Contemporary Mathematics

Singular and supersingular perturbations: Hilbert space methods

P. Kurasov

Abstract. These lecture notes are devoted to recent developments in the theory of generalized finite rank perturbations of self–adjoint operators. Both singular and supersingular perturbations are considered. We concentrate our attention to resolvent formulas describing such interactions. Developed methods are applied to obtain point interaction model in $\mathbf{R}^3$ leading to not spherically symmetric eigenfunctions, but having p−symmetry instead.

Contents

1. Introduction

These lecture notes were prepared for the Workshop on Schrödinger operators at the Instituto de Matematicas de la Universidad Nacional de Mexico, Unidad Cuernavaca, held in December 2001. I would like to thank Jaime Cruz Sampedro, Rafael del Rio and Carlos Villegas Blas for a kind invitation to Mexico.

1.1. Historical remarks. Finite rank perturbations of self-adjoint operators are widely used in modern mathematical physics, since they lead to exactly solvable models preserving important features of realistic physical problems. From the extended list of literature the following books and reviews should be mentioned:

Yu.Demkov and V.N.Ostrovsky, *Zero-range potentials and their applications in atomic physics*, [**15**], describing applications of these methods to atomic physics;

2000 *Mathematics Subject Classification.* Primary 47A55, 81Q10; Secondary 47A20, 81Q15.

S.Albeverio, F.Gesztesy, R.Høegh-Krohn and H.Holden, *Solvable models in quantum mechanics*, [**3**], where mathematical theory of point interactions is described;

B.Pavlov, *The theory of extensions and explicitly solvable models*, [**48, 49**], presenting the method of generalized point interactions;

B.Simon, *Spectral analysis of rank one perturbations and applications*, [**58**], where the mathematical theory of form bounded singular rank one perturbations is presented;

S.Albeverio and P.Kurasov, *Singular perturbations of differential operators*, [**8**], containing detailed description of singular and generalized finite rank interactions with applications to problems from quantum mechanics.

These publications contain extended reference lists describing development of these methods in the recent years. All mentioned publications have been grown up from two main sources

E.Fermi, *Sul moto dei neutroni nelle sostanze idrogenate*, [**23**], presenting the first physical model with point interaction;

F.A.Berezin and L.D.Faddeev, *Remark on the Schrödinger equation with singular potential*, [**11**], giving the first mathematically rigorous presentation of Fermi model using the extension theory for self–adjoint operators.

Since the appearance of these articles the methods described there have been used in numerous physical problems which showed their efficiency. It is natural that the method of point interactions is limited and cannot demonstrate all features of real problems. But it is important to be able to obtain more and more realistic models. Let us consider classical example studied in [**11**]: the Laplace operator with point interaction in $\mathbf{R}^3$ given formally by

$$L_\alpha = -\Delta + \alpha\delta, \tag{1}$$

where $-\Delta$ is the Laplace operator, δ is Dirac's delta function, α - real coupling parameter. The perturbation term is so singular that the perturbed operator cannot be defined using standard methods from perturbation theory of self–adjoint operators. It has been proved by F.A.Berezin and L.D.Faddeev that the family of self–adjoint operators corresponding to the formal expression (1) is described by one real parameter and every operator from this family differs from the Laplace operator on the subspace of spherically symmetric functions only. This is natural since the potential δ-function vanishes on every function from $L^2(\mathbf{R}^3)$ having symmetry different from the s-type.

It is useful to consider the operator L_α as a rank one singular perturbation of $-\Delta$. On regular functions u the following equality holds

$$\delta u = u(0)\delta = \langle \delta, u\rangle \delta,$$

where the brackets $\langle \cdot, \cdot\rangle$ denote both the scalar product in $L^2(\mathbf{R}^3)$ and the action of functionals. Then the operator L_α can at least formally be written as

$$L_\alpha = -\Delta + \alpha\langle \delta, \cdot\rangle \delta. \tag{2}$$

The last formula describes the relation between the operators with singular potentials supported by a discrete set of points (like formal operator (1)) and finite rank singular perturbations of operators (like formal operator (2)). Point interaction method is described in detail in [**3, 15, 48**]. Generalized finite rank perturbations were studied in [**5, 6, 7, 8, 25, 32, 58**].

The operator L_α is a point perturbation of the Laplace operator. Operator B is called *point perturbation* of a differential operator A in $L^2(\mathbf{R}^d)$ if the restriction of the operators A and B to the set of functions vanishing on a certain set of points in $\mathbf{R}^3$ coincide. The perturbation term in (1) vanishes on the functions equal to zero at the origin. Therefore any operator corresponding to L_α is a point perturbation of the Laplace operator. For point interactions one can easily calculate the resolvent of the perturbed operator using Krein's formula [**1**] originally derived in [**33, 34, 35, 46**]. Generalizations of Krein's formula which include operators with infinite defficiency indices and therefore can connect any two self-adjoint operators are presented in [**24, 41, 54**]. This formula is especially effective if the number of points supporting the perturbation is finite. This makes these problems exactly solvable. In fact this resolvent formula plays crucial role in investigation of the spectral structure of all operators described in the current review. We are going to concentrate our attention to different generalizations of this resolvent formula to the case of highly singular perturbations. Four different resolvent formulas will be presented.

It is possible to ask the opposite question: How to describe the set of all point perturbations at the origin of the Laplace operator? Consider the restriction of the operator $-\Delta$ to the set of functions vanishing at the origin. The deficiency elements for this operator are spherically symmetric and any its self–adjoint extension differs from the original operator $-\Delta$ on the subspace of spherically symmetric functions only. In other words any point perturbation of the Laplace operator is spherically symmetric. Moreover the perturbed operator has at most one eigenvalue and the corresponding eigenfunction is spherically symmetric. This fact restricts possible applications of the point interaction method to physics enormously. For example in atomic physics one cannot apply this method to get point models of atoms with non-spherically symmetric eigenfunctions.

To overcome this obstacle it was suggested to use operators in certain Pontryagin-Krein extensions of the original Hilbert space. If singular perturbations are described by Nevanlinna functions, the supersingular perturbations correspond to generalized Nevanlinna functions. Such functions appear naturally considering extensions of symmetric operators in Krein spaces [**22, 36, 37, 38, 19**]. This was probably the original motivation to use extensions with indefinite scalar product. Models of that type were first suggested by Yu.Shondin [**55, 56**] and later by J.van Diejen and A.Tip [**18**]. See also recent articles [**19, 21, 57**] for further developments. Similar models were analyzed in [**31, 50, 51, 52**]. To apply these models to quantum mechanics one restricts these operators to a certain positive definite subspace of the Pontryagin space.

It was discovered in [**9, 43**] that it is possible to determine such perturbations as self–adjoint operators in a certain extended Hilbert space. This method has been generalized to include so-called singular and supersingular perturbations of arbitrary order [**40, 44**]. The operator corresponding to supersingular interactions were defined in the class of regular operators - densely defined operators having the same domain as their adjoints. The spectrum of such operators is not necessarily real, but all operators appearing in these models have real spectrum. It was shown later that by considering more general models self-adjoint operators can be obtained [**20**].

Yu.N.Demkov and G.F.Drukarev suggested an alternative approach to construct models with point-like interactions allowing not spherically symmetric eigenfunctions [**16, 17**]. The equations used there are not of operator type. Another approach due to Yu.Karpeshina leads to an operator which is defined through its spectral decomposition [**30**] and therefore is not given explicitly which limits possible applications of the model drastically.

1.2. Regular, singular and supersingular rank one perturbations. Let A be a self-adjoint operator acting in a certain Hilbert space $\mathcal{H}$. To avoid unessential difficulties we suppose that A is positive. The case of semibounded operators can be considered without any serious modification of the method. This assumption is usually fulfilled in physical applications. To cover not-semibounded operators one needs to modify our approach which can easily be done. We confine our consideration here to the case of positive operators to avoid additional difficulties only.

In standard perturbation theory one considers the operator sum

$$A + V.$$

The operator V has to be subordinated in some sense by the operator A in order to get an efficient theory. This operator sum can be considered in the sense of quadratic forms, then the quadratic form of V has to be bounded with respect to the quadratic form of A. It has been known for a long time that for finite rank perturbations one can go beyond the standard perturbation theory. The condition on the quadratic forms of the operator and perturbation is substituted by the condition that the quadratic form of the perturbation is bounded with respect to the quadratic form of the operator $(A+1)^n$, where n is a certain natural number. Therefore the language of the scale of Hilbert spaces associated with the positive operator A is useful

$$\begin{array}{ccccccccccc} & & \mathrm{Dom}\,(A) & & & \mathcal{H} & & & (\mathrm{Dom}\,(A))^* & & \\ & & \| & & & \| & & & \| & & \\ \ldots \subset \mathcal{H}_3 & \subset & \mathcal{H}_2 & \subset & \mathcal{H}_1 & \subset \mathcal{H}_0 & \subset & \mathcal{H}_{-1} \subset & \mathcal{H}_{-2} & \subset & \mathcal{H}_{-3} \subset \ldots \end{array} \tag{3}$$

The spaces $\mathcal{H}_{-s}$, $s = 1, 2, ...$ can be considered as completion of $\mathcal{H} = \mathcal{H}_0$ with respect to the following norm

$$\| U \|^2_{\mathcal{H}_n} = \langle U, (A+1)^n U \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the original Hilbert space $\mathcal{H}$. Then the spaces with positive indexes are just adjoint spaces

$$\mathcal{H}_s = \mathcal{H}^*_{-s},$$

so that the spaces $\mathcal{H}_n \subset \mathcal{H} \subset \mathcal{H}_{-n}$, $n = 1, 2, ...$ form Gelfand triplet of Hilbert spaces. The operator $A + 1$ acts as isometric shift in the scale of Hilbert spaces mapping $\mathcal{H}_n$ onto $\mathcal{H}_{n-2}$. Therefore the space $\mathcal{H}_2$ coincides with the domain of the operator A.

Consider the formal operator expression

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \quad \alpha \in \mathbf{R}, \tag{4}$$

where φ is an element from the scale of Hilbert spaces. We say that the interaction is from the class $\mathcal{H}_{-n}$ if and only if $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}$.

The following definitions will be useful in what follows:

Regular perturbations - perturbations defined by vectors φ from $\mathcal{H}$. The perturbation term is bounded and the perturbed operator is defined on the domain of the original operator.

Singular perturbations - perturbations defined by vectors φ from $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ but not from $\mathcal{H}$. These perturbations can be defined in the original Hilbert space, but the domain of the perturbed operator does not coincide with the domain of the original one.

Supersingular perturbations - perturbations defi6ned by vectors φ from $\mathcal{H}_{-n}, n \geq 3$ but not from $\mathcal{H}_{-2}$. To define these perturbations one needs to extend the original Hilbert space.

Rank one regular, singular and supersingular perturbations are described in the following sections. As application of our methods we consider the model of point interaction in $\mathbf{R}^3$ leading to eigenfunctions having p-symmetry. We call this interaction p-symmetric point interaction (even if the interaction itself commutes with the rotations).

2. Regular rank one perturbations and the first resolvent formula

In this section we consider rank one perturbations determined by vectors from the Hilbert space. Such perturbations are bounded and therefore do not change the domain of the original operator.

Let A be a self-adjoint operator acting in the Hilbert space $\mathcal{H}$ with the domain $\mathrm{Dom}\,(A)$, and φ be a vector from this space $\varphi \in \mathcal{H}$. Then the operator

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \;\; \varphi \in \mathcal{H}, \;\; \alpha \in \mathbf{R}, \tag{5}$$

is called rank one perturbation of the operator A. The perturbation term is given by a bounded (rank one) self-adjoint operator $\alpha\langle\varphi,\cdot\rangle\varphi$. Hence the perturbed operator is uniquely defined and has the same domain $\mathrm{Dom}\,(A)$ as the original operator A.

The relation between the resolvents of the original and perturbed operators are described by

PROPOSITION 2.1. *Let A be a self–adjoint operator acting in the Hilbert space H and let φ be arbitrary vector from the Hilbert space, $\varphi \in \mathcal{H}$. Then the resolvents of the original operator A and its rank one perturbation $A_\alpha = A + \alpha\langle\varphi,\cdot\rangle\varphi$, $\alpha \in \mathbf{R}$, are related as follows for arbitrary $z, \Im z \neq 0$,*

$$\frac{1}{A_\alpha - z} - \frac{1}{A - z} = -\frac{\alpha}{1 + \alpha F(z)} \left\langle \frac{1}{A - \bar{z}}\varphi, \cdot \right\rangle \frac{1}{A - z}\varphi, \tag{6}$$

where

$$F(z) = \left\langle \varphi, \frac{1}{A - z}\varphi \right\rangle. \tag{7}$$

Proof To calculate the resolvent of the self–adjoint operator A_α we have to solve the following equation

$$h = (A_\alpha - z) f,$$

for a given $h \in H$ and $f \in \mathrm{Dom}\,(A_\alpha) = \mathrm{Dom}\,(A)$. We assume that the imaginary part of the spectral parameter z is positive $\Im z > 0$. We apply the operator $A_\alpha - z$

to the last equality

$$\begin{aligned} h &= (A + \alpha\langle\varphi,\cdot\rangle\varphi - z) f \\ &= Af - zf + \alpha\langle\varphi, f\rangle\varphi. \end{aligned}$$

By applying the resolvent of the original operator we get

$$\frac{1}{A-z}h = f + \alpha\langle\varphi, f\rangle \frac{1}{A-z}\varphi.$$

Projection on the vector φ leads to the following formula for $\langle\varphi, f\rangle$

$$\langle\varphi, f\rangle = \frac{\langle\varphi, \frac{1}{A-z}h\rangle}{1 + \alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}.$$

It follows that

$$f = \frac{1}{A-z}h - \frac{\alpha}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\left\langle\varphi, \frac{1}{A-z}g\right\rangle \frac{1}{A-z}\varphi,$$

which is exactly formula (6). The theorem is proven.

□

All spectral properties of the perturbed operator are described by the real coupling parameter α and the bordered resolvent

$$F(z) = \left\langle \varphi, \frac{1}{A-z}\varphi \right\rangle. \tag{8}$$

The function F is a Nevanlinna function, i.e. a holomorphic function in $\mathbf{C} \setminus \mathbf{R}$ satisfying the following conditions

$$\overline{F(z)} = F(\overline{z}); \tag{9}$$

$$\frac{\Im F(z)}{\Im z} \geq 0, \quad z \in \mathbf{C} \setminus \mathbf{R}. \tag{10}$$

Such functions are also called Herglotz , or R-functions. Every Nevanlinna function R possesses the representation

$$R(z) = a + bz + \int_{\mathbf{R}} \frac{1+\lambda z}{\lambda - z}\frac{1}{\lambda^2+1} d\sigma(\lambda), \tag{11}$$

where $a \in \mathbf{R}, b \geq 0$ and the positive measure $d\sigma(\lambda)$ satisfies

$$\int_{\mathbf{R}} \frac{d\sigma(\lambda)}{\lambda^2+1} < \infty. \tag{12}$$

Good presentation of the theory of Nevanlinna functions can be found in [**29**].

3. Singular perturbations and the second resolvent formula

Let us consider rank one perturbations determined by vectors from the spaces $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ from the scale of Hilbert spaces associated with the operator A. The perturbed operator is formally defined by the following expression

$$A_\alpha = A + \alpha\langle\varphi,\cdot\rangle\varphi, \ \varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}, \alpha \in \mathbf{R}. \tag{13}$$

If the singular vector φ belongs to $\mathcal{H}_{-1} \setminus \mathcal{H}$ then the perturbation term is form bounded (but not bounded) with respect to the operator A. Such perturbations are called *singular form bounded.* Perturbations determined by $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ are called *singular form unbounded.* The main difference between the form bounded and form unbounded singular perturbations is related to the question whether the formal expression (13) determines the perturbed operator uniquely or not. If $\varphi \in \mathcal{H}_{-1}$ then the perturbation term is form bounded with respect to the original operator A with the relative bound less than 1 and the perturbed operator can be determined using the standard perturbation theory (for example using KLMN theorem from [**53**]). In the case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ the perturbation term is not form bounded and the perturbed operator is not determined uniquely by (13). Instead of one operator expression (13) determines a one-parameter family of self-adjoint operators, described by a ceratin real parameter γ. The same one-parameter family corresponds to all formal operators with different coupling constants $\alpha \neq 0$. To determine a unique self-adjoint operator one needs to establish the relation between the coupling constant α and the parameter γ describing the family of self-adjoint operators. This relation cannot be established without taking into account additional restrictions like the homogeneity of the original operator and the perturbation. Such considerations can be applied to many operators appearing in applications and the corresponding abstract approach has been developed in [**5**].

3.1. Form bounded singular perturbations. Let us describe the unique self-adjoint operator corresponding to (13) in the case $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$. Consider the restriction A^0 of the operator A to the following domain

$$\mathrm{Dom}\,(A^0) = \{\psi \in \mathrm{Dom}\,(A) : \langle \varphi, \psi \rangle = 0\}.$$

One can easily show that the operator A^0 is a symmetric operator with the deficiency indices $(1,1)$. The restriction of the formal linear operator A_α to this domain coincides with the operator A^0, since the perturbation term simply vanishes on functions orthogonal to φ. Thus the self-adjoint operators A and A_α are two (different) self-adjoint extensions of the symmetric operator A^0. The operator A^{0*} adjoint to A^0 is defined on the domain

$$\mathrm{Dom}\,(A^{0*}) = \{\psi = \tilde{\psi} + \psi_1 g_1,\ \tilde{\psi} \in \mathrm{Dom}\,(A), \psi_1 \in \mathbf{C}\}, \tag{14}$$

where $g_1 = \frac{1}{A+1}\varphi$. All self-adjoint extensions of the operator A^0 can be described by one real parameter γ as restrictions of the operator A^{0*} to the domain of functions satisfying the following boundary condition

$$\langle \varphi, \tilde{\psi} \rangle = \gamma \psi_1,\ \gamma \in \mathbf{R} \cup \{\infty\}. \tag{15}$$

To determine the operator A_α one needs to establish the relation between the coupling constant α and the extension parameter γ. This relation is described by the following theorem.

THEOREM 3.1. *Let $\varphi \in \mathcal{H}_{-1}(A) \setminus H$. Then the operator $A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi$ is a self-adjoint extension of the operator A^0 to the domain*

$$\mathrm{Dom}\,(A_\alpha) = \left\{ \psi \in \mathrm{Dom}\,(A^{0*}) : \langle \varphi, \tilde{\psi} \rangle = - \left(\frac{1}{\alpha} + \langle \varphi, \frac{1}{A+1} \varphi \rangle \right) \psi_1 \right\}. \tag{16}$$

In particular for $\alpha = 0$ we have $A_0 = A$.

Proof The linear operator A_α can be considered as a linear operator between the vector spaces $\mathcal{H}_1(A)$ and $\mathcal{H}_{-1}(A)$. Let ψ be an element from $\mathcal{H}_1(A)$. Let us study the question: Under what conditions is the distribution $A_\alpha \psi$ an element from the Hilbert space $\mathcal{H}$? Consider an arbitrary vector η from the domain $\mathrm{Dom}\,(A^0) \subset$

$\mathcal{H}$. Then $\langle \eta, A_\alpha \psi\rangle$ is a bounded linear functional on η only if $\psi \in \mathrm{Dom}\,(A^{0*})$, since the following equalities hold

$$\begin{aligned} \langle \eta, A_\alpha \psi\rangle &= \langle \eta, A + \alpha\langle \varphi, \psi\rangle \varphi\rangle \\ &= \langle \eta, A\psi\rangle + \alpha \langle \varphi, \psi\rangle\langle \eta, \varphi\rangle \\ &= \langle A\eta, \psi\rangle. \end{aligned}$$

We have taken into account that $\langle \eta, \varphi\rangle = 0$ (as an element from $\mathrm{Dom}\,(A^0)$) and the operator A is defined in the generalized sense on the vectors from $\mathcal{H}_1(A)$.

Let $\psi \in \mathrm{Dom}\,(A^{0*})$ now. Then the representation (15) is valid and the linear operator acts as follows

$$\begin{aligned} A_\alpha \psi &= (A + \alpha\langle \varphi, \cdot\rangle\varphi)\left(\tilde{\psi} + \psi_1 \frac{1}{A+1}\varphi\right) \\ &= A\tilde{\psi} + \alpha\langle\varphi,\tilde{\psi}\rangle\varphi + \psi_1 \tfrac{A}{A+1}\varphi + \alpha\psi_1\left\langle \varphi, \tfrac{1}{A+1}\varphi\right\rangle\varphi \qquad (17) \\ &= \left\{A\tilde{\psi} - \psi_1\frac{1}{A+1}\varphi\right\} + \left[\alpha\langle\varphi,\tilde{\psi}\rangle + \psi_1 + \alpha\psi_1\langle\varphi, \frac{1}{A+1}\varphi\rangle\right]\varphi. \end{aligned}$$

The expression in the braces { } belongs to the original Hilbert space $\mathcal{H}$. Therefore the element $A_\alpha\psi$ belongs to $\mathcal{H}$ if and only if the expression in the square brackets [ ] is equal to zero, i.e. if the following equality holds

$$\langle\varphi,\tilde{\psi}\rangle = -\left(\frac{1}{\alpha} + \left\langle\varphi, \frac{1}{A+1}\varphi\right\rangle\right)\psi_1. \qquad (18)$$

The parameter

$$\gamma = -\frac{1}{\alpha} - \left\langle\varphi, \frac{1}{A+1}\varphi\right\rangle \qquad (19)$$

is real and the adjoint operator A^{0*} restricted to the domain of functions from $\mathrm{Dom}\,(A^{0*})$ satisfying the boundary condition (18) is self–adjoint. The restrictions of the operators A_α and A^{0*} to this domain are identical since the expression in the square brackets [ ] in formula (17) vanishes for the elements satisfying the boundary conditions (18). Thus we have proven that the self–adjoint operator defined by the formal expression (13) is a self–adjoint extension of the operator A^0 described by the parameter γ given by (19).

If $\alpha = 0$ then the parameter $\gamma = \infty$ and corresponding operator coincides with the original operator A. □

This theorem establishes a one-to-one correspondence between the coupling parameter α and the extension parameter γ. Taking formally all $\alpha \in \mathbf{R} \cup \{\infty\}$ one gets all self-adjoint extensions of the symmetric operator A^0.

3.2. Form unbounded singular perturbations. Consider form unbounded perturbations i.e. perturbations determined by vectors

$$\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}.$$

The perturbation term is not form bounded with respect to the original operator in formula (13) and the perturbed operator cannot be determined using standard methods of perturbation theory. One can try to carry out the restriction-extension procedure developed for form bounded singular perturbations. Really the restriction A^0 of the operator A can be defined using the same formula (14), since the scalar product $\langle \varphi, \psi \rangle$ is well defined if $\psi \in \mathrm{Dom}\,(A) = \mathcal{H}_2$ and $\varphi \in \mathcal{H}_{-2}$. Therefore any self-adjoint operator corresponding to the formal expression A_α given by (13) must be an extension of the operator A^0 and again any such extension is described by one real parameter γ. The relation between the coupling constant α appearing in (13) and the extension parameter γ from (15) cannot be established without taking into account any additional information. The proof of the Theorem 3.1 cannot be repeated, since the scalar product $\langle \varphi, \frac{1}{A+1}\varphi \rangle$ is not defined if $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$. In other words the corresponding integral is diverging. One of the possibilities to overcome this difficulty is to renormalize this expression. One can formally define

$$\langle \varphi, \frac{1}{A+1}\varphi \rangle = c, \tag{20}$$

where c is a real constant. This formula determines an extension φ_c of the distribution φ originally defined on $\mathrm{Dom}\,(A) = \mathcal{H}_2$ to a larger domain

$$\mathrm{Dom}\,(A^{0*}) = \mathrm{Dom}\,(A) \dot{+} \mathcal{L}\{g_1\}.$$

This extension is defined by the formula

$$\langle \varphi_c, \tilde{\psi} + \psi_1 g_1 \rangle = \langle \varphi, \tilde{\psi} \rangle + \psi_1 \langle \varphi_c, g_1 \rangle = \langle \varphi, \tilde{\psi} \rangle + c\psi_1, \ \tilde{\psi} \in \mathcal{H}_2, \psi_1 \in \mathbf{C}. \tag{21}$$

Consider the formal expression

$$A_\alpha = A + \alpha \langle \varphi_c, \cdot \rangle \varphi, \tag{22}$$

which determines a linear operator from $\mathrm{Dom}\,(A^{0*})$ to $\mathcal{H}_{-2}$. Then the following theorem describes the unique self-adjoint operator corresponding to the formal expression (22).

THEOREM 3.2. *Let $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}$ and φ_c be arbitrary extension of φ defined by (21). Then the operator $A_\alpha = A + \alpha \langle \varphi_c, \cdot \rangle \varphi$ is a self-adjoint extension of the operator A^0 to the domain*

$$\mathrm{Dom}\,(A_\alpha) = \left\{ \psi \in \mathrm{Dom}\,(A^{0*}) : \langle \varphi, \tilde{\psi} \rangle = -\left(\frac{1}{\alpha} + c \right) \psi_1 \right\}. \tag{23}$$

In particular for $\alpha = 0$ we have $A_0 = A$.

The proof of this theorem follows the same lines as for Theorem 3.1.

This theorem establishes the one-to-one correspondence between the coupling constant α and the extension parameter γ

$$\gamma = -\frac{1}{\alpha} - c. \tag{24}$$

This relation contains arbitrary real parameter c, which appears as a result of the regularization of the scalar product $\langle \varphi, g_1 \rangle$ given by (20). In special cases when both the operator A and the singular vector φ are homogeneous with respect to a certain unitary semigroup in $\mathcal{H}$ the parameter c is defined uniquely if one requires that the extended distribution φ_c is again homogeneous with respect to the same semigroup. This approach was first developed in [**5**] and is described in [**8**]. It was noticed later that this approach is similar to so-called dimensional regularization in quantum field theory [**27, 28, 12, 14**]. These relations are described in [**42**]. These methods were effectively applied to point interactions in [**2, 5, 39, 13**] and to finite rank perturbations in [**7, 13**].

3.3. The second resolvent formula. In this section we are going to derive the formula connecting the resolvents of the original and perturbed operators for both form bounded and form unbounded singular perturbations. Our consideration is based on Krein's resolvent formula, since the operators A_α and A are two different self-adjoint extensions of the symmetric operator A^0 with deficiency indices $(1,1)$ [**33, 34, 35, 46**]. Instead of using Krein's formula we are going to calculate the resolvent of the operator A_α explicitly using its definition given by Theorems 3.1 and 3.2.

THEOREM 3.3. *Let A be a positive self-adjoint operator and $\varphi \in \mathcal{H}_{-2}$. Then the resolvent of the operator $A_\alpha = A + \alpha\langle\varphi,\cdot\rangle\varphi$ defined by Theorems 3.1 and 3.2 is given by*

$$\frac{1}{A_\alpha-\lambda} = \frac{1}{A-\lambda} - \frac{1}{1/\alpha + c + \left\langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi\right\rangle}\left\langle \frac{1}{A-\bar{\lambda}}\varphi,\cdot\right\rangle \frac{1}{A-\lambda}\varphi, \tag{25}$$

where the real parameter c is equal to $\langle\varphi, \frac{1}{A+1}\varphi\rangle$ in case $\varphi \in \mathcal{H}_{-1}$ and to the regularized value of $\langle\varphi, \frac{1}{A+1}\varphi\rangle$ in case $\varphi \in \mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$. For $\alpha = 0$ the resolvent is given by

$$\frac{1}{A_0-\lambda} = \frac{1}{A-\lambda}. \tag{26}$$

Proof Let us prove formula in the case $\varphi \notin \mathcal{H}$. We are going to use the following notation $\langle\varphi,(A+1)^{-1}\varphi\rangle = c$ in the case $\varphi\in\mathcal{H}_{-1}$ in order to treat all vectors φ from $\mathcal{H}_{-2}$ at once. Consider the resolvent equation

$$(A_\alpha - z)^{-1} f = \psi, \tag{27}$$

where $f\in\mathcal{H}, \psi \in \mathrm{Dom}\,(A_\alpha)$ and z is in the resolvent set of A_α. This equation can be re-written as

$$f = (A_\alpha - z)\psi = (A_\alpha - z)\left(\tilde{\psi} + \psi_1\frac{1}{A+1}\varphi\right) = (A-z)\tilde{\psi} - (z+1)\psi_1\frac{1}{A+1}\varphi. \tag{28}$$

By applying the resolvent $(A-z)^{-1}$ of the original operator A we obtain

$$\frac{1}{A-z}f = \tilde{\psi} - (z+1)\psi_1\frac{1}{A-z}\frac{1}{A+1}\varphi.$$

Projection on φ then gives the equation

$$\left\langle\varphi, \frac{1}{A-z}f\right\rangle = \langle\varphi,\tilde{\psi}\rangle - (z+1)\psi_1\left\langle\frac{1}{A-\bar{z}}\varphi, \frac{1}{A+1}\varphi\right\rangle,$$

which implies together with (16) and (23) that

$$\langle\varphi,\tilde{\psi}\rangle = \frac{\left\langle\varphi,\frac{1}{A-z}f\right\rangle}{1+\frac{\alpha(z+1)}{1+\alpha c}\left\langle\frac{1}{A-\bar{z}}\varphi,\frac{1}{A+1}\varphi\right\rangle}$$

and

$$\tilde{\psi} = \frac{1}{A-z}f - \frac{\alpha(z+1)\left\langle\varphi,\frac{1}{A-z}f\right\rangle}{1+\alpha c+\alpha(z+1)\left\langle\frac{1}{A-\bar{z}}\varphi,\frac{1}{A+1}\varphi\right\rangle}\frac{1}{A-z}\frac{1}{A+1}\varphi,$$

where c is equal to $\left\langle \varphi, \frac{1}{A+1}\varphi \right\rangle$ if $\varphi \in \mathcal{H}_{-1}$ and to the renormalization constant (20) if $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$. We are getting the second resolvent formula (25).

In the special case $\varphi \in \mathcal{H}$ the second resolvent formula (25) coincides with already proven formula (6). $\square$

In the case $\varphi \in \mathcal{H}_{-1}$ the second resolvent formula just proven can be written as

$$\frac{1}{A_\alpha - z} = \frac{1}{A-z} - \frac{\alpha}{1+\alpha \left\langle \varphi, \frac{1}{A-z}\varphi \right\rangle} \left\langle \frac{1}{A-\bar{z}}\varphi, \cdot \right\rangle \frac{1}{A-z}\varphi, \tag{29}$$

which shows that the second resolvent formula is just an extension of the first resolvent formula to the set $\varphi \in \mathcal{H}_{-2}$.

4. Supersingular perturbations I: $\mathcal{H}_{-3}$-case

Our aim in this section is to define the self–adjoint operator corresponding to the formal expression

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \quad \varphi \in \mathcal{H}_{-3} \setminus \mathcal{H}_{-2}, \alpha \in \mathbf{R}. \tag{30}$$

In this section we are going to follow the main lines of [**43**], where such operator was defined as a self-adjoint operator in a certain extended Hilbert space.

The restriction-extension procedure developed for the operator A_α with $\varphi \in \mathcal{H}_{-2}$ cannot be applied without any serious modification to the case $\varphi \in \mathcal{H}_{-3} \setminus \mathcal{H}_{-2}$ for the following simple reason:
The restriction of the operator A to the set of functions $u \in \mathrm{Dom}\,(A) = \mathcal{H}_2$ satisfying additional condition $\langle \varphi, u \rangle = 0$ is not well-defined, since the scalar product does not make sense for all $u \in \mathrm{Dom}(A)$. One need to introduce additional restricting condition $u \in \mathcal{H}_3 \subset \mathcal{H}_2$. But the operator $A_{\min}$ being the restriction of A to the set of functions satisfying these two conditions

$$u \in \mathcal{H}_3, \ \ \langle \varphi, u \rangle = 0, \tag{31}$$

is essentially self-adjoint in $\mathcal{H}$. The operator $A_{\min}$ so defined is an analog of the operator A^0 introduced for $\varphi \in \mathcal{H}_{-2}$. We conclude that no self-adjoint operator corresponds to the formal expression (30) in the original Hilbert space. One needs to modify the original Hilbert space in order to be able to determine a self-adjoint operator.

4.1. The extended Hilbert space and the maximal operator. To determine the Hilbert space suitable for the operator A_α we make two observations:

(1) The operator $A_{\min}$ is a densely defined symmetric operator in the Hilbert space $\mathcal{H}_1$. Really the operator A considered in $\mathcal{H}_1$ is self-adjoint on the domain $\mathcal{H}_3$. [1] Then the operator $A_{\min}$ is a restriction of the operator A acting in $\mathcal{H}_1$ determined by only one restricting condition $\langle \varphi, u \rangle = 0$.
(2) The range of the formal resolvent of the operator A_α contains elements of the form $\frac{1}{A-\lambda}\varphi \in \mathcal{H}_{-1}, \ \lambda \in \mathbf{C} \setminus \mathbf{R}$. Hence the Hilbert space $\mathcal{H}_1$ must

[1] The operator A acting in $\mathcal{H}_1$ is a positive self-adjoint operator and it determines a new scale of Hilbert spaces which differs from the scale $\mathcal{H}_n$ introduced earlier by a shift. To avoid complicated notations we are going to use in what follows the scale of Hilbert spaces associated with the original operator A in $\mathcal{H}$ only.

be extended to include such elements. In fact such an extension is one-dimensional due to Hilbert's identity

$$\frac{1}{A-\lambda}\varphi - \frac{1}{A-\mu}\varphi = (\lambda-\mu)\frac{1}{A-\lambda}\frac{1}{A-\mu}\varphi \in \mathcal{H}_1,$$

which implies that the difference between any two such vectors is an element from $\mathcal{H}_1$.

Let us introduce the extended Hilbert space

$$\mathbf{H} = \mathbf{C} \oplus \mathcal{H}, \tag{32}$$

together with the natural embedding

$$\begin{aligned} \rho: \quad & \mathbf{H} \to \mathcal{H}_{-1} \\ & \mathbf{U} = (u_1, U) \mapsto U + u_1 g_1, \end{aligned} \tag{33}$$

where $g_1 = \frac{1}{A+1}\varphi \in \mathcal{H}_{-1}$.

Let us discuss now how to define the self-adjoint operator corresponding to (30) acting in $\mathbf{H}$. In case $\varphi \in \mathcal{H}_{-2}$ such operator was defined as a restriction of the adjoin operator A^{0*} to a certain domain. In case $\varphi \in \mathcal{H}_{-3} \setminus \mathcal{H}_{-2}$ the restricted minimal operator $A_{\rm min}$ has two adjoint operators:

(1) The operator $A^*_{\rm min}$ adjoint to the symmetric operator $A_{\rm min}$ as a densely defined symmetric operator in $\mathcal{H}_1$. This is the operator adjoint to $A_{\rm min}$ with respect to the scalar product in $\mathcal{H}_1$.
(2) The triplet adjoint operator $A^\dagger_{\rm min}$ with respect to the scalar product in the original Hilbert space $\mathcal{H}$.

Let us remind the definition of the triplet adjoint operator [**10, 40**]. Consider arbitrary Gelfand triplet $\mathcal{K} \subset \mathcal{H} \subset \mathcal{K}^*$. Let B be a densely defined operator in the space $\mathcal{K}$ then the **triplet adjoint** operator $B^\dagger$ acting in $\mathcal{K}^*$ is defined on the domain

$$\mathrm{Dom}\,(B^\dagger) = \{f \in \mathcal{K}^* : g \in \mathrm{Dom}\,(B) \Rightarrow |\langle Bg, f\rangle| \le C_f \parallel g \parallel_{\mathcal{K}}\}$$

by the following equality

$$\langle Bg, f\rangle = \langle g, B^\dagger f\rangle.$$

Note that the scalar product appearing in the last definition should be understood as pairing defined by the original scalar product in $\mathcal{H}$. The triplet adjoint operator coincides with the standard adjoint operator in the case $\mathcal{K} = \mathcal{H} = \mathcal{K}^*$. Otherwise the triplet adjoint operator $B^\dagger$ is different from the adjoint operator B^* - operator adjoint to B considered as an operator in the Hilbert space $\mathcal{H} \supset \mathcal{K}$.

In what follows the triplet adjoint operator $A^\dagger_{\rm min}$ will be called the maximal operator, since it will play the role of the adjoint operator A^{0*} used in constructing singular perturbations. The following lemma describes the maximal operator.

LEMMA 4.1. *The maximal operator $A_{\rm max} = A^\dagger_{\rm max}$ is defined on the domain*

$$\mathrm{Dom}\,(A_{\rm max}) = \left\{ U = \tilde U + u_1 g_1 \in \mathcal{H}_{-1}, \tilde U \in \mathcal{H}_1, u_1 \in \mathbf{C} \right\}. \tag{34}$$

by the following formula

$$A_{\rm max}(\tilde U + u_1 g_1) = A\tilde U - u_1 g_1. \tag{35}$$

Proof The domain of the triplet adjoint operator $A_{\min}^{\dagger}$ consists of all elements $U \in \mathcal{H}_{-1}$ such that the sesquilinear form $\langle (A+1)V, U\rangle = \langle V, (A+1)U\rangle$ can be estimated as follows

$$|\langle V, (A+1)U\rangle| \leq C_U \parallel V \parallel_{\mathcal{H}_1}$$

for all $V \in \mathrm{Dom}\,(A_{\min})$, since the operator $A_{\min}$ is a restriction of the operator A. The last estimate holds for all $V \in \mathcal{H}_3$, $\langle V, \varphi\rangle = 0$ if and only if

$$(A+1)U = \hat{U} + u_1 \varphi,$$

where $\hat{U} \in \mathcal{H}_{-1}$, $u_1 \in \mathbf{C}$. It follows that the function U possesses representation (34).

Suppose now that the representation (34) holds. Then the sesquilinear form can be written as follows

$$\begin{aligned} \langle (A+1)V, U\rangle &= \langle (A+1)V, \tilde{U}\rangle + \langle (A+1)V, \frac{1}{A+1}\varphi\rangle u_1 \\ &= \langle V, (A+1)\tilde{U}\rangle. \end{aligned}$$

It follows that

$$(A+1)^{\dagger}(\tilde{U} + u_1 g_1) = (A+1)\tilde{U}$$

and hence (63) holds. □

The maximal operator $A_{\max}$ acts in the Hilbert space $\mathcal{H}_{-1}$. This operator can be lifted to the operator $\mathbf{A}_{\max}$ in the space $\mathbf{H}$ so that the embedding ρ intertwins these two operators

$$\rho \mathbf{A}_{\max} = A_{\max} \rho. \tag{36}$$

The maximal operator $\mathbf{A}_{\max}$ is described by

Lemma 4.2. *The maximal operator $\mathbf{A}_{\max}$ in $\mathbf{H}$ is defined on the domain*

$$\mathrm{Dom}\,(\mathbf{A}_{\max}) = \left\{ \mathbf{U} = (u_1, U_r + u_2 g_2) \in \mathbf{H}, u_1, u_2 \in \mathbf{C}, U_r \in \mathcal{H}_3 \right\}, \tag{37}$$

where $g_2 = \frac{1}{A+1} g_1 \equiv \frac{1}{(A+1)^2}\varphi$, by the formula

$$\mathbf{A}_{\max} \left(\begin{array}{c} u_1, U_r + u_2 g_2 \end{array} \right) = \left(\begin{array}{c} u_2 - u_1 \\ AU_r - u_2 g_2 \end{array} \right). \tag{38}$$

Proof. Let $\mathbf{U} = (u_1, U)$ be an arbitrary vector from the domain of the operator $\mathbf{A}_{\max}$ defined by (36). Let us denote its image $\mathbf{A}_{\max}\mathbf{U}$ by $\mathbf{W} = (w_1, W)$. Then equality (36) reads as follows

$$w_1 g_1 + W = (u_2 - u_1) g_1 + AU, \tag{39}$$

and implies that

$$W + U + (w_1 + u_1) g_1 = (A+1)U.$$

Applying the resolvent $\frac{1}{A+1}$ to the last equality we obtain

$$U = \frac{1}{A+1}(W+U) + (w_1 + u_1) g_2,$$

which implies that representation

$$U = U_r + u_2 g_2$$

necessarily holds with

$$U_r = \frac{1}{A+1}(W+U) \in \mathcal{H}_3, \; u_2 = w_1 + u_1 \in \mathbf{C}.$$

Taking into account the representation just proven, equation(39) can be written as follows

$$w_1 g_1 + W = -u_1 g_1 + AU_r + u_2 \frac{A}{A+1} g_1 = (u_2 - u_1) g_1 + AU_r - u_2 g_2,$$

which implies (38). $\square$

The minimal operator $A_{\min}$ can be lifted to the space $\mathbf{H}$ using similar method. The corresponding operator $\mathbf{A}_{\min}$ is defined on the domain:

$$\text{Dom}\,(\mathbf{A}_{\min}) = \{\mathbf{U} = (0, U_r), U_r \in \mathcal{H}_3, \langle \varphi, U_r \rangle = 0\} \tag{40}$$

by the fomula

$$\mathbf{A}_{\min}(0, U_r) = (0, AU_r). \tag{41}$$

Obviously the maximal operator $\mathbf{A}_{\max}$ is an extension of the minimal operator $\mathbf{A}_{\min}$.

The operator $\mathbf{A}_{\max}$ plays the role of the adjoint operator in our construction and therefore is not symmetric.

LEMMA 4.3. *The boundary form of the maximal operator* $\mathbf{A}_{\max}$ *is a sesquilinear form in the space of boundary values* $(u_1, u_2, \langle \varphi, U_r \rangle) \in \mathbf{C}^3$

$$\langle \mathbf{U}, \mathbf{A}_{\max} \mathbf{V} \rangle_{\mathbf{H}} - \langle \mathbf{A}_{\max} \mathbf{U}, \mathbf{V} \rangle_{\mathbf{H}} = \tag{42}$$

$$= \left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \langle \varphi, U_r \rangle \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \langle \varphi, V_r \rangle \end{pmatrix} \right\rangle.$$

Proof Consider two arbitrary vectors $\mathbf{U}, \mathbf{V}$ from the domain Dom $(\mathbf{A}_{\max}$. Then the following calculations prove the lemma

$$\begin{aligned}
&\langle \mathbf{U}, \mathbf{A}_{\max} \mathbf{V} \rangle_{\mathbf{H}} - \langle \mathbf{A} \mathbf{U}, \mathbf{V} \rangle_{\mathbf{H}} \\
= \;& \left\langle \begin{pmatrix} u_1 \\ U_r + u_2 g_2 \end{pmatrix}, \begin{pmatrix} v_2 - v_1 \\ AV_r - v_2 g_2 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} u_2 - u_1 \\ AU_r - u_2 g_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ V_r + v_2 g_2 \end{pmatrix} \right\rangle \\
= \;& \bar{u}_1 (v_2 - v_1) + \langle (A+1)(U_r + u_2 g_2), (AV_r - v_2 g_2) \rangle \\
& -(\bar{u}_2 - \bar{u}_1) v_1 - \langle (A+1)(AU_r - u_2 g_2), (V_r + v_2 g_2) \rangle \\
= \;& \bar{u}_1 v_2 - \bar{u}_2 v_1 + \langle u_2 g_1, AV_r \rangle - \langle U_r, v_2 g_1 \rangle + \langle u_2 g_1, V_r \rangle - \langle AU_r, v_2 g_1 \rangle \\
= \;& \bar{u}_1 v_2 - \bar{u}_2 v_1 + \bar{u}_2 \langle \varphi, V_r \rangle - \langle U_r, \varphi \rangle v_2.
\end{aligned}$$

The last formula can be written in matrix notations as (42). $\square$

4.2. Self-adjoint operator. Every symmetric restriction of the operator $\mathbf{A}_{\max}$ can be defined by certain boundary conditions imposed on the functions from the domain of the operator. These boundary conditions connect the boundary values $u_1, u_2, \langle \varphi, U_r \rangle$ of the functions from the domain of the restricted operator. The problem of defining a symmetric restriction of $\mathcal{A}$ is equivalent to the problem of finding a Lagrangian plane of the boundary form.

The boundary conditions can be written in the form

$$au_1 + bu_2 + c < \varphi, U_r >= 0, \tag{43}$$

where $a, b, c \in \mathbf{C}$ are certain complex parameters, not all equal to zero simultaneously. Suppose that the parameter c is different from zero. Then the boundary form of the operator restricted to the linear set of functions satisfying the boundary conditions is given by

$$(44) \qquad \langle \mathbf{AU}, \mathbf{V} \rangle_{\mathbf{H}} - \langle \mathbf{U}, \mathbf{AV} \rangle_{\mathbf{H}} = \bar{u}_1 v_2 \left(1 + \tfrac{\bar{a}}{\bar{c}}\right) - \bar{u}_2 v_1 \left(1 + \tfrac{a}{c}\right) - \bar{u}_2 v_2 2\Re(\tfrac{b}{c}).$$

This expression vanishes for arbitrary $u_{1,2}, v_{1,2}$ if and only if the following conditions are satisfied:

$$(45) \qquad c = -a, \;\; \Re(b/c) = 0.$$

These conditions imply that the complex parameters a, b, c have the same phase and therefore without loss of generality can be chosen real. Then the boundary form vanishes if the vector (a, b, c) is orthogonal to the vector $(1, 0, 1)$. The case when $c = 0$ can be considered in a similar way.

Since the length of the three-dimensional vector (a, b, c) orthogonal to $(1, 0, 1)$ does not play any role, all of the Lagrangian planes can be parameterized by one real parameter $\theta \in [0, 2\pi)$ as follows:

$$(46) \qquad (a, b, c) = (-\sin\theta, \cos\theta, \sin\theta).$$

Definition 4.1. *The operator $\mathbf{A}_\theta$, $\theta \in [0, 2\pi)$, is the restriction of the operator $\mathbf{A}_{\max}$ defined by (38) to the domain of functions $\mathcal{U} = (u_1, U) \in \mathbf{H}$ possessing the representation*

$$(u_1, U) = (u_1, U_r + u_2 g_2), \; U_r \in H_3, u_{1,2} \in \mathbf{C}$$

and satisfying the boundary condition

$$(47) \qquad -\sin\theta \, u_1 + \cos\theta \, u_2 + \sin\theta \, \langle \varphi, u_r \rangle = 0.$$

The following theorem proves that the operator $\mathbf{A}_\theta$ so defined is not only symmetric but self-adjoint as well.

Theorem 4.1. *The operator $\mathbf{A}_\theta$ is a self–adjoint operator in $\mathbf{H}$.*

Proof. It has already been proven that the operator $\mathcal{A}_\theta$ is symmetric. We are going to prove that it is self–adjoint by calculating its resolvent for small negative λ, $\lambda \ll -1$.

We prove that the range of the operator $\mathbf{A}_\theta - \lambda$ coincides with the Hilbert space $\mathbf{H}$

$$\mathcal{R}(\mathbf{A}_\theta - \lambda) = \mathbf{H},$$

i.e. that for for any $\mathbf{V} = (v_1, V) \in \mathbf{H}$ there exits an element $\mathbf{U} = (u_1, U_r + u_2 g_2) \in \mathrm{Dom}\,(\mathbf{A}_\theta)$ such that

$$(48) \qquad (\mathbf{A}_\theta - \lambda)\mathbf{U} = \mathbf{V}.$$

The last equation can be written as

$$\begin{cases} (A - \lambda) U_r - (1 + \lambda) u_2 g_2 & = V; \\ \\ u_2 - (1 + \lambda) u_1 & = v_1. \end{cases}$$

The first of these equations can be rewritten as

$$U_r - (1 + \lambda) \frac{1}{A - \lambda} g_2 u_2 = \frac{1}{A - \lambda} V,$$

which implies

$$\langle \varphi, U_r \rangle - (1+\lambda)\langle \varphi, \frac{1}{A-\lambda} g_2 \rangle u_2 = \langle \varphi, \frac{1}{A-\lambda} V \rangle.$$

It will be convenient in what follows to introduce special notation for the following Nevanlinna function

$$Q_3(\lambda) = \left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_2 \right\rangle \equiv \left\langle g_1, \frac{1+\lambda}{A-\lambda} g_1 \right\rangle . \tag{49}$$

The vector $\mathcal{U} = (u_1, U_r + u_2 g_2)$ should satisfy the boundary condition (47). Hence the vector $(u_1, u_2, \langle \varphi, U_r \rangle) \in \mathbf{C}^3$ solves the system of linear equations

$$\begin{pmatrix} -(1+\lambda) & 1 & 0 \\ 0 & -Q_3(\lambda) & 1 \\ -\sin\theta & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \langle \varphi, u_r \rangle \end{pmatrix} = \begin{pmatrix} v_1 \\ \langle \varphi, \frac{1}{A-\lambda} V \rangle \\ 0 \end{pmatrix} . \tag{50}$$

The determinant of this system is given by

$$D(\lambda, \theta) = \sin\theta (1+\lambda) \left(Q_3(\lambda) - \frac{1}{1+\lambda} + \cot\theta \right) , \tag{51}$$

and is different from zero for all nonreal λ, since the expression in the brackets is a sum of Nevanlinna functions, one having nontrivial imaginary part for those λ. Therefore the system (50) always is solvable and hence the resolvent equation (48) is solvable as well for arbitrary $\mathbf{V} \in \mathbf{H}$.[2] □

The family of self-adjoint operators $\mathbf{A}_\theta$ will be considered as a rigorous mathematical interpretation for the formal operator (30) in the case $\varphi \in \mathcal{H}_{-3}$. As in the case of $\mathcal{H}_{-2}$-perturbations the exact relation between the coupling parameter α appearing in (30) and the extension parameter θ cannot be established without taking into account any additional assumptions. The parameter θ is chosen in such a way that the operator $\mathbf{A}_0$ is the unique operator from the family $\mathbf{A}_\theta$ possessing the orthogonal decomposition

$$\mathbf{A}_\theta = (-1) \oplus A.$$

The last operator can be identified with the original operator A. (More precisely with the operator A acting in the Hilbert space $\mathcal{H}_1$ and self-adjoint on the domain $\mathcal{H}_3$.) Any other operator from the family can be considered as a rank two perturbation of this new unperturbed operator.

Similar problem for operator relations was investigated in [**26**].

Precise relation between the coupling and extension parameters can be established sometimes if the operator and the singular vector φ are homogeneous with respect to a certain unitary semigroup. This approach has been developed in [**42**].

4.3. The third resolvent formula. The resolvent of the operator $\mathbf{A}_\theta$ can be calculated using Krein's resolvent formula, since this operator is a rank two perturbation of the operator $\mathbf{A}_0 = (-1) \oplus A$. We are going to calculate this resolvent

[2] The resolvent of the operator $\mathbf{A}_\theta$ will be calculated in the following section explicitly by solving this equation.

by solving the linear system (50)

$$\begin{aligned} u_1 &= -\tfrac{1}{D(\lambda,\theta)} \left((\sin\theta \left\langle g_1, \tfrac{1}{A-\lambda} g_1 \right\rangle + \cos\theta) v_1 + \sin\theta \left\langle \varphi, \tfrac{1}{A-\lambda} V \right\rangle \right), \\ u_2 &= -\tfrac{1}{D(\lambda,\theta)} \left(\sin\theta v_1 + \sin\theta (1+\lambda) \left\langle \varphi, \tfrac{1}{A-\lambda} V \right\rangle \right), \\ \langle \varphi, U_r \rangle &= -\tfrac{1}{D(\lambda,\theta)} \left(\sin\theta \left\langle g_1, \tfrac{1}{A-\lambda} g_1 \right\rangle v_1 + (\sin\theta - \cos\theta(1+\lambda)) \left\langle \varphi, \tfrac{1}{A-\lambda} V \right\rangle \right). \end{aligned}$$

Then the function $\mathbf{U} = (u_1, U)$ can be calculated using the following formulas (52)

$$\begin{aligned} u_1 &= -\tfrac{1}{-1-\lambda} v_1 - \tfrac{\sin\theta}{D(\lambda,\theta)} \left[\tfrac{1}{1+\lambda} v_1 + \left\langle \varphi, \tfrac{1}{A-\lambda} V \right\rangle \right], \\ U &= \tfrac{1}{A-\lambda} V + \tfrac{1}{A-\lambda} g_1 u_2 = \tfrac{1}{A-\lambda} V - \tfrac{1}{A-\lambda} g_1 \tfrac{\sin\theta}{D(\lambda,\theta)} \left[v_1 + (1+\lambda) \left\langle \varphi, \tfrac{1}{A-\lambda} V \right\rangle \right]. \end{aligned}$$

This formula can be written in operator notations as follows

$$\frac{1}{\mathbf{A}_\theta - \lambda} = \frac{1}{\mathbf{A}_0 - \lambda} - \frac{\sin\theta}{D(\lambda,\theta)} \begin{pmatrix} \frac{1}{1+\lambda} & \left\langle \varphi, \frac{1}{A-\lambda} \cdot \right\rangle \\ \frac{1}{A-\lambda} g_1 & \frac{1+\lambda}{A-\lambda} g_1 \left\langle \varphi, \frac{1}{A-\lambda} \cdot \right\rangle \end{pmatrix}. \tag{53}$$

It is natural to consider the restriction of the resolvent $(\mathbf{A}_\theta - \lambda)^{-1}$ to the infinite dimensional subspace $\mathcal{H}_1 \subset \mathbf{H}$ combined with the embedding ρ

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda} |_{\mathcal{H}_1} = \frac{1}{A-\lambda} - \frac{\sin\theta}{D(\lambda,\theta)} \left\langle \frac{1}{A-\bar\lambda} \varphi, \cdot \right\rangle \frac{1}{A-\lambda} \varphi. \tag{54}$$

This formula is the third resolvent formula which is valid for supersingular perturbations of class $\mathcal{H}_{-3}$. The last formula can be written as follows to illustrate the similarity to the first two resolvent formulas

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda} |_{\mathcal{H}_1} = \frac{1}{A-\lambda} - \frac{1}{(1+\lambda)\left[Q_3(\lambda) - \frac{1}{1+\lambda} + \cot\theta \right]} \left\langle \frac{1}{A-\bar\lambda} \varphi, \cdot \right\rangle \frac{1}{A-\lambda} \varphi. \tag{55}$$

In the case $\varphi \in \mathcal{H}_{-2}$ the RHS of this formula is similar to (25).

Formula (55) determines a linear operator acting between the spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$. The denominator appearing in this formula can be considered as renormalized bordered resolvent.

5. Supersingular perturbations II: general case

In this section we are going to extend our construction developed in the previous section for $\mathcal{H}_{-3}$-perturbations to the general case of supersingular perturbations from the class $\mathcal{H}_{-n}$, $n \geq 4$. The main difference between these two cases is that instead of a self-adjoint operator we construct operator which we call regular. The operator constructed is very close to a self-adjoint operator: the domain of the original and adjoint operators coincide, the spectrum of the operator is purely real. But this operator is not similar to a self-adjoint. To get a self-adjoint model one needs to consider a generalization of our model which contains extra parameters: one can choose a different scalar product in the finite dimensional extension space and use different points to regularize the diverging integral. This approach has been developed in [**55**] and is described in more details in the following section.

5.1. The Extended Hilbert Space and the Maximal Operator. Consider as in the previous section the formal expression

$$A_\alpha = A + \alpha\langle\varphi,\cdot\rangle\varphi, \quad \alpha \in \mathbf{R},\ \varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}. \tag{56}$$

Consider the extended Hilbert space $\mathbf{H} \equiv \mathbf{H}_{-n} = \mathbf{C}^{(n-2)} \oplus \mathcal{H}_{n-2}$ equipped with the scalar product

$$\begin{aligned} \ll \mathbf{U},\mathbf{V} \gg &= \bar{u}_1 v_1 + \ldots + \bar{u}_{n-3}v_{n-3} + \bar{u}_{n-2}v_{n-2} + \langle U, V\rangle_{\mathcal{H}_{n-2}} \\ &= \langle \vec{u}, \vec{v}\rangle_{\mathbf{C}^{n-2}} + \langle U, (1+A)^{(n-2)}V\rangle, \end{aligned} \tag{57}$$

where we used the following vector notation

$$\begin{aligned} \mathbf{U} = (\vec{u}, U), \quad \vec{u} = (u_1, u_2, \ldots, u_{n-2}) \\ \mathbf{V} = (\vec{v}, V), \quad \vec{u} = (v_1, v_2, \ldots, v_{n-2}). \end{aligned}$$

The natural embedding of the space $\mathbf{H}$ into the space $\mathcal{H}_{-n+2}$ is defined by

$$\begin{aligned} \rho\mathbf{U} &= u_1 g_1 + u_2 g_2 + \ldots + u_{n-2} g_{n-2} + U \\ &= \sum_{k=1}^{n-2} u_k g_k + U, \end{aligned} \tag{58}$$

where the vectors $g_k, k = 1, 2, \ldots, n-2$ are defined by

$$g_0 = \varphi, \quad g_k = \frac{1}{A+1} g_{k-1} = \frac{1}{(A+1)^k}\varphi,\ k = 1, 2, \ldots, n-2. \tag{59}$$

The minimal operator $A_{\min}$ corresponding to the formal expression (56) is defined as the restriction of the operator A acting in the space $\mathcal{H}_{n-2}$ (having the domain $\mathcal{H}_n$) to the set of function orthogonal to φ

$$\mathrm{Dom}\,(A_{\min}) = \{\psi \in \mathcal{H}_n : \langle \varphi, \psi\rangle = 0\}. \tag{60}$$

The operator $A_{\min}$ is a densely defined symmetric operator in $\mathcal{H}_{n-2}$ with deficiency indices $(1,1)$. The maximal operator $A_{\max}$ is defined as triplet adjoint to $A_{\min}$ with respect to the Gelfand triplet $\mathcal{H}_{n-2} \subset \mathcal{H} \subset \mathcal{H}_{-n+2}$

$$A_{\max} = A^{\dagger}_{\min}. \tag{61}$$

The proof of the following Lemma just coincides with the proof of Lemma 4.1:

LEMMA 5.1. *The maximal operator $A_{\max}$ is defined on the domain*

$$\mathrm{Dom}\,(A_{\max}) = \left\{ U = \tilde{U} + u_1 g_1 \in \mathcal{H}_{-n+2}, \tilde{U} \in \mathcal{H}_{-n+4}, u_1 \in \mathbf{C} \right\}. \tag{62}$$

by the following formula

$$A_{\max}(\tilde{U} + u_1 g_1) = A\tilde{U} - u_1 g_1. \tag{63}$$

The maximal operator $\mathbf{A}_{\max}$ in $\mathbf{H}$ is defined using formula (36). Let us use the following definition.

DEFINITION 5.1. *The* **maximal operator $\mathbf{A}_{\max}$** *acting in the Hilbert space* $\mathbf{H}$ *is the restriction of the operator $A_{\max}$ to the Hilbert space* $\mathbf{H}$ *defined by the following equality*

$$A_{\max}\rho = \rho\mathbf{A}_{\max} \tag{64}$$

on the following domain

$$\mathrm{Dom}\,(\mathbf{A}_{\max}) = \{\mathbf{U} \in \mathbf{H} : A_{\max}\rho(\mathbf{U}) \in \mathcal{R}ange(\rho)\}.$$

LEMMA 5.2. *The maximal operator* $\mathbf{A_{max}}$ *determined by Definition 5.1 is defined on the domain*

$$\begin{aligned} \mathrm{Dom}(\mathbf{A_{max}}) \;\; = \;\; & \{\mathbf{U} = (u_1, u_2, \ldots, u_{n-2}, U_r + u_{n-1} g_{n-1}), \\ & u_1, u_2, \ldots, u_{n-2}, u_{n-1} \in \mathbf{C}, U_r \in \mathcal{H}_n\} \end{aligned} \tag{65}$$

by the formula

$$\mathbf{A_{max}} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{n-2} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix} = \begin{pmatrix} u_2 - u_1 \\ u_3 - u_2 \\ \cdots \\ u_{n-1} - u_{n-2} \\ AU_r - u_{n-1} g_{n-1} \end{pmatrix}. \tag{66}$$

Proof Consider any vector $\mathbf{U} = (u_1, u_2, ..., u_{n-2}, U)$ from the domain of the operator $\mathbf{A_{max}}$ and let us denote its image by $\mathbf{W} = (w_1, w_2, \ldots, w_{n-2}, W)$. Then equality (64) can be written as follows

$$\begin{aligned} & w_1 g_1 + w_2 g_2 + \ldots + w_{n-3} g_{n-3} + w_{n-2} g_{n-2} + W \\ =& (u_2 - u_1) g_1 + (u_3 - u_2) g_2 + \ldots + (u_{n-2} - u_{n-3}) g_{n-3} - u_{n-2} g_{n-2} + AU. \end{aligned} \tag{67}$$

We conclude that

$$\begin{aligned} w_1 &= u_2 - u_1; \\ w_2 &= u_3 - u_2; \\ & \cdots \\ w_{n-3} &= u_{n-2} - u_{n-3}; \\ W + w_{n-2} g_{n-2} &= AU - u_{n-2} g_{n-2}. \end{aligned} \tag{68}$$

The last equality can be written as

$$W + U + w_{n-2} g_{n-2} + u_{n-2} g_{n-2} = (A+1)U$$

and therefore

$$U = \frac{1}{A+1}(W+U) + (w_{n-2} + u_{n-2}) \frac{1}{A+1} g_{n-2}.$$

It follows that the element U possesses the following representation

$$U = U_r + u_{n-1} g_{n-1},$$

where $U_r \in \mathcal{H}_n$, $u_{n-1} \in \mathbf{C}$. Then equality (67) can be written as

$$\begin{aligned} & w_1 g_1 + w_2 g_2 + \ldots + w_{n-2} g_{n-2} + W \\ = \;\; & (u_2 - u_1) g_1 + (u_3 - u_2) g_2 + \ldots + (u_{n-1} - u_{n-2}) g_{n-2} + AU_r - u_{n-1} g_{n-1}, \end{aligned}$$

and one can deduce that formula (66) holds. □

One can prove that the domain of the operator operator adjoint to $\mathbf{A_{max}}$ is contained in the domain of the maximal operator $\mathbf{A_{max}}$, but the action of the this operator is different from the action of $\mathbf{A_{max}}$. It is easy to see, since the restriction of the maximal operator to the set of functions from the subspace $\{(\vec{u},0)\} \subset \mathrm{Dom}\,(\mathbf{A_{max}})$ is given by non Hermitian matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & ... \\ 0 & -1 & 1 & 0 & ... \\ 0 & 0 & -1 & 1 & ... \\ 0 & 0 & 0 & -1 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{69}$$

Therefore no densely defined restriction of the maximal operator $\mathbf{A_{max}}$ is symmetric and therefore is not self–adjoint, like it was observed for $\mathcal{H}_{-3}$-perturbations. We conclude that no self-adjoint operator is associated with the formal expression (56).

The boundary form of the maximal operator can be calculated using the same method as we used in Lemma 4.3

LEMMA 5.3. *The boundary form of the maximal operator* $\mathbf{A}_{\max}$ *is given by*

$$\ll \mathbf{A_{max}U}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{A_{max}V} \gg$$

$$= \left\langle \begin{pmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ u_{n-3} \\ \langle \varphi, U_r \rangle \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_{n-2} \\ v_{n-3} \\ \langle \varphi, V_r \rangle \end{pmatrix} \right\rangle \tag{70}$$

The matrix describing the boundary form

$$\mathbf{B} \equiv \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

is symplectic and has rank n for even n and $n-1$ for odd n. Hence any symmetric restriction of the operator $\mathbf{A_{max}}$ is described by at least $[\frac{n}{2}]$ boundary conditions.[3]

The restriction $\mathbf{A}^0$ of the maximal operator $\mathbf{A_{max}}$ to the set of functions satisfying $u_{n-2} = 0$ leads to the operator, which is reduced by the orthogonal decomposition of the Hilbert space $\mathbf{H} = \mathbf{C}^{n-2} \oplus \mathcal{H}_{n-2}$. The domain of $\mathbf{A}^0$ is equal to the orthogonal sum $\mathbf{C}^{n-2} \oplus \mathcal{H}_n \subset \mathbf{H}$. The adjoint operator has just the same domain, since the operator A is self-adjoint in $\mathcal{H}_{n-2}$ defined on $\mathcal{H}_n$ and the part of the operator in $\mathbf{C}^{n-2}$ is just a multiplication by the matrix (69). Closed operators such that the domains of the operator and its adjoint coincide will be called **regular**. The difference between regular and self-adjoint operators is that the later ones are in addition symmetric. The operator $\mathbf{A}^0$ just defined is regular, but not symmetric. It is obvious that all regular restrictions, and hence all possible self-adjoint restrictions of $\mathbf{A_{max}}$ are defined by one boundary condition only. To make the operator $\mathbf{A_{max}}$ symmetric one needs to impose at least $[\frac{n}{2}]$ boundary conditions. We conclude that no restriction of the maximal operator $\mathbf{A_{max}}$ is self-adjoint.

[3] $[\cdot]$ denotes the integer part here.

Let us remind that in the case $n = 3$ the rank of the matrix $\mathbf{B}$ is 2 and all Lagrangian planes of the boundary form are described by one condition. Thus the restrictions of $\mathbf{A_{max}}$ to the corresponding subspaces are self-adjoint operators as described in the previous section.

In the following section we are going to describe all regular restrictions of the operator $\mathbf{A_{max}}$.

5.2. Supersingular perturbations as regular operators. It has been proven in [**40**] that all regular restrictions of the operator $\mathbf{A_{max}}$ are described by the boundary conditions (71) below. These conditions are similar to the boundary conditions (47), therefore we are not going to prove the necessity of these conditions but introduce the following definition instead:

DEFINITION 5.2. *The operator $\mathbf{A}_\theta$ is the restriction of the maximal operator $\mathbf{A_{max}}$ defined on the domain* $\mathrm{Dom}\,(\mathbf{A_{max}})$ *(given by (65)) to the set of functions satisfying the boundary condition*

$$\sin\theta\langle\varphi, U_r\rangle + \cos\theta u_{n-1} - \sin\theta u_{n-2} = 0,\ \theta\in[0,\pi). \tag{71}$$

The action of the operator $\mathbf{A}_\theta$ is given by (66).

Like in the case of $\mathcal{H}_{-2}$- and $\mathcal{H}_{-3}$-perturbations a one-dimensional family of operators corresponds to the formal expression (56). The difference is that the operator $\mathbf{A}_\theta$ is not self-adjoint but regular. It is easy to see from the following decomposition

$$\mathbf{A}_\theta = \Re\mathbf{A}_\theta + i\Im\mathbf{A}_\theta, \tag{72}$$

where

$$(\Re\mathbf{A}_\theta)\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-3} \\ u_{n-2} \\ U_r + u_{n-1}g_{n-1}\end{pmatrix} = \begin{pmatrix} \frac12 u_2 - u_1 \\ \frac12(u_3+u_1) - u_2 \\ \dots \\ \frac12(u_{n-2}+u_{n-4}) - u_{n-3} \\ u_{n-1} + \frac12 u_{n-3} - u_{n-2} \\ AU_r - u_{n-1}g_{n-1}\end{pmatrix};$$

$$\Im\mathbf{A}_\theta = \frac12\begin{pmatrix} 0 & -i & 0 & \dots & 0 & 0 & 0 \\ i & 0 & -i & \dots & 0 & 0 & 0 \\ 0 & i & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -i & 0 \\ 0 & 0 & 0 & \dots & i & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \tag{73}$$

The real part of $\mathbf{A}_\theta$ is a self-adjoint operator. The imaginary part is a non-trivial bounded self-adjoint operator. Hence the operator $\mathbf{A}_\theta$ is regular but not self-adjoint.

Let us study the operator $\mathbf{A}_0$ in more details. This operator is equal to the orthogonal sum of two operators acting in the spaces $\mathbf{C}^{n-2}$ and $\mathcal{H}_{n-2}$. Indeed the domain of the operator $\mathbf{A}_0$ can be decomposed as follows

$$\mathrm{Dom}\,(\mathbf{A}_0) = \mathbf{C}^{n-2}\oplus\mathcal{H}_n \subset \mathbf{C}^{n-2}\oplus\mathcal{H}_{n-2} \equiv \mathbf{H}.$$

The two operators appearing in the corresponding decomposition of the operator $\mathbf{A}_0$

$$\mathbf{A}_0 = \mathbf{T} \oplus A,$$

are the operator in $\mathbf{C}^{n-2}$ given by the upper triangular matrix

$$\mathbf{T} = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

and the operator A in $\mathcal{H}_{n-2}$ with the domain $\mathcal{H}_n$. The resolvent of the operator $\mathbf{A}_0$ for arbitrary nonreal λ can easily be calculated

$$\frac{1}{\mathbf{A}_0 - \lambda} = \begin{pmatrix} \frac{-1}{1+\lambda} & \frac{-1}{(1+\lambda)^2} & \frac{-1}{(1+\lambda)^3} & \dots & \frac{-1}{(1+\lambda)^{n-2}} \\ 0 & \frac{-1}{(1+\lambda)} & \frac{-1}{(1+\lambda)^2} & \dots & \frac{-1}{(1+\lambda)^{n-3}} \\ 0 & 0 & \frac{-1}{(1+\lambda)} & \dots & \frac{-1}{(1+\lambda)^{n-4}} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{-1}{1+\lambda} \end{pmatrix} \oplus \frac{1}{A-\lambda}. \tag{74}$$

Let us study now the spectrum of the operator $\mathbf{A}_\theta$. The following theorem implies that the spectrum is real, since the resolvent of $\mathbf{A}_\theta$ exists and is a bounded operator for nonreal values of the spectral parameter.

THEOREM 5.1. *The resolvent of the operator $\mathbf{A}_\theta$ for all nonreal λ is given by the $(n-1)\times(n-1)$ bounded matrix operator*

$$\begin{aligned} \frac{1}{\mathbf{A}_\theta - \lambda} = & \frac{1}{\mathbf{A}_0 - \lambda} \\ & - \frac{\sin\theta}{D(\lambda)} \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^{n-1}} & \frac{1}{(1+\lambda)^{n-2}} \langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot \rangle \\ 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^{n-2}} & \frac{1}{(1+\lambda)^{n-3}} \langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot \rangle \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^2} & \frac{1}{1+\lambda} \langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot \rangle \\ 0 & 0 & \dots & 0 & \frac{1}{1+\lambda}\frac{1}{A-\lambda} g_{n-2} & \left(\frac{1}{A-\lambda} g_{n-2}\right) \langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot \rangle \end{pmatrix} \end{aligned} \tag{75}$$

where the function $D(\lambda,\theta)$ is the following Nevanlinna function

$$D(\lambda,\theta) = \left(Q_n(\lambda) - \frac{1}{1+\lambda}\right)\sin\theta + \cos\theta, \quad Q_n(\lambda) = \langle \varphi, \frac{1+\lambda}{A-\lambda} g_{n-1} \rangle. \tag{76}$$

Proof Consider arbitrary $\mathbf{F} = (f_1, f_2, \ldots, f_{n-2}, F) \in \mathbf{H}$. Then the resolvent equation

$$(\mathbf{A}-\lambda)\begin{pmatrix} u_1 \\ u_2 \\ \ldots \\ u_{n-2} \\ U_r + u_{n-1}g_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \ldots \\ f_{n-2} \\ F \end{pmatrix} \tag{77}$$

together with the boundary condition (71) imply that

$$\begin{pmatrix} -(1+\lambda) & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -(1+\lambda) & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -(1+\lambda) & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -(1+\lambda) & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -\langle \varphi, \dfrac{1+\lambda}{A-\lambda} g_{n-1}\rangle & 1 \\ 0 & 0 & 0 & \ldots & -\sin\theta & \cos\theta & \sin\theta \end{pmatrix}$$

$$\times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_{n-2} \\ u_{n-1} \\ \langle \varphi, U_r\rangle \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \ldots \\ f_{n-2} \\ \langle \varphi, \frac{1}{A-\lambda} F\rangle \\ 0 \end{pmatrix}. \tag{78}$$

To derive this equation we used the following transformation of the last equation in the system (77)

$$(A-\lambda)U_r - (1+\lambda)u_{n-1}g_{n-1} = F \tag{79}$$

$$\Rightarrow \langle \varphi, U_r\rangle - (1+\lambda)u_{n-1}\langle \varphi, \frac{1}{A-\lambda} g_{n-1}\rangle = \langle \varphi, \frac{1}{A-\lambda} F\rangle.$$

The determinant of the matrix appearing in the last equation is equal to $(-1)^{n-1}(1+\lambda)^{n-2}D(\lambda,\theta)$ and it does not vanish for nonreal λ since D is a Nevanlinna function with nontrivial imaginary part (if $\theta \neq 0$). We conclude that the linear system (78) has unique solution for all nonreal λ. It follows that the spectrum of the operator $\mathbf{A}_\theta$ is real.

To calculate the resolvent exactly consider the system of equations for $u_{n-2}, u_{n-1}, \langle \varphi, U_r\rangle$

$$\begin{pmatrix} -(1+\lambda) & 1 & 0 \\ 0 & -\langle \varphi, \dfrac{1+\lambda}{A-\lambda} g_{n-1}\rangle & 1 \\ -\sin\theta & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} u_{n-2} \\ u_{n-1} \\ \langle \varphi, U_r \end{pmatrix} = \begin{pmatrix} f_{n-2} \\ \langle \varphi, \frac{1}{A-\lambda} F\rangle \\ 0 \end{pmatrix} \tag{80}$$

The solution to this linear system reads as follows

$$
(81)\quad \begin{aligned}
u_{n-2} &= -\frac{(\sin\theta\langle\varphi, \frac{1+\lambda}{A-\lambda}g_{n-1}\rangle + \cos\theta)f_{n-2} + \sin\theta\langle\varphi, \frac{1}{A-\lambda}F\rangle}{(1+\lambda)D(\lambda,\theta)};\\
u_{n-1} &= -\frac{\Big((1+\lambda)\langle\varphi, \frac{1}{A-\lambda}F\rangle + f_{n-2}\Big)\sin\theta}{(1+\lambda)D(\lambda,\theta)};\\
\langle\varphi, U_r\rangle &= -\frac{\sin\theta\langle\varphi, \frac{1+\lambda}{A-\lambda}g_{n-1}\rangle f_{n-2} + (\sin\theta - (1+\lambda)\cos\theta)\,\langle\varphi, \frac{1}{A-\lambda}F\rangle}{(1+\lambda)D(\lambda,\theta)}.
\end{aligned}
$$

Then all other components of the vector $\vec{u}$ can be calculated from the recursive relations

$$
u_l = \frac{1}{1+\lambda}u_{l+1} - \frac{1}{1+\lambda}f_l, \quad l = 1, 2, \ldots, n-3,
$$

which coincide with the first $n-3$ equations of the system (78). The following formula holds

$$
u_l = \frac{1}{(1+\lambda)^{n-2-l}}u_{n-2} - \sum_{m=l}^{n-3}\frac{1}{(1+\lambda)^{m+1-l}}f_m.
$$

The component U can be calculated from (79)

$$
\begin{aligned}
U &= U_r + u_{n-1}g_{n-1}\\
&= \frac{1}{A-\lambda}F + (1+\lambda)u_{n-1}\frac{1}{A-\lambda}g_{n-1} + u_{n-1}g_{n-1}\\
&= \frac{1}{A-\lambda}F + u_{n-1}\frac{1}{A-\lambda}g_{n-2}.
\end{aligned}
$$

This completes the calculation of the resolvent of the operator $\mathbf{A}_\theta$ given by formula (75) for all nonreal λ. □

The theorem implies that the spectrum of the operator $\mathbf{A}_\theta$ is real. This is a very important property of the regular operators constructed in this section. One can prove that these operators are similar to self-adjoint ones but this question needs further investigations.

5.3. The fourth resolvent formula. Consider the restriction of the resolvent to the infinite dimensional subspace $\mathcal{H}_{n-2} \subset \mathbf{H}$ combined with the embedding ρ

$$
(82)\quad \begin{aligned}
\rho\frac{1}{\mathbf{A}_\theta - \lambda}|_{\mathcal{H}_{n-2}} &= \frac{1}{A-\lambda}\\
&- \frac{1}{(\lambda+1)^{n-2}\Big\{\cot\theta + Q_n(\lambda) - \frac{1}{1+\lambda}\Big\}}\left\langle\frac{1}{A-\bar\lambda}\varphi, \cdot\right\rangle\frac{1}{A-\lambda}\varphi,
\end{aligned}
$$

This formula is a generalization of the resolvent formulas (6), (25), and (55). It is a natural generalization of Krein's resolvent formula to the case of rank one perturbations determined by $\varphi \in \mathcal{H}_n$. This formula determines a linear operator acting between the Hilbert spaces $\mathcal{H}_{n-2}$ and $\mathcal{H}_{-n+2}$. The denominator appearing in this formula is a generalized Nevanlinna function, which usually appears in extension problems in Pontryagin and Krein spaces. This function can be obtained by regularizing the bordered resolvent $\left\langle\varphi, \frac{1}{A-\lambda}\varphi\right\rangle$ sufficiently many times. In general this regularization contains several arbitrary parameters. In the case when the operator A and the singular vector φ are homogeneous with respect to a certain unitary semigroup the normalization constants can be determined uniquely. This approach developed first in [**5**] for singular interactions was generalized recently [**42**] to include supersingular perturbations. It was shown that this procedure is similar to dimensional regularization of G.'t Hooft and M.Veltman [**27, 28**] and gives the same result as Pauli-Villars regularization [**47**] in quantum field theory (see also [**12**]).

Advantage of the model presented is that it uses Hilbert space methods only. The model presented originally in [**40**] has been generalized in [**20**]. The new model suggested uses instead of repeated regularizations at point $\lambda = -1$, different regularization points. In addition the metric in the extensions space $\mathcal{H}^{n-2}$ is given by a non-diagonal Gram matrix. This allowed to obtain model operators corresponding to the formal expression (56) which are self-adjoint with respect to the new scalar product.

6. Point interaction in $\mathbf{R}^3$ with p-symmetric eigenfunctions

In this section we are going to describe how to construct point interactions with p-symmetry for the Laplace operator in $\mathbf{R}^3$. The Laplace operator $-\Delta$ in $L_2(\mathbf{R}^3)$ is self-adjoint with the domain $\mathrm{Dom}\,(-\Delta) = W_2^2(\mathbf{R}^3)$. The restriction of $-\Delta$ to the set of functions vanishing at the origin is a symmetric operator with the deficiency indices $(1,1)$. The corresponding deficiency element $\frac{e^{ik|x|}}{4\pi|x|}$, $k^2 = \lambda$ is spherically symmetric. Therefore any self-adjoint extension of this operator differs from the unperturbed Laplace operator on the subspace of spherically symmetric functions only. The extended operator corresponds to the following formal operator

$$-\Delta + \alpha\delta, \tag{83}$$

where the coupling parameter α is related in some way to the extension parameter [**11**]. This formal operator can be studied using methods of singular rank one perturbations, since the multiplication operator by Dirac's delta function δ formally coincides with the generalized projector on the Dirac's delta function $\langle \delta, \cdot \rangle \delta$. This is an example of rank one $\mathcal{H}_{-2}$-perturbation treated in detail in [**5**], since $\delta \in \mathcal{H}_{-2}(-\Delta) = W_2^{-2}(\mathbf{R}^3)$.

The simplest formal differential operator leading to non-trivial point interactions for non-spherically symmetric function can be written as

$$L_{\vec{\alpha}} = -\Delta + \alpha_x \langle \delta_x, \cdot \rangle \delta_x + \alpha_y \langle \delta_y, \cdot \rangle \delta_y + \alpha_z \langle \delta_z, \cdot \rangle \delta_z, \tag{84}$$

where α_j, $j = x, y, z$ are real coupling constants and $\delta_x, \delta_y, \delta_z$ denote the derivatives of the three-dimensional delta function

$$\delta_x = \frac{\partial}{\partial x}\delta, \;\; \delta_y = \frac{\partial}{\partial y}\delta, \;\; \delta_z = \frac{\partial}{\partial z}\delta.$$

The perturbation term in (84) is a rank 3 operator between the Hilbert space $\mathcal{H}_3$ and $\mathcal{H}_{-3}$, i.e. the operator $L_{\vec{\alpha}}$ cannot be defined using standard methods developed for finite rank singular perturbations [**7**]. We are going to use an obvious modification of the method to treat $\mathcal{H}_{-3}$-perturbations described in the previous section. This modification is easy, since the subspaces generated by the three generalized functions $\delta_x, \delta_y, \delta_z$ are orthogonal having different symmetries. The scale of Hilbert spaces $\mathcal{H}_s(-\Delta)$ just coincides with the scale of Sobolev spaces $W_2^s(\mathbf{R}^3) = \mathcal{H}_3(-\Delta)$.

Consider the three pairwise orthogonal functions

$$
\begin{array}{rclclcl}
g_1^x & = & \frac{1}{L+1}\delta_x & = & -\frac{e^{-r}}{4\pi r^3}(1+r)x & = & -\frac{e^{-r}}{4\pi r^2}(1+r)\cos\varphi\sin\theta; \\
g_1^y & = & \frac{1}{L+1}\delta_y & = & -\frac{e^{-r}}{4\pi r^3}(1+r)y & = & -\frac{e^{-r}}{4\pi r^2}(1+r)\sin\varphi\sin\theta; \\
g_1^z & = & \frac{1}{L+1}\delta_z & = & -\frac{e^{-r}}{4\pi r^3}(1+r)z & = & -\frac{e^{-r}}{4\pi r^2}(1+r)\cos\theta.
\end{array}
\tag{85}
$$

These three functions belong to the space $W_2^{-1}(\mathbf{R}^3)$ but have different symmetries and therefore are pairwise orthogonal with respect the scalar product in $W_2^{-1}(\mathbf{R}^3)$.

Let us introduce the Hilbert space $\mathbf{H} = \mathbf{C}^3 \oplus W_2^1(\mathbf{R}^3)$ and the natural embedding

$$
\rho : \begin{array}{ccc} \mathbf{H} & \to & \mathcal{H}_{-1} \\ \mathbf{U} = (u_1^x, u_1^y, u_1^z, U) & \mapsto & u_1^x g_1^x + u_1^y g_1^y + u_1^z g_1^z + U. \end{array}
\tag{86}
$$

It will be convenient to use the following vector notation $\vec{u}_1 = (u_1^x, u_1^y, u_1^z)$ and the convention of summation over repeated indices

$$
u_1^\gamma g_1^\gamma \equiv u_1^x g_1^x + u_1^y g_1^y + u_1^z g_1^z.
$$

The scalar product in $\mathbf{H}$ can be chosen equal to

$$
\begin{array}{rcl}
\ll \mathbf{U}, \mathbf{V} \gg_{\mathbf{H}} & = & \langle \vec{u}_1, \vec{v}_1 \rangle_{C^3} + \langle U, (-\Delta+1)V \rangle_{L^2(\mathbf{R}^3)} \\
& = & \overline{u_1^\gamma} v_1^\gamma + \langle U, (-\Delta+1)V \rangle_{L^2(\mathbf{R}^3)}.
\end{array}
\tag{87}
$$

The minimal operator $L_{\min}$ is the operator $-\Delta$ in $W_2^1(\mathbf{R}^3)$ restricted to the domain of functions from $W_2^3(\mathbf{R}^3)$ satisfying the following condition at the origin

$$
\nabla u|_{(x,y,z)=(0,0,0)} = \vec{0}.
\tag{88}
$$

Then Lemma 4.1 implies that the triplet adjoint operator $L_{\max} = L_{\min}^\dagger$ acting in the space $W_2^{-1}(\mathbf{R}^3)$ is defined on the domain

$$
\mathrm{Dom}\,(L_{\max}) = \left\{ U = \tilde{U} + u_1^\gamma g_1^\gamma \in W_2^{-1}(\mathbf{R}^3), \tilde{U} \in W_2^1(\mathbf{R}^3), \vec{u}_1 \in \mathbf{C}^3 \right\}
\tag{89}
$$

by the formula

$$
L_{\max}\left(\tilde{U} + u_1^\gamma g_1^\gamma\right) = -\Delta \tilde{U} - u_1^\gamma g_1^\gamma.
\tag{90}
$$

Then the maximal operator $\mathbf{L}_{\max}$ in $\mathbf{H}$ is uniquely determined by the equality

$$
L_{\max}\rho = \rho \mathbf{L}_{\max}.
\tag{91}
$$

This equality implies that the operator $\mathbf{L}_{\max}$ acts on the functions in the three-dimensional space as the Laplace operator just ignoring the singularities of the functions at the origin. The operator $\mathbf{L}_{\max}$ is defined on the domain

$$
\mathrm{Dom}\,(\mathbf{L}_{\max}) = \left\{ \mathbf{U} = (\vec{u}_1, U_r + u_2^\gamma g_2^\gamma), \vec{u}_1, \vec{u}_2 \in \mathbf{C}^3, U_r \in W_2^3(\mathbf{R}^3) \right\}
\tag{92}
$$

by the formula

$$
\mathbf{L}_{\max} \begin{pmatrix} \vec{u}_1 \\ U_r + u_2^\gamma g_2^\gamma \end{pmatrix} = \begin{pmatrix} \vec{u}_2 - \vec{u}_1 \\ -\Delta U_r - u_2^\gamma g_2^\gamma \end{pmatrix}
\tag{93}
$$

Let us calculate the boundary form of the maximal operator $\mathbf{L}_{\max}$

$$
\begin{aligned}
& \ll \mathbf{U}, \mathbf{L}_{\max}\mathbf{V} \gg - \ll \mathbf{L}_{\max}\mathbf{U}, \mathbf{V} \gg \\
= & \ll \begin{pmatrix} \vec{u}_1 \\ U_r + u_2^\gamma g_2^\gamma \end{pmatrix}, \begin{pmatrix} \vec{v}_2 - \vec{v}_1 \\ -\Delta V_r - v_2^\gamma g_2^\gamma \end{pmatrix} \gg_{\mathbf{H}} \\
& - \ll \begin{pmatrix} \vec{u}_2 - \vec{u}_1 \\ -\Delta U_r - u_2^\gamma g_2^\gamma \end{pmatrix}, \begin{pmatrix} \vec{v}_1 \\ V_r + v_2^\gamma g_2^\gamma \end{pmatrix} \gg_{\mathbf{H}} \\
= & \langle \vec{u}_1, \vec{v}_2 \rangle_{\mathbf{C}^3} - \langle \vec{u}_1, \vec{v}_1 \rangle_{\mathbf{C}^3} \\
& + \langle U_r, (-\Delta+1)(-\Delta)V_r \rangle_{L^2} + \langle u_2^\gamma g_2^\gamma, (-\Delta+1)(-\Delta)V_r \rangle_{L^2} \\
& - \langle U_r, (-\Delta+1) v_2^\beta g_2^\beta \rangle_{L^2} - \langle u_2^\gamma g_2^\gamma, (-\Delta+1) v_2^\beta g_2^\beta \rangle_{L^2} \\
& - \langle \vec{u}_2, \vec{v}_1 \rangle_{\mathbf{C}^3} + \langle \vec{u}_1, \vec{v}_1 \rangle_{\mathbf{C}^3} \\
& - \langle (-\Delta) U_r, (-\Delta+1) V_r \rangle_{L^2} + \langle u_2^\gamma g_2^\gamma, (-\Delta+1) V_r \rangle_{L^2} \\
& - \langle (-\Delta) U_r, (-\Delta+1) v_2^\beta g_2^\beta \rangle_{L^2} + \langle u_2^\gamma g_2^\gamma, (-\Delta+1) v_2^\beta g_2^\beta \rangle_{L^2} \\
= & \langle \vec{u}_1, \vec{v}_2 \rangle_{\mathbf{C}^3} - \langle \vec{u}_2, \vec{v}_1 \rangle_{\mathbf{C}^3} \\
& - \langle \vec{u}_2, \nabla V_r(0) \rangle_{\mathbf{C}^3} + \langle \nabla U_r(0), \vec{v}_2 \rangle_{\mathbf{C}^3} \\
= & \langle \vec{u}_1 + \nabla U_r, \vec{v}_2 \rangle_{\mathbf{C}^3} - \langle \vec{u}_2, \vec{v}_1 + \nabla V_r \rangle_{\mathbf{C}^3}.
\end{aligned} \tag{94}
$$

The boundary form is given by 6×6 symplectic matrix and all self-adjoint restrictions of the operator $\mathbf{L}_{\max}$ are given by Lagrangian planes corresponding to this symplectic form. All such subspaces can be described by the following boundary conditions

$$
\mathbf{a}(\vec{u}_1 + \nabla U_r(0)) = \mathbf{b}\vec{u}_2, \tag{95}
$$

where $\mathbf{a}$ and $\mathbf{b}$ are certain 3×3 matrices such that

(1) the rank of the 3×6 matrix $(\mathbf{a}, \mathbf{b})$ is equal to 3;
(2) $\mathbf{ab}^* = \mathbf{ba}^*$.

These conditions guarantee that the restricted operator is symmetric and maximal. The following lemma describes all boundary conditions that are invariant under rotations around the origin in $\mathbf{R}^3$ and reflections in planes passing through the origin. Rotations R and reflections J of the original space $\mathbf{R}^3$ determine unique transformations of functions on $\mathbf{R}^3$ as follows $Rf(x) = f(R^{-1}x), Jf(x) = f(Jx)$. These transformations can be lifted to unitary transformations $\mathbf{R}$ and $\mathbf{J}$ in the Hilbert space $\mathbf{H}$ using the embedding ρ

$$
R\rho = \rho\mathbf{R}, \quad J\rho = \rho\mathbf{J}.
$$

The rotation $\mathbf{R}$ and reflection $\mathbf{J}$ operators so defined are reduced by the orthogonal decomposition of $\mathbf{H} = W_2^1(\mathbf{R}^3) \oplus \mathbf{C}^3$ and are given by rotation respectively reflection operators in $L_2(\mathbf{R}^3)$ and $\mathbf{C}^3$.

LEMMA 6.1. *The operator $\mathbf{L}$ restricted to the domain of functions satisfying the boundary conditions (95) commutes with the rotations $\mathbf{R}$ around the origin in $\mathbf{R}^3$ and reflections $\mathbf{J}$ in planes passing through the origin if and only if these boundary conditions can be written in the form*

$$
a(\vec{u}_1 + \nabla U_r(0)) = b\vec{u}_2, \tag{96}
$$

where a and b are two real parameters not equal to zero simultaneously.

Proof. The operator $\mathbf{L}_{\max}$ commutes with the rotations and reflections, therefore the restricted operator is commuting with these transformations if and only if the boundary conditions (95) are invariant under $\mathbf{R}$ and $\mathbf{J}$. The pairs of matrices $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}R, \mathbf{b}R)$ describe the same boundary conditions only if the kernels of the matrices $\mathbf{a}$ and $\mathbf{a}R$ coincide, but this is possible only if this kernel is trivial or equal to the whole space $\mathbf{R}^3$. Consider these two cases separately.

1). Suppose that the matrix $\mathbf{a}$ has rank 3, i.e. it is invertible. The boundary conditions (95) can be written in the form

$$\vec{u}_1 + \nabla U_r(0) = \mathbf{b}\vec{u}_2,$$

with a certain Hermitian matrix $\mathbf{b}$. These conditions are invariant under the rotations and inversions if and only if the matrix $\mathbf{b}$ satisfies

$$R\mathbf{b} = \mathbf{b}R, \quad J\mathbf{b} = \mathbf{b}J.$$

It follows that this matrix is a multiple of the unit matrix and necessarily coincides with the multiplication by a certain real constant b. The boundary conditions are then of the form (96).

2). Suppose that the rank of $\mathbf{a}$ is zero. The boundary conditions takes the form

$$0 = \mathbf{b}\vec{u}_2$$

where $\mathbf{b}$ is a certain invertible matrix. This condition is equivalent to

$$\vec{u}_2 = 0,$$

which is of the form (96). □

In what follows only restrictions of the operator $\mathbf{L}_{\max}$ commuting with the rotations and inversions will be studied. All such restrictions can be parameterized by one real parameter $\gamma \in \mathbf{R} \cup \{\infty\}$.

DEFINITION 6.1. *The operator $\mathbf{L}_\gamma$ is the restriction of the maximal operator $\mathbf{L}_{\max}$ to the domain of functions satisfying the boundary conditions*

$$\vec{u}_1 + \nabla U_r(0) = \gamma \vec{u}_2, \quad \gamma \in \mathbf{R} \cup \{\infty\}. \tag{97}$$

The operator L_γ is a self-adjoint operator with the point interaction affecting non-spherically symmetric functions. As in the case of Berezin-Faddeev operator (83) the relation between the coupling parameter $\alpha = \alpha_x = \alpha_y = \alpha_z$ appearing in (84) and the extension parameter γ is not straightforward. It is obvious that $\gamma = \infty$ corresponds to $\alpha = 0$, but precise relation between these parameters cannot be established without bringing into consideration any new assumption, like homogeneity properties of the operators under investigation (see [**5, 40, 4**], where this approach has been developed for Berezin-Faddeev operator).

THEOREM 6.1. *The operator $\mathbf{L}_\gamma$ in $\mathbf{H}$ is a bounded from below self-adjoint operator. Its resolvent is given by*

$$\begin{array}{c}\frac{1}{\mathbf{L}_\gamma-\lambda}=\begin{pmatrix}\frac{1}{-1-\lambda} & 0\\ 0 & \frac{1}{L-\lambda}\end{pmatrix}\\ -\begin{pmatrix}\frac{1}{(1+\lambda)^2\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)} & \frac{-1}{(1+\lambda)^2\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\left(\nabla\frac{1}{L-\lambda}\cdot\right)(0)\\ \frac{1}{(1+\lambda)\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\frac{1}{L-\lambda}g_1^\beta & \frac{-1}{\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\left(\frac{\partial}{\partial\beta}\frac{1}{L-\lambda}\cdot\right)(0)\frac{1}{L-\lambda}g_1^\beta\end{pmatrix},\end{array} \tag{98}$$

for any nonreal λ, where $D(\lambda)=\left(\frac{\partial}{\partial x}\frac{1+\lambda}{L-\lambda}g_2^x\right)(0)$ is a Nevanlinna function.

Proof To prove the theorem it is enough to show that the range of the operator $\mathbf{L}_\gamma-\lambda$ coincides with $\mathbf{H}$ for a certain nonreal λ, since we already proved that the operator is symmetric. Therefore let us calculate the resolvent of the operator $\mathbf{L}_\gamma$.

The resolvent equation

$$(\mathbf{L}_\gamma-\lambda)^{-1}\mathbf{F}=\mathbf{U},\quad \mathbf{F}\in\mathbf{H},\,\mathbf{U}\in\mathrm{Dom}\,(\mathbf{L}_\phi\subset\mathbf{H}$$

implies the following system of equations

$$\left\{\begin{array}{l}\vec{u}_2-\vec{u}_1-\lambda\vec{u}_1=\vec{f}_1,\\ \\ LU_r-u_2^\beta g_2^\beta-\lambda U_r-\lambda u_2^\beta g_2^\beta=F.\end{array}\right. \tag{99}$$

Then the second equation (99) implies

$$U_r=u_2^\beta\frac{1+\lambda}{L-\lambda}g_2^\beta+\frac{1}{L-\lambda}F \tag{100}$$

and

$$\nabla U_r(0)-D(\lambda)\vec{u}_2=\left(\nabla\frac{1}{L-\lambda}F\right)(0).$$

Using the boundary condition one excludes $\nabla U_r(0)$ to get the following 2×2 linear system

$$\left\{\begin{array}{l}-(1+\lambda)\vec{u}_1+\vec{u}_2=\vec{f}_1\\ \\ -\vec{u}_1+(\gamma-D(\lambda))\vec{u}_2=\left(\nabla\frac{1}{L-\lambda}F\right)(0).\end{array}\right. \tag{101}$$

Solution to this system reads as follows

$$\begin{array}{rcl}\vec{u}_1&=&\frac{-\gamma+D(\lambda)}{(1+\lambda)\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\vec{f}_1+\frac{1}{(1+\lambda)\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\left(\nabla\frac{1}{L-\lambda}F\right)(0);\\ \vec{u}_2&=&\frac{-1}{(1+\lambda)\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\vec{f}_1+\frac{1}{\left(\gamma-\frac{1}{1+\lambda}-D(\lambda)\right)}\left(\nabla\frac{1}{L-\lambda}F\right)(0).\end{array} \tag{102}$$

Using (100) we get component U of the function $\mathbf{U}$ which completes calculation of the resolvent

$$U=U_r+u_2^\beta g_2^\beta=\frac{1}{L-\lambda}F+u_2^\beta\frac{1}{L-\lambda}g_1^\beta. \tag{103}$$

The theorem is proved. □

The model operator $\mathbf{L}_\gamma$ can have eigenfunctions having p-symmetry. The corresponding eignvalue has multiplicity 3 corresponding to three spherical functions with p-symmetry. At the same moment the model is constructed in such a way that the total operator commutes with the rotations and reflections in $\mathbf{R}^3$. This model can be extended to include interactions of higher order.

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Dept. of Mathematics, Lund Institute of Technology, Box 118, 22100 Lund, Sweden; Dept. of Physics, St.Petersburg University, 198904, St.Petersburg, Russia.

E-mail address: kurasov@maths.lth.se