (X_t^{f,k})_{t=1}^T$, $k = 1, \dots, m$ of available financial instruments and denote by X^f the $\mathbb{R}^m$-valued process such that X_t^f denotes the (column) vector of time-t discounted cash flows of the m instruments. A portfolio with portfolio-weight vector $v \in \mathbb{R}^m$, representing the number of units of the m instruments, generates the discounted cash flow $v^{\mathrm{T}} X_t^f$ at time t.

Various criteria for selection of replicating portfolio have been considered in the literature. The optimization problem

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}} \Big[(X_t^o - v^{\mathrm{T}} X_t^f)^2 \Big]^{1/2} \tag{11}$$

is referred to as cash flow matching in [16]. Under mild conditions, it is shown in Theorems 1 (and 2) in [16] that an optimal (unique optimal) solution exists. An alternative cash-flow-matching problem is

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}} \Big[(X_t^o - v^{\mathrm{T}} X_t^f)^2 \Big]. \tag{12}$$

Comparisons between (11) and (12) are found in [14]. The optimization problem

$$\inf_{v \in \mathbb{R}^m} \mathbb{E}_0^{\mathbb{Q}} \Big[\Big(\sum_{t=1}^{T} (X_t^o - v^{\mathrm{T}} X_t^f) \Big)^2 \Big]^{1/2} \tag{13}$$

is referred to as terminal-value matching in [14], [15] and [16]. It is a standard quadratic optimization problem with explicit solution

$$\widehat{v} = \mathbb{E}_0^{\mathbb{Q}} \left[\begin{pmatrix} X_\cdot^{f,1} X_\cdot^{f,1} & \dots & X_\cdot^{f,1} X_\cdot^{f,m} \\ \vdots & & \vdots \\ X_\cdot^{f,m}, X_\cdot^{f,1} & \dots & X_\cdot^{f,m} X_\cdot^{f,m} \end{pmatrix} \right]^{-1} \mathbb{E}_0^{\mathbb{Q}} \left[\begin{pmatrix} X_\cdot^o X_\cdot^{f,1} \\ \vdots \\ X_\cdot^o X_\cdot^{f,m} \end{pmatrix} \right]$$

provided that the matrix inverse exists, where the subscript $\cdot$ means summation over the index t.

A replicating portfolio selection criterion should have the property that if perfect replication is possible, then the discounted optimal replicating portfolio cash flow $\widehat{v}^{\mathrm{T}} X^f$ satisfies $X^o = \widehat{v}^{\mathrm{T}} X^f$. This requirement ensures market-consistent liability values: $L_0 = \sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X_t^o]$ for a replicable liability cash flow.

Remark 2. *The versions of the optimization problems* (11), (12) *and* (13) *obtained by replacing the expectation* $\mathbb{E}_0^{\mathbb{Q}}$ *by* $\mathbb{E}_0^{\mathbb{P}}$ *may also be reasonable. Notice that if the only available replication instruments are zero-coupon bonds* *in the numéraire asset of all maturities $t = 1, \dots, T$ (or, equivalently, European call options on the numéraire asset with maturities $t = 1, \dots, T$ and common strike price 0), then $m = T$ and X^f is the $T \times T$ identity matrix. In this case,*

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}\Big[(X_t^o - v^{\mathrm{T}} X_t^f)^2\Big] = \inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}\Big[(X_t^o - v_t)^2\Big],$$

and the unique optimal solution is $\widehat{v} = \mathbb{E}_0^{\mathbb{P}}[X^o]$ which is referred to as the actuarial best-estimate reserve.

Notice also that, given the above restricted set of replication instruments, any $\widehat{v}$ satisfying $\sum_{t=1}^{T} \widehat{v}_t = \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}[X_t^o]$ is an optimal solution to the version of the terminal value problem (13) obtained by replacing the expectation $\mathbb{E}_0^{\mathbb{Q}}$ by $\mathbb{E}_0^{\mathbb{P}}$.

In our setting, the capital provider provides the capital

$$C_t := R_t(-X_{t+1} - V_{t+1}) - V_t = \mathbb{E}_t^{\mathbb{Q}}[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+]$$

at time t ensuring that the solvency capital requirement is met. Good initial replication should make the need for external capital small. Therefore, it is reasonable to select a replicating portfolio that minimizes the need for external funding of the liability runoff. We may consider the optimization problem

$$\inf_{v \in \mathbb{R}^m} \psi(v), \quad \psi(v) := \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{Q}}[C_t^v], \tag{14}$$

where $X^v := X^o - v^{\mathrm{T}} X^f$, $V_{t+1}^v := V_{t+1}(X^v)$, $R_t^v := R_t(-X_{t+1}^v - V_{t+1}^v)$, and

$$C_t^v := \mathbb{E}_t^{\mathbb{Q}}\Big[(R_t^v - X_{t+1}^v - V_{t+1}^v)_+\Big].$$

Notice that, due to the properties (2) and (7), the objective function in the optimization problem (14) is invariant under translations of X^v by constant vectors. Consequently, (14) will not have a unique optimal solution if risk-free cash flows in the numéraire asset are included as replication instruments. See Theorem 5 below for a more precise statement.

The optimization problems (11)-(14) can all be expressed as

$$\inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f)$$

for a mapping $\Psi : L^p((\mathcal{F}_t)_{t=1}^T, \mathbb{Q}) \to \mathbb{R}_+$ satisfying $\Psi(0) = 0$, i.e. optimality of perfect replication. Existence of a minimizer $\widehat{X}^r := \widehat{v}^{\mathrm{T}} X^f$ can be expressed as

$$\Psi(X^o - \widehat{X}^r) = \inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f).$$

Conditions for existence of a minimizer $\widehat{v}$ in (14) are presented in Theorem 8 below.

Now we define the value of the liability as the market price of a particular portfolio of financial instruments: the Ψ-optimal replicating portfolio and a position V_0 in the numéraire asset.

Definition 3. *$\widehat{X}^r := \widehat{v}^{\mathrm{T}} X^f$ is said to be an optimal discounted replicating portfolio cash flow with respect to the criterion Ψ if $\Psi(X^o - \widehat{X}^r) = \inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f)$, and then*

$$L_0 := \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}\big[\widehat{X}_t^r\big] + V_0(X^o - \widehat{X}^r)$$

is the value of the liability with replicating portfolio chosen with respect to the criterion Ψ.

Remark 3. *Notice from* (1) *that*

$$V_t = R_t - \mathbb{E}_t^{\mathbb{Q}}\big[\big(R_t - X_{t+1} - V_{t+1}\big)_+\big] \leq \mathbb{E}_t^{\mathbb{Q}}\big[X_{t+1}\big] + \mathbb{E}_t^{\mathbb{Q}}\big[V_{t+1}\big].$$

In particular, $V_0 \leq \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[X_t]$ and $L_0 \leq \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[X_t^o]$ regardless of the criterion for choosing the replicating portfolio.

The following result explains two properties of the valuation of a liability cash flow when the replicating portfolio is chosen as the one minimizing the expected sum of capital injections. First, if a nonrandom discounted cash flow is added to the discounted cash flow of the replicating portfolio, then neither the value of the liability nor the capital injections C_t change. Secondly, if the sum of the discounted residual liability cash flows equals of nonrandom constant, then there is no need for any capital injections: $C_t = 0$ for all t.

Theorem 5. *(i) If $\widetilde{X}^r = X^r + b$, where b_t is $\mathcal{F}_0$-measurable for all $t \in \{1, \dots, T\}$, then, with $\widetilde{X} := X^o - \widetilde{X}^r$*

$$\sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}\big[\widetilde{X}_t^r\big] + V_0(\widetilde{X}) = \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}\big[X_t^r\big] + V_0(X)$$

and, for all $t \in \{0, \dots, T-1\}$,

$$\begin{aligned} \widetilde{C}_t :=& \mathbb{E}_t^{\mathbb{Q}}\big[(R_t(-\widetilde{X}_{t+1} - V_{t+1}(\widetilde{X})) - \widetilde{X}_{t+1} - V_{t+1}(\widetilde{X}))_+\big] \\ =& \mathbb{E}_t^{\mathbb{Q}}\big[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+\big] =: C_t. \end{aligned}$$

(ii) If there is an $\mathcal{F}_0$-measurable K such that $\sum_{t=1}^{T} X_t = K$, then, for all $t \in \{0, \dots, T-1\}$,

$$C_t := \mathbb{E}_t^{\mathbb{Q}}\big[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+\big] = 0.$$

Remark 4. *Notice that if* $\sum_{t=1}^{T} X_t^o = K$ *for some constant* K*, then by Theorem 5,* $\widehat{X}^r = 0$ *and* $L_0 = V_0(X^o) = K$*. Such a cash flow is perfectly replicable by an initial numéraire position* K *by adjusting it by the amount* X_t^o *in the numéraire at time* t *and rolling the remaining position forward.*

Remark 5. *The deterministic replicating portfolio cash flow* $\widehat{X}^r = \mathbb{E}_0^{\mathbb{P}}[X^o]$ *corresponds to a classical actuarial best-estimate reserve, and solves a cash-flow-matching problem with only risk-free cash flows in the numéraire asset as replication instruments, see Remark 2. In this case, by Theorem 4,*

$$L_0 = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}[X_t^o] + V_0\Big(X^o - \mathbb{E}^{\mathbb{P}}[X^o]\Big) = V_0(X^o).$$

In particular, if $V_0(X^o) \geq \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}[X_t^o]$*, then* $L_0 \geq \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}[X_t^o]$*. As noted in Remark 2, any deterministic cash flow* $\widehat{X}^r$ *with* $\sum_{t=1}^{T} \widehat{X}_t^r = \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}[X_t^o]$ *is a optimal solution to the (alternative) terminal value problem*

$$\inf_{v \in \mathbb{R}^m} \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sum_{t=1}^{T} (X_t^o - v^{\mathrm{T}} X_t^f)\Big)^2\Big],$$

with only risk-free cash flows in the numéraire asset as replication instruments. In this case, by Theorem 4 or Theorem 5,

$$L_0 = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{Q}}[\widehat{X}_t^r] + V_0\Big(X^o - \widehat{X}^r\Big) = V_0(X^o).$$

We now address the questions of existence of an optimal replicating portfolio according to the portfolio selection criterion (14), and continuity of the value of the liability cash flow as a function of the portfolio weights of the replicating portfolio.

For $t \in \{1, \dots, T\}$, define $Z_t := (X_t^o, -(X_t^f)^{\mathrm{T}})^{\mathrm{T}}$ and, for $w \in \mathbb{R}^{m+1}$, $\widetilde{X}_t^w := w^{\mathrm{T}} Z_t$. Notice that a residual liability cash corresponds to $\widetilde{X}^w$ with $w_1 = 1$. The reason for introducing this notation is primarily that it allows us to formulate sufficient conditions for coerciveness that will lead to sufficient conditions for the existence of an optimal replicating portfolio, see Theorem 7 below.

Theorem 6. *Let* $(D_t)_{t=0}^{T}$ *satisfy either of the conditions (i) or (ii) in Theorem 3. Suppose that, for each* $t \in \{0, \dots, T-1\}$*,* $R_t : L^p(\mathcal{F}_{t+1}, \mathbb{P}) \to L^p(\mathcal{F}_t, \mathbb{P})$ *in (6) is a conditional monetary risk measure in the sense of Definition 1 for every* $p \in [1, \infty]$ *that is* L^1*-Lipschitz continuous in the sense*

$$|R_t(-Y) - R_t(-\widetilde{Y})| \leq K \mathbb{E}_t^{\mathbb{P}}[|Y - \widetilde{Y}|], \quad Y, \widetilde{Y} \in L^1(\mathcal{F}_{t+1}, \mathbb{P}).$$

for some $K \in (0, \infty)$. *If* $(Z_t)_{t=1}^T \in L^p((\mathcal{F}_t)_{t=1}^T, \mathbb{P})$ *for some* $p > 1$, *then*

$$\mathbb{R}^{m+1} \ni w \mapsto W_0 \circ \cdots \circ W_{T-1}\Big(\sum_{t=1}^T \widetilde{X}_t^w\Big)$$

and $\mathbb{R}^m \ni v \mapsto V_0(X^v)$ *are Lipschitz continuous.*

For $t = 0, \dots, T-1$, set

$$\begin{aligned}
\widetilde{V}_t^w &:= W_t \circ \cdots \circ W_{T-1}\Big(\sum_{s=t+1}^T \widetilde{X}_s^w\Big), \\
\widetilde{R}_t^w &:= R_t\Big(-\widetilde{X}_{t+1}^w - \widetilde{V}_{t+1}^w\Big), \\
\widetilde{C}_t^w &:= \mathbb{E}_t^{\mathbb{Q}}\Big[(\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{V}_{t+1}^w)_+\Big], \\
\widetilde{\psi}(w) &:= \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{Q}}[\widetilde{C}_t^w].
\end{aligned}$$

Under mild conditions it can be shown that $\widetilde{\psi}$ and ψ, given by (14), are coercive, i.e.

$$\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty, \quad \lim_{|v|\to\infty} \psi(v) = \infty.$$

Theorem 7. *Suppose, for* $t = 0, \dots, T-1$, *that* R_t *is positively homogeneous in the sense* $R_t(\lambda Y) = \lambda R_t(Y)$ *for* $\lambda \in \mathbb{R}_+$. *Suppose further that* $\inf_{|w|=1} \widetilde{\psi}(w) > 0$. *Then* $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ *and* $\lim_{|v|\to\infty} \psi(v) = \infty$, *where* ψ *is given by* (14).

Remark 6. *Notice that the condition* $\inf_{|w|=1} \widetilde{\psi}(w) > 0$ *means that perfect replication is not possible. It also disqualifies risk-free cash flows as replication instruments. The argument is as follows. If one of the replication instruments has a risk-free cash flow* x *so that* $X^{f,k} = x$ $\mathbb{P}$*-a.s., then* $X^{f,k} = x$ $\mathbb{Q}$*-a.s. and* $w^{\mathrm{T}} Z = x$ *for some* $w \in \mathbb{R}^{m+1}$ *with* $|w| = 1$. *Then* $\widetilde{\psi}(w) = 0$.

For $t \in \{0, \dots, T-1\}$, set

$$R_{t,T-1}^\circ := \begin{cases} R_t, & t = T-1, \\ R_t \circ (-R_{t+1}) \circ \cdots \circ (-R_{T-1}), & t < T-1. \end{cases}$$

Theorem 8. *Suppose, for* $t = 0, \dots, T-1$, *that* R_t *is positively homogeneous in the sense* $R_t(\lambda Y) = \lambda R_t(Y)$ *for* $\lambda \in \mathbb{R}_+$. *Suppose further that* ψ *in* (14) *is continuous, and for all* $w \in \mathbb{R}^{m+1} \setminus \{0\}$ *there exists* $t \in \{0, \dots, T-1\}$ *such that*

$$\mathbb{P}\Big(\big(R_{t,T-1}^\circ - R_{t+1,T-1}^\circ\big)\big(-w^{\mathrm{T}}(Z_{t+1} + \cdots + Z_T)\big) > 0\Big) > 0. \qquad (15)$$

Then there exists an optimal solution $\widehat{v} \in \mathbb{R}^m$ *to* (14).

Remark 7. *The conditions of Theorem 8 are sufficient but not necessary for the existence of an optimal solution to* (14). *For instance, including risk-free cash flows as replication instrument would violate the condition that* (15) *holds for some t and all w without affecting either the optimal portfolio weights in the original replication instruments or the value of the liability cash flow.*

3 Gaussian cash flows

Let $(\epsilon_t)_{t=1}^T$ be a sequence of n-dimensional independent random vectors that are standard normally distributed under $\mathbb{P}$. For, $t = 1, \dots, T$ and nonrandom $A_t \in \mathbb{R}^n$, $B_{t,1}, \dots, B_{t,t} \in \mathbb{R}^{n\times n}$, let

$$G_t := A_t + \sum_{s=1}^{t} B_{t,s}\epsilon_s.$$

Let $(\mathcal{G}_t)_{t=0}^T$, with $\mathcal{G}_0 = \{\emptyset, \Omega\}$, be the filtration generated by the Gaussian process $(G_t)_{t=1}^T$. In what follows, $\mathbb{E}_t^{\mathbb{P}}$ and $\mathbb{E}_t^{\mathbb{Q}}$ mean conditional expectations with respect to $\mathcal{G}_t$. $(G_t)_{t=1}^T$, seen as a column vector, is the result of applying an affine transformation $x \mapsto A+Bx$ to $(\epsilon_t)_{t=1}^T$, where B is a lower-triangular block matrix with blocks $B_{i,j}$ and determinant $\prod_{t=1}^T \det(B_{t,t})$. In order to avoid unnecessary technicalities we assume that $\det(B_{t,t}) \neq 0$ for all t. This implies that the filtration generated by $(\epsilon_t)_{t=1}^T$ equals the filtration generated by $(G_t)_{t=1}^T$.

A natural interpretation of the Gaussian model is as follows: $X^o = G^{(1)}$ is the discounted liability cash flow, $G^{(2)}, \dots, G^{(m+1)}$ represent discounted cash flows of replication instruments, and $G^{(m+2)}, \dots, G^{(n)}$ represent insurance technical information flows.

For a nonrandom sequence $(\lambda_t)_{t=1}^T$, $\lambda_t \in \mathbb{R}^n$, let

$$D_t := \exp\Big\{\sum_{s=1}^{t}\big(\lambda_s^{\mathrm{T}}\epsilon_s - \frac{1}{2}\lambda_s^{\mathrm{T}}\lambda_s\big)\Big\}, \quad t = 1, \dots, T.$$

We let the measure $\mathbb{Q}$ be defined in terms of the $(\mathbb{P}, \mathbb{G})$-martingale $(D_t)_{t=1}^T$: For a $\mathcal{G}_t$-measurable sufficiently integrable Z and $s < t$, in accordance with Section 2, $\mathbb{E}_s^{\mathbb{Q}}[Z] = D_s^{-1}\mathbb{E}_s^{\mathbb{P}}[D_t Z]$. This choice has several pleasant consequences: for arbitrary vectors $g_s \in \mathbb{R}^n$ and $u > t$,

$$\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big] - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big] \in \mathcal{G}_0,$$

$$\mathrm{Var}_t^{\mathbb{Q}}\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big) = \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big) \in \mathcal{G}_0,$$

i.e. the conditional expectations with respect to $\mathbb{Q}$ and $\mathbb{P}$ only differ by a constant and the conditional variances with respect to $\mathbb{Q}$ and $\mathbb{P}$ are equal and nonrandom.

Definition 4. *The triple* $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ *is called a Gaussian model.*

The Gaussian model allows for explicit valuation formulas when combined with conditional monetary risk measures in Definition 2. The following properties considerably simplify computations. For $u > t$,

$$\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big]$$

is independent of $\mathcal{G}_t$, and, whenever $\operatorname{Var}_t^{\mathbb{P}}\big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\big) \neq 0$,

$$\operatorname{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big)^{-1/2}\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big]\Big)$$

is standard normally distributed with respect to $\mathbb{P}$. Since a risk measure R_t in the sense of Definition 2 has the additional property $R_t(\lambda Y) = \lambda R_t(Y)$ if $\lambda \in \mathbb{R}_+$ and $Y \in L^p(\mathcal{F}_{t+1}, \mathbb{P})$ (positive homogeneity), it follows that

$$R_t\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big) = -\mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big] + \operatorname{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^{u} g_s^{\mathrm{T}} G_s\Big)^{1/2} r_0,$$

where

$$r_0 := \int_0^1 \Phi^{-1}(u) dM(u). \tag{16}$$

We will first derive an explicit expression for the value of a general Gaussian liability cash flow, where the generality lies in that X_t is allowed to be an arbitrary linear combination $g_t^{\mathrm{T}} G_t$, where $g_t \in \mathbb{R}^n$ may be time dependent. Then we will return to the relevant special case when $g_t = g$ for all t and $g^{(1)} = 1$, $(g^{(k)})_{k=2}^{m+1} = v \in \mathbb{R}^m$ and $g^{(k)} = 0$ for $k > m+1$.

Theorem 9. *Let* $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ *be a Gaussian model and, for* $t = 1, \dots, T$*, set* $X_t := g_t^{\mathrm{T}} G_t$*. For* $t = 0, \dots, T-1$*, let* R_t *be conditional monetary risk measures in the sense of Definition 2 for a common probability distribution* M*. Let* r_0 *be given by* (16)*. Then*

$$V_t = \sum_{s=t+1}^{T} \mathbb{E}_t^{\mathbb{Q}}[X_s] + K_t^{\mathbb{Q}} = \sum_{s=t+1}^{T} \mathbb{E}_t^{\mathbb{P}}[X_s] + K_t^{\mathbb{P}},$$

where, with e_1 standard normally distributed with respect to $\mathbb{P}$,

$$
\begin{aligned}
K_t^{\mathbb{Q}} = \sum_{s=t+1}^{T} \Big(&\sigma_s r_0 - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s \\
&- \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s\big(r_0 - e_1\big) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big), \\
K_t^{\mathbb{P}} = \sum_{s=t+1}^{T} \Big(&\sigma_s r_0 - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s\big(r_0 - e_1\big) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big), \\
\sigma_s^2 = \mathrm{Var}_{s-1}^{\mathbb{P}}&\Big(\sum_{u=s}^{T} X_u\Big) - \mathrm{Var}_s^{\mathbb{P}}\Big(\sum_{u=s}^{T} X_u\Big) = \sum_{j=s}^{T}\sum_{k=s}^{T} g_j^{\mathrm{T}} B_{j,s} B_{k,s}^{\mathrm{T}} g_k.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
C_t &:= R_t(-X_{t+1} - V_{t+1}) - V_t \\
&= \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}\big(r_0 - e_1\big) - \sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1}\Big)_+\Big].
\end{aligned}
$$

Remark 8. *Notice that*

$$
\begin{aligned}
C_t &= \mathbb{E}_t^{\mathbb{Q}}\Big[\Big(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1}\Big)_+\Big] \\
&= \frac{1}{1+\eta_t}\mathbb{E}_t^{\mathbb{P}}\Big[\Big(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1}\Big)_+\Big],
\end{aligned}
$$

where, given the setting in Theorem 9,

$$
\frac{1}{1+\eta_t} = \frac{\mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}\big(r_0 - e_1\big) - \sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1}\Big)_+\Big]}{\mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}\big(r_0 - e_1\big)\Big)_+\Big]}.
$$

In particular, $\eta_t \geq 0$ for every t if $\sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1} \geq 0$ for every t. Since

$$
\begin{aligned}
\sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{Q}}[X_u] - \sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{P}}[X_u] &= \sum_{u=t+1}^{T}\sum_{s=t+1}^{u} g_u^{\mathrm{T}} B_{u,s}\lambda_s \\
&= \sum_{s=t+1}^{T}\sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s
\end{aligned}
$$

we see that $\eta_t \geq 0$ for every t holds if $\sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{Q}}[X_u] \geq \sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{P}}[X_u]$ for every t. This is completely in line with Remark 1.

The following result presents the value of the liability cash flow when the replicating portfolio is a static portfolio with portfolio weights solving (14).

Theorem 10. *Let $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ be a Gaussian model. Let $X^o = G^{(1)}$ be the discounted liability cash flow, let $X^{f,1} := G^{(2)}, \dots, X^{f,m} := G^{(m+1)}$ represent discounted cash flows of replication instruments, and let $G^{(m+2)}, \dots, G^{(n)}$ represent arbitrary information flows. For $t = 0, \dots, T-1$, let R_t be conditional monetary risk measures in the sense of Definition 2 for a common probability distribution M. Let r_0 be given by* (16)*. Then there exists an optimal solution to* (14) *and the value of the liability is given by*

$$L_0 = \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[X_t^o] + \widehat{K}_0^{\mathbb{Q}},$$

where, with $S_t := \sum_{u=t}^T B_{u,t}$ and e_1 standard normally distributed with respect to $\mathbb{P}$,

$$\begin{aligned}\widehat{K}_0^{\mathbb{Q}} &= \sum_{t=1}^{T} \Big(\widehat{\sigma}_t r_0 - \widehat{g}^{\mathrm{T}} S_t \lambda_t - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\widehat{\sigma}_t\big(r_0 - e_1\big) - \widehat{g}^{\mathrm{T}} S_t \lambda_t\Big)_+\Big]\Big),\\ \widehat{\sigma}_t^2 &= \widehat{g}^{\mathrm{T}} S_t S_t^{\mathrm{T}} \widehat{g},\end{aligned}$$

where $\widehat{g}$ is the minimizer in $\big\{g \in \mathbb{R}^n : g_1 = 1, g_k = 0 \text{ for } k > m+1\big\}$ of

$$g \mapsto \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\big(g^{\mathrm{T}} S_{t+1} S_{t+1}^{\mathrm{T}} g\big)^{1/2}\big(r_0 - e_1\big) - g^{\mathrm{T}} S_{t+1}\lambda_{t+1}\Big)_+\Big].$$

4 Proofs

Proof of Theorem 1. We prove the statement by proving the equivalence of (i) and (ii) backward recursively in s. For $s = T-1$:

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{T,T}} \mathbb{E}_{T-1}^{\mathbb{Q}}[H_{T-1,\tau}] &= \mathbb{E}_{T-1}^{\mathbb{Q}}[H_{T-1,T}]\\ &= -(R_{T-1} - V_{T-1}) + \mathbb{E}_{T-1}^{\mathbb{Q}}[(R_{T-1} - X_T)_+].\end{aligned}$$

Hence, (i) and (ii) are equivalent for $s = T-1$. Now assume the equivalence of (i) and (ii) holds for $s = t+1$. Then

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}\left[H_{t,\tau}\right] \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{I}\{\tau = t+1\}H_{t,t+1} + \mathbb{I}\{\tau > t+1\}H_{t,\tau}\right] \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t+1,T}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{\tau = t+1\}H_{t,t+1} + \mathbb{I}\{\tau > t+1\}\mathbb{E}_{t+1}^{\mathbb{Q}}\left[H_{t,\tau}\right]\Big]\Big) \\
&= \operatorname*{ess\,sup}_{A\in\mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\}H_{t,t+1} + \mathbb{I}\{A^C\}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t+2,T}} \mathbb{E}_{t+1}^{\mathbb{Q}}\big[H_{t,\tau}\big]\Big]\Big) \\
&= -R_t + V_t + \operatorname*{ess\,sup}_{A\in\mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\}(R_t - X_{t+1} - V_{t+1})_+ \\
&\quad + \mathbb{I}\{A^C\}\Big(R_t - X_{t+1} - V_{t+1} + \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t+2,T}} \mathbb{E}_{t+1}^{\mathbb{Q}}\big[H_{t+1,\tau}\big]\Big)\Big]\Big) \\
&= -R_t + V_t + \operatorname*{ess\,sup}_{A\in\mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\}(R_t - X_{t+1} - V_{t+1})_+ \\
&\quad + \mathbb{I}\{A^C\}(R_t - X_{t+1} - V_{t+1})\Big]\Big) \\
&= -R_t + V_t + \mathbb{E}_t^{\mathbb{Q}}\big[(R_t - X_{t+1} - V_{t+1})_+\big].
\end{aligned}$$

Hence, (i) and (ii) are equivalent for $s = t$ and by the induction principle (i) and (ii) are equivalent for all $s \in \{0, \dots, T-1\}$. Furthermore, we see that the optimal stopping time is indeed given by τ_{t+1}^*. The proof is complete. □

Proof of Theorem 3. We prove the more involved statement (ii). Statement (i) is proved with the same arguments.

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y) - Y)_+\Big]^p\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{P}}\Big[\frac{D_{t+1}}{D_t}(R_t(-Y) - Y)_+\Big]^p\Big] \\
&\le \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p (R_t(-Y) - Y)_+^p\Big]\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p (R_t(-Y) - Y)_+^p\Big],
\end{aligned}$$

where the inequality is due to Jensen's inequality for conditional expectations. Moreover, for every $r > 1$, by Hölder's inequality,

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p (R_t(-Y) - Y)_+^p\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^{pr}\Big]^{\frac{1}{r}} \mathbb{E}^{\mathbb{P}}\Big[(R_t(-Y) - Y)_+^{p\frac{r}{r-1}}\Big]^{\frac{r-1}{r}}.$$

For $r > 1$ sufficiently large, it follows from the assumptions that the two expectations exist finitely. Finally, it follows from Minkowski's inequality that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(R_t(-Y) - \mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y) - Y)_+\Big]\Big)^p\Big]^{\frac{1}{p}} \\ \leq \mathbb{E}^{\mathbb{P}}\Big[|R_t(-Y)|^p\Big]^{\frac{1}{p}} + \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y) - Y)_+\Big]^p\Big]^{\frac{1}{p}}.$$

The finiteness of the first terms follows from the assumptions and the finiteness of the second term has been proven above. This proves that the mapping is well-defined.

The remaining part of statement (ii) follows, upon minor modifications, from Proposition 1 in [9]. □

Proof of Theorem 5. (i) From Theorem 4 (i), $V_t(\widetilde{X}) = V_t(X) - \sum_{s=t+1}^{T} b_s$ from which both statements follow immediately. (ii) Notice that, for all $t \in \{1, \dots, T\}$, due to (7) and (9),

$$\begin{aligned}\sum_{s=1}^{t} X_s + V_t &= \sum_{s=1}^{t} X_s + W_t \circ \cdots \circ W_{T-1}\Big(\sum_{s=t+1}^{T} X_s\Big) \\ &= W_t \circ \cdots \circ W_{T-1}\Big(\sum_{s=1}^{T} X_s\Big) \\ &= K.\end{aligned}$$

Hence, for all $t \in \{1, \dots, T\}$, $X_t + V_t = K - \sum_{s=1}^{t-1} X_s$ is $\mathcal{F}_{t-1}$-measurable. This in turn, using (2), implies that, for all t,

$$C_t := \mathbb{E}_t^{\mathbb{Q}}\big[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+\big] = 0.$$

□

Proof of Theorem 6. For $w \in \mathbb{R}^{m+1}$ and $t \in \{0, \dots, T-1\}$, define

$$V_t^w := W_t \circ \cdots \circ W_{T-1}\Big(\sum_{s=t+1}^{T} w^{\mathrm{T}} Z_s\Big).$$

We prove the statement inductively. Assume that for some nonnegative $B_{t+2} \in L^1(\mathcal{F}_{t+2}, \mathbb{P})$,

$$|V_{t+1}^w - V_{t+1}^v| \leq ||v - w||_1 \mathbb{E}_{t+1}^{\mathbb{P}}[B_{t+2}],$$

where $||\cdot||_p$ denotes the Euclidean p-norm in $\mathbb{R}^{m+1}$. We start by showing the induction step, noting that verifying the induction base is trivial since $V_T^w = 0$.

Defining $Y^w_{t+1} := w^{\mathrm{T}} Z_{t+1} + V^w_{t+1}$ and applying Hölder's inequality,

$$\begin{aligned}|Y^w_{t+1} - Y^v_{t+1}| &\leq |V^w_{t+1} - V^v_{t+1}| + |w^{\mathrm{T}} Z_{t+1} - v^{\mathrm{T}} Z_{t+1}| \\ &\leq ||v-w||_1 \mathbb{E}^{\mathbb{P}}_{t+1}[B_{t+2}] + |w^{\mathrm{T}} Z_{t+1} - v^{\mathrm{T}} Z_{t+1}| \\ &\leq ||v-w||_1 \mathbb{E}^{\mathbb{P}}_{t+1}[||Z_{t+1}||_\infty + B_{t+2}]\end{aligned}$$

Now, due to the L^1-Lipschitz continuity of R_t,

$$\begin{aligned}|R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| &\leq K \mathbb{E}^{\mathbb{P}}_t[|Y^w_{t+1} - Y^v_{t+1}|] \\ &\leq K ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t[||Z_{t+1}||_\infty + B_{t+2}]\end{aligned}$$

With $C^w_t := \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1})_+]$, due to subadditivity of $x \mapsto x_+ := \max(x,0)$,

$$\begin{aligned}C^w_t - C^v_t &= \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1})_+ - (R_t(-Y^v_{t+1}) - Y^v_{t+1})_+] \\ &\leq \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1})_+] \\ &\leq \mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|], \\ C^w_t - C^v_t &\geq \mathbb{E}^{\mathbb{Q}}_t[-(R_t(-Y^v_{t+1}) - Y^v_{t+1} - R_t(-Y^w_{t+1}) + Y^w_{t+1})_+] \\ &\geq -\mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|]\end{aligned}$$

from which it follows that

$$\begin{aligned}|C^w_t - C^v_t| &\leq \mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|] \\ &\leq |R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| + \mathbb{E}^{\mathbb{Q}}_t[|Y^w_{t+1} - Y^v_{t+1}|].\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_t[|Y^w_{t+1} - Y^v_{t+1}|] &\leq \mathbb{E}^{\mathbb{Q}}_t\Big[||v-w||_1 \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big] \\ &= ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t\Big[\frac{D_{t+1}}{D_t} \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big].\end{aligned}$$

Hence,

$$\begin{aligned}|V^w_t - V^v_t| &\leq |R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| + |C^w_t - C^v_t| \\ &\leq 2K ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t[||Z_{t+1}||_\infty + B_{t+2}] \\ &\quad + ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t\Big[\frac{D_{t+1}}{D_t} \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big] \\ &= ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big)\Big] \\ &= ||v-w||_1 \mathbb{E}^{\mathbb{P}}_t\Big[B_{t+1}\Big],\end{aligned}$$

where $B_{T+1} = 0$ and otherwise

$$B_{t+1} := \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big).$$

In particular,

$$|V_0^w - V_0^v| \leq ||v - w||_1 \mathbb{E}^{\mathbb{P}}_0[B_1],$$

Now what remains is to show that $\mathbb{E}^{\mathbb{P}}_0[B_1] < \infty$. For the Euclidean norms, the inequality $||x||_p \leq ||x||_1$ holds for $p \in [1, \infty]$. In particular, for each $t = 1, \ldots, T$, $0 \leq B_t \leq \widetilde{B}_t$, where

$$\widetilde{B}_{t+1} := \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big), \quad \widetilde{B}_{T+1} = 0.$$

Recall that for $t = 1, \ldots, T$, $Z_t^{(k)} \in L^{p_t}(\mathcal{F}_t, \mathbb{P})$ for all k and some $p_t > 1$. Also notice that if $\widetilde{B}_{t+2} \in L^{q_{t+2}}(\mathcal{F}_{t+2}, \mathbb{P})$ for $q_{t+2} > 1$, then $\mathbb{E}^{\mathbb{P}}_{t+1}[\widetilde{B}_{t+2}] \in L^{q_{t+2}}(\mathcal{F}_{t+1}, \mathbb{P})$ and, for $r_{t+1} = \min(p_{t+1}, q_{t+2})$,

$$\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big] \in L^{r_{t+1}}(\mathcal{F}_{t+1}).$$

Hence, for any $\epsilon > 0$,

$$\widetilde{B}_{t+1} = \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big) \in L^{r_{t+1}-\epsilon}(\mathcal{F}_{t+1}).$$

Since $\widetilde{B}_{T+1} = 0$ we may choose $\epsilon > 0$ small enough so that $\widetilde{B}_t \in L^1(\mathcal{F}_t, \mathbb{P})$ for $t = 1, \ldots, T$. Hence, also $B_t \in L^1(\mathcal{F}_t, \mathbb{P})$ for $t = 1, \ldots, T$.

Finally, notice that

$$X_t^v := X_t^o - v^{\mathrm{T}} X_t^f = w^{\mathrm{T}} Z_t$$

if $w \in \mathbb{R}^{m+1}$ is chosen so that $w_1 = 1$ and $(w_k)_{k=2}^{m+1} = v$. Therefore, we have also shown that $v \mapsto V_0(X^v)$ is Lipschitz continuous. □

Proof of Theorem 7. From positive homogeneity of the R_ts follows positive homogeneity of the W_ts which implies $\widetilde{V}_t^w(\lambda \widetilde{X}^w) = \lambda \widetilde{V}_t^w(\widetilde{X}^w)$ and further that $\widetilde{\psi}(\lambda w) = \lambda \widetilde{\psi}(w)$. In particular,

$$\widetilde{\psi}(w) = |w| \widetilde{\psi}(w/|w|) \geq |w| \inf_{|w|=1} \widetilde{\psi}(w)$$

from which $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ follows from the assumption $\inf_{|w|=1} \widetilde{\psi}(w) > 0$. For the second statement, notice that

$$X_t^v := X_t^o - v^{\mathrm{T}} X_t^f = w^{\mathrm{T}} Z_t$$

if $w \in \mathbb{R}^{m+1}$ is chosen so that $w_1 = 1$ and $(w_k)_{k=2}^{m+1} = v$. Therefore, $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ implies $\lim_{|v|\to\infty} \psi(v) = \infty$. □

Proof of Theorem 8. Take $w \in \mathbb{R}^{m+1} \setminus \{0\}$. Suppose that $\widetilde{C}_t^w = 0$ $\mathbb{Q}$-a.s. for all t. Then $\widetilde{V}_t^w = \widetilde{R}_t^w$ for all t and $\widetilde{C}_t^w = 0$ is equivalent to $\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{R}_{t+1}^w \leq 0$ $\mathbb{Q}$-a.s. which is equivalent to $\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{R}_{t+1}^w \leq 0$ $\mathbb{P}$-a.s. since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. Notice that

$$\begin{aligned}
\widetilde{R}_t^w &= R_t(-\widetilde{X}_{t+1}^w - \widetilde{R}_{t+1}^w) \\
&= R_t(-\widetilde{X}_{t+1}^w - R_{t+1}(-\widetilde{X}_{t+2}^w - \widetilde{R}_{t+2}^w)) \\
&= R_t \circ (-R_{t+1}) \circ \cdots \circ (-R_{T-1})\Big(- \sum_{s=t+1}^{T} \widetilde{X}_s^w \Big)
\end{aligned}$$

The inequality $\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{R}_{t+1}^w \leq 0$ $\mathbb{P}$-a.s. can thus be expressed as

$$\big(R_{t,T-1}^{\circ} - R_{t+1,T-1}^{\circ}\big)\big(- w^{\mathrm{T}}(Z_{t+1} + \cdots + Z_T)\big) \leq 0 \quad \mathbb{P}\text{-a.s.}$$

However, this is contradicting the assumption in the statement of the theorem. Therefore we conclude that $\widetilde{C}_t^w > 0$ $\mathbb{Q}$-a.s. for some t which implies that $\widetilde{\psi}(w) > 0$. Therefore, by Theorem 7, ψ is coercive so if a minimum exists it exists in some compact set in $\mathbb{R}^m$. However, a continuous function on a compact set attains its infimum. □

Lemma 1. *For $u < v$, $\mathbb{E}_u^{\mathbb{Q}}[G_v] = \mathbb{E}_u^{\mathbb{P}}[G_v] + \sum_{s=u+1}^{v} B_{v,s}\lambda_s$.*

Proof.

$$\begin{aligned}
\mathbb{E}_u^{\mathbb{Q}}[G_v] &= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\mathbb{E}_u^{\mathbb{P}}\Big[\frac{D_v}{D_u}\epsilon_s\Big] \\
&= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\mathbb{E}_0^{\mathbb{P}}\Big[\exp\Big\{\lambda_s^{\mathrm{T}}\epsilon_1 - \frac{1}{2}\lambda_s^{\mathrm{T}}\lambda_s\Big\}\epsilon_1\Big] \\
&= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\lambda_s \\
&= \mathbb{E}_u^{\mathbb{P}}[G_v] + \sum_{s=u+1}^{v} B_{v,s}\lambda_s.
\end{aligned}$$

□

Lemma 2. *If $X_s := g_s^{\mathrm{T}} G_s$, then*

$$\mathbb{E}_t^{\mathbb{P}}\Big[\mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] = \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}.$$

Proof. For $s \geq t+1$, with an empty sum defined as 0, it follows from Lemma 1 that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t+1}[X_s] &= \mathbb{E}^{\mathbb{P}}_{t+1}[X_s] + g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u, \\
\mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] &= \sum_{s=t+1}^{T} \Big(\mathbb{E}^{\mathbb{P}}_t[\mathbb{E}^{\mathbb{P}}_{t+1}[X_s]] + g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u\Big) \\
&= \mathbb{E}^{\mathbb{P}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] + \sum_{s=t+1}^{T} g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u, \\
\mathbb{E}^{\mathbb{P}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] &= \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} \sum_{u=t+1}^{s} B_{s,u}\lambda_u.
\end{aligned}$$

□

Proof of Theorem 9. We will prove inductively that

$$V_t = \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] + K^{\mathbb{Q}}_t, \tag{17}$$

and derive the recursive form of the constant term $K^{\mathbb{Q}}_t$ via induction. The induction base is trivial: $V_T = 0$. Now assume that (17) holds for $t+1$. Notice that

$$\begin{aligned}
V_t &= W_t\Big(X_{t+1} + \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+2}^{T} X_s\Big] + K^{\mathbb{Q}}_{t+1}\Big) \\
&= W_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big] + K^{\mathbb{Q}}_{t+1}\Big) \\
&= K^{\mathbb{Q}}_{t+1} + R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) \\
&\quad - \mathbb{E}^{\mathbb{Q}}_t\Big[\Big(R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) - \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)_+\Big]
\end{aligned}$$

We first evaluate the risk measure part.

$$
\begin{aligned}
&R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)\\
&\quad= \mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] + \mathrm{Var}^{\mathbb{P}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)^{1/2} r_0\\
&\quad= \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} + \mathrm{Var}^{\mathbb{P}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)^{1/2} r_0,
\end{aligned}
$$

where in the final step we used Lemma 2. Moreover,

$$
\begin{aligned}
\mathrm{Var}^{\mathbb{P}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) &= \mathrm{Var}^{\mathbb{Q}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)\\
&= \mathrm{Var}^{\mathbb{Q}}_t\Big(\sum_{s=1}^{T} X_s\Big) - \mathrm{Var}^{\mathbb{Q}}_{t+1}\Big(\sum_{s=1}^{T} X_s\Big)\\
&=: \sigma_{t+1}^2.
\end{aligned}
$$

The remaining term: if $\sigma_{t+1} \neq 0$, then there exists a random variable e^*_{t+1} that is independent of $\mathcal{G}_t$ and standard normally distributed with respect to $\mathbb{Q}$ such that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_t\Big[\Big(R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big] - K^{\mathbb{Q}}_{t+1}\Big) - \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big] - K^{\mathbb{Q}}_{t+1}\Big)_+\Big]\\
&\quad= \mathbb{E}^{\mathbb{Q}}_t\Big[\Big(\sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - \sigma_{t+1} e^*_{t+1}\Big)_+\Big]\\
&\quad= \mathbb{E}^{\mathbb{P}}_0\Big[\Big(\sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - \sigma_{t+1} e_1\Big)_+\Big].
\end{aligned}
$$

Putting the pieces together now yields

$$
\begin{aligned}
V_t = \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] + K^{\mathbb{Q}}_{t+1} + \sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}\\
- \mathbb{E}^{\mathbb{P}}_0\Big[\Big(\sigma_{t+1}(r_0 - e_1) - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}\Big)_+\Big]
\end{aligned}
$$

which proves the induction step and from which it follows that

$$
K^{\mathbb{Q}}_t = \sum_{s=t+1}^{T}\Big(\sigma_s r_0 - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s - \mathbb{E}^{\mathbb{P}}_0\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big).
$$

Finally,

$$\begin{aligned}
V_t &= \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] + K_t^{\mathbb{Q}} \\
&= \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=t+1}^{T} X_s\Big] + \sum_{s=t+1}^{T}\sum_{u=t+1}^{s} g_s^{\mathrm{T}} B_{s,u}\lambda_u + K_t^{\mathbb{Q}} \\
&= \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=t+1}^{T} X_s\Big] + K_t^{\mathbb{P}},
\end{aligned}$$

where

$$K_t^{\mathbb{P}} = \sum_{s=t+1}^{T}\Big(\sigma_s r_0 - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big).$$

We now derive an expression for σ_{t+1}. Recall that $X_s := g_s^{\mathrm{T}} G_s$.

$$\begin{aligned}
\mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) &= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T}\sum_{u=t+1}^{s} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{u=t+1}^{T}\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \sum_{u=t+1}^{T} \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \sum_{u=t+1}^{T}\Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\Big)\Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\Big)^{\mathrm{T}} \\
&= \sum_{u=t+1}^{T}\sum_{j=u}^{T}\sum_{k=u}^{T} g_j^{\mathrm{T}} B_{j,u} B_{k,u}^{\mathrm{T}} g_k
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{t+1}^2 &:= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) - \mathrm{Var}_{t+1}^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) \\
&= \sum_{j=t+1}^{T}\sum_{k=t+1}^{T} g_j^{\mathrm{T}} B_{j,t+1} B_{k,t+1}^{\mathrm{T}} g_k
\end{aligned}$$

We now derive the expression for C_t. Using the same arguments as earlier in the proof,

$$\begin{aligned}C_t &= R_t(-X_{t+1} - V_{t+1}) - V_t \\ &= R_t\Big(- \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big] - K^{\mathbb{Q}}_{t+1}\Big) - \mathbb{E}^{\mathbb{Q}}_{t}\Big[\sum_{s=t+1}^{T} X_s\Big] - K^{\mathbb{Q}}_{t} \\ &= \sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - K^{\mathbb{Q}}_{t} + K^{\mathbb{Q}}_{t+1} \\ &= \mathbb{E}^{\mathbb{P}}_0\Big[\Big(\sigma_{t+1}(r_0 - e_1) - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}\Big)_+\Big].\end{aligned}$$

□

Proof of Theorem 10. We will prove that there exists an optimal solution to (14). The remaining part then follows from Theorem 9.

From Theorem 9 we immediately see that $\psi(w)$ is continuous. Once we show that for all $w \in \mathbb{R}^{m+1} \setminus \{0\}$ there exists $t \in \{0, \dots, T-1\}$ such that (15) holds, existence of an optimal solution to (14) follows. We prove this statement by first proving that there is no $w \in \mathbb{R}^{m+1} \setminus \{0\}$ such that $\sum_{t=1}^{T} w^{\mathrm{T}} Z_t \in \mathcal{G}_0$, where $Z_t := (X^o_t, -(X^f_t)^{\mathrm{T}})^{\mathrm{T}}$. Notice that, for $g \in \mathbb{R}^n$,

$$g^{\mathrm{T}} \sum_{t=1}^{T} G_t = g^{\mathrm{T}} \sum_{t=1}^{T} A_t + g^{\mathrm{T}} \sum_{s=1}^{T-1}\sum_{t=s}^{T} B_{s,t}\epsilon_s + g^{\mathrm{T}} B_{T,T}\epsilon_T.$$

The ϵ_s are independent and $g^{\mathrm{T}} B_{T,T} \neq 0$ for all $g \neq 0$. Hence, there is no $g \in \mathbb{R}^n \setminus \{0\}$ such that $g^{\mathrm{T}} \sum_{t=1}^{T} G_t \in \mathcal{G}_0$ which in turn implies that here is no $w \in \mathbb{R}^{m+1} \setminus \{0\}$ such that $\sum_{t=1}^{T} w^{\mathrm{T}} Z_t \in \mathcal{G}_0$. We now prove that the latter statement implies that for all $w \in \mathbb{R}^{m+1} \setminus \{0\}$ there exists $t \in \{0, \dots, T-1\}$ such that (15) holds.

Notice that

$$\begin{aligned}&\big(R^{\circ}_{t,T-1} - R^{\circ}_{t+1,T-1}\big)\big(- w^{\mathrm{T}}(Z_{t+1} + \dots + Z_T)\big) \\ &\quad= \big(R^{\circ}_{t,T-1} - R^{\circ}_{t+1,T-1}\big)\big(- w^{\mathrm{T}}(Z_{1} + \dots + Z_T)\big) \\ &\quad= \mathbb{E}^{\mathbb{P}}_t[w^{\mathrm{T}}(Z_1 + \dots + Z_T)] - \mathbb{E}^{\mathbb{P}}_{t+1}[w^{\mathrm{T}}(Z_1 + \dots + Z_T)] + c\end{aligned}$$

for some constant c, where the last equality follows from calculations completely analogous to the proof of Theorem 9. Now assume that for some $w \in \mathbb{R}^{m+1} \setminus \{0\}$, (15) does not hold. In the current Gaussian setting, the support of a Gaussian distribution is either infinite or a singleton, this implies that

$$\big(R^{\circ}_{t,T-1} - R^{\circ}_{t+1,T-1}\big)\big(- w^{\mathrm{T}}(Z_{1} + \dots + Z_T)\big) = 0 \quad \mathbb{P}\text{-a.s. for all } t$$

or, equivalently, that

$$\mathbb{E}^{\mathbb{P}}_t[w^{\mathrm{T}}(Z_1+\cdots+Z_T)] - \mathbb{E}^{\mathbb{P}}_{t+1}[w^{\mathrm{T}}(Z_1+\cdots+Z_T)] \in \mathcal{G}_0 \quad \text{for all } t. \tag{18}$$

For $t=0$, (18) implies that $\mathbb{E}^{\mathbb{P}}_1[w^{\mathrm{T}}(Z_1+\cdots+Z_T)] \in \mathcal{G}_0$ which together with (18) for $t=1$ implies that $\mathbb{E}^{\mathbb{P}}_2[w^{\mathrm{T}}(Z_1+\cdots+Z_T)] \in \mathcal{G}_0$. By repeating this argument we have shown that

$$w^{\mathrm{T}}(Z_1+\cdots+Z_T) = \mathbb{E}^{\mathbb{P}}_T[w^{\mathrm{T}}(Z_1+\cdots+Z_T)] \in \mathcal{G}_0$$

which contradicts the assumption $w^{\mathrm{T}}(Z_1+\cdots+Z_T) \notin \mathcal{G}_0$. Hence, we conclude that there exists an optimal solution to (14). The remaining part follows immediately from Theorem 9. □

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