Inverse scattering problem on the half line
and positon solutions of the KdV equation. 1

Kurasov P.

Alexander von Humboldt fellow, Dept. of Math., Ruhr-Univ. Bochum,
44780 Bochum, GERMANY; Dept. of Math. and Comp. Physics, St.Petersburg
Univ., 198904 St.Petersburg, RUSSIA and Dept. of Math., Lulea
Univ., 97187 Lulea, SWEDEN

Abstract. The inverse scattering problem for the Schrödinger operator
on the half line is studied for potentials of positon type with long range
oscillating tails at infinity. The inverse problem can be solved for the scatter-
ing matrices with arbitrary finite phase shift. Solution of the inverse
problem is unique if the following scattering data are given: scattering ma-
trix, energies of the bound states and corresponding normalizing constants,
zeroes of the spectral density on the real line.

1. Introduction

The Schrödinger operator $H u = -\frac{d^2 u}{dx^2} + V(x)u$ defined on the functions
on the half axis satisfying the Dirichlet boundary condition at the origin
$u(0) = 0$ is investigated in the present paper. We study the inverse scattering
problem for potentials with long range oscillating tails at infinity violating the Faddeev condition
$\int_0^\infty (1 + x)|V(x)|dx < \infty$. Such potentials were studied first in relation with the positive bound states [1,3,9]. Correspond-
ing solutions of the KdV equation were named positons [11,12]. Uniqueness
results for the inverse problem can be established using the inverse spectral
problem originally studied by I.M.Gelfand, B.M.Levitan, V.A.Marchenko
and L.D.Faddeev [4,2,13] and recently by F.Gesztesy and B.Simon [5]. The
existence of the potential has been proven only for the scattering matrices $S(k)$ with the negative phase shift on the real axis: $\arg S(k)|_{-\infty}^\infty \leq 0$,
which is related to the number of the bound states through the Levinson
theorem. We prove that the inverse problem can be solved for the scatter-
ing matrices with any finite phase shift on the real line (not only negative
one). It is related to the fact that the spectral density corresponding to
long range oscillating potentials can have zeroes on the real axis. See [6,7]
where examples of such potentials are considered. We note, that potentials
considered have only negative bound states. We continue here investigation
carried out in [6-8].

2. Scattering matrices and Jost functions.

Def 1. The set $S$ of scattering matrices consists of all continuous
functions on the real line with the following properties: $|S(k)| = S(\infty) = 1$;

\[ S(-k) = \overline{S(k)}; \quad S(k) - 1 = \int_{-\infty}^{\infty} \sigma(t)e^{ikt}dt, \quad \int_{-\infty}^{\infty} |\sigma(t)|dt < \infty. \]

The set of Jost functions for potentials with the finite first momentum will be denoted by \( \mathcal{F}_F \). Let \( F(k) \in \mathcal{F}_F \) then
1. \( F(k) \) is analytical in the open upper half plane \( \Im k > 0; \)
2. \( F(k) \) is continuous in the closed upper half plane \( \Im k \geq 0; \)
3. \( \lim_{k \to \infty} F(k) = 1; \)
4. \( F(k) \) has representation
   \[ F(k) = 1 + \int_{0}^{\infty} f(t)e^{ikt}dt; \quad \int_{0}^{\infty} |f(t)|dt < \infty. \]

**Def 2.** The set \( \mathcal{F} \) is the set of functions \( F(k) \) which possess the following representation
\[
F(k) = \frac{k + ia_0}{k} \left( \prod_{j=1}^{N_{\text{sing}}} \frac{k + ia_j k + i\bar{a}_j}{k - b_j k + b_j} \right) F_F(k), \tag{1}
\]
where function \( F_F(k) \in \mathcal{F}_F, a_0 \geq 0, \Re a_j > 0. \) If \( F_F(0) = 0 \), then \( a_0 = 0. \)

The spectral density \( \rho(k) = \frac{2\pi}{k} \frac{k^2}{|F(k)|^2} \) corresponding to the Jost functions from \( \mathcal{F} \) vanishes at the points \( \pm b_j \).

**Def 3.** The set \( \mathcal{D} \) of scattering data consists of the following data:
1. scattering matrix \( S(k) \) from \( \mathcal{S} \);
2. natural numbers \( N_{bs}, N_{\text{sing}} \in \mathbb{N} \) and \( \alpha = -1, 0, 1 \) such that \( \arg S(k)|_{-\infty}^{\infty} = -2\pi(2N_{bs} - 2N_{\text{sing}} + \alpha) \);
3. the energies of the bound states and corresponding normalizing constants \( E_j < 0, s_j > 0, j = 1, 2, ..., N_{bs} \);
4. positions of the zeroes of the spectral density on the real line \( b_j > 0, j = 1, 2, ..., N_{\text{sing}} \).

**Lemma 1.** Let \( F(k) \) be any function from \( \mathcal{F} \), then the quotient
\[
S(k) = \frac{F(-k)}{F(k)} \tag{2}
\]
is a function from \( \mathcal{S} \). Let \( S(k) \) be any function from \( \mathcal{S} \), then there exists \( F(k) \in \mathcal{F} \), such that (2) holds. The function \( F(k) \) from \( \mathcal{F} \) is defined uniquely by the scattering data from \( \mathcal{D} \).

**Lemma 2.** The scattering data from \( \mathcal{D} \) and the spectral measures corresponding to the Jost functions from \( \mathcal{F} \) are in one-to-one correspondence.

2. The inverse problem

The following class of potentials will be considered

**Def 5.** The set \( \mathcal{V} \) of potentials is the set of all locally integrable potentials \( V(x) \) on the half axis having the following representation
\[
V(x) = \sum_{j=1}^{N} c_j \sin \left( \frac{2(b_j x + \beta_j)}{x + 1} \right) + V_2(x) + V_F(x),
\]
that function $F$ are real.

**Lemma 3.** Let us assume that potential $\tilde{V} \in \mathcal{V}_F$ and function $S_1(k) = \prod_{j=1}^{N_{sing}} k^{-ia_j} k^{-ia_j}$ are given, then there exists potential $V_1(x) \in \mathcal{V}$, such that $S_1(k)$ is a scattering matrix for the pair of operators $\tilde{H} = -\frac{d^2}{dx^2} + \tilde{V}$, $H_1 = -\frac{d^2}{dx^2} + \tilde{V} + V_1$. **Proof.** Solution of the problem is given by the following formula

$$V_1(x) = -2 \frac{\partial^2}{\partial x^2} \log \det Q(x) = -2 \frac{(\det Q)'' \det Q - (\det Q')^2}{(\det Q)^2},$$

where $Q(x)$ is $2N_{sing} \times 2N_{sing}$ matrix with the following elements

$$Q_{2l-1,2j-1}(x) = \frac{2b_{l-j}}{a_l^2 + b_j^2} W[f(ia_l, x), \varphi(b_j, x)] - \frac{i}{a_l^2 + b_j^2} W[f(ia_l, x), \partial_x \varphi(b_j, x)];$$

$$Q_{2l-1,2j}(x) = \frac{1}{a_l^2 + b_j^2} W[f(ia_l, x), \varphi(b_j, x)].$$

$$Q_{2l,2j-1}(x) = -Q_{2l-1,2j-1}(x); \quad Q_{2l,2j}(x) = Q_{2l,2j-1}(x).$$

The regular solution for potential $\tilde{V} + V_1$ is equal to

$$\varphi_1(k, x) = \frac{\det \begin{bmatrix} Q(x) & f(x) \\ \beta_1(k, x) & \varphi(k, x) \end{bmatrix}}{\det Q(x)}$$

with the following vectors

$$\beta_{1,2j-1} = -i \int_0^x \left( \frac{\partial}{\partial k} \varphi(b_j, t) \right) \varphi(k, t) dt, \quad \beta_{1,2j} = \int_0^x \varphi(b_j, t) \varphi(k, t) dt$$

$$f(x)_{2l-1} = f(ia_l, x); \quad f(x)_{2l} = f(i\bar{a}_l, x).$$

Functions $f(k, x)$ and $\varphi(k, x)$ denote the Jost and regular solutions for the potential $\tilde{V}$. The scattering matrix is defined by the asymptotics of the regular solution.

**Lemma 4.** Let potential $V$ be from the set $\mathcal{V}$, such that the Jost function $F_V(k)$ does not vanish at the origin $F_V(0) \neq 0$ and the operator does not have zero energy bound state. Then for any positive $0 < a_0 < \sqrt{-E_1}$ ($E_1$ is the energy of the highest bound state) there exists potential $V_0 \in \mathcal{V}_C$ such, that function $S_0(k) = \frac{k^{-ia_0}}{k+i\alpha_0}$ is a scattering matrix for the pair of operators $H_1 = -\frac{d^2}{dx^2} + V$, $H_0 = -\frac{d^2}{dx^2} + V + V_0$. 

3
Proof. Solution of the inverse problem is given by the following formula

\[
V_0(x) = 2a^2 \frac{f_1^2(ia_0, x) \varphi_1^2(0, x) - f_1^2(ia_0, x) \varphi_1^2(0, x) + a^2 f_1^2(ia_0, x) \varphi_1^2(0, x)}{(f_1(ia_0, x) \varphi_1'(0, x) - f_1'(ia_0, x) \varphi_1(0, x))^2},
\]

where \( \varphi_1(k, x) \), \( f_1(k, x) \) are regular and Jost solutions for the Schrödinger operator \( H_1 \). The asymptotics of the potential is equal to \( V_0(x) \sim_{x \to \infty} \frac{2}{(1+x)^2} \). It follows, that \( V_0 \in V \).

Theorems 1. Let the scattering data

\[
S(k); N_{bs}, N_{sing}, \alpha; E_j, s_j, j = 1, 2, \ldots, N_{bs}; b_j, j = 1, 2, \ldots, N_{sing}
\]

from \( \mathcal{D} \) be given, then there exists potential \( V \in \mathcal{V} \), corresponding to these scattering data.

Proof. Any function \( S(k) \in \mathcal{S} \) can be presented by a quotient of two Jost functions from \( \mathcal{F} \). The constant \( a_0 \) in the representation can be chosen in such a way, that \( 0 < a_0 < \sqrt{-E_1} \), where \( E_1 \) is the highest bound state of the operator.

The theorem can be proven in three steps. One can construct first potential \( V_F \in \mathcal{V}_F \) corresponding to the scattering matrix \( S_F(k) \) and having \( N_{bs} \) bound states with the energies \( E_j \) and normalizing constants \( s_j \), \( j = 1, 2, \ldots, N_{bs} \). On the second step the inverse problem should be solved for the scattering matrix \( S_1(k) = \prod_{j=1}^{N_{sing}(k-ia_j, k+ia_j)} \), considering potential \( V_F \) as a background potential. Solution of this problem - potential \( V_1 \in \mathcal{V}_C \) - is given by Lemma 3. In the last step if it is necessary the zero energy singularity has to be introduced. We note, that the constant \( a_0 \) is not equal to zero only if the function \( S_F(k) \) is equal to one at the origin. Thus Lemma 4 can be applied to find potential \( V \in \mathcal{V}_C \), which solves the inverse problem for the scattering matrix \( S_0(k) = k - ia_0 \) with the background potential \( V_1 \).

Acknowledgements.

The author wishes to thank professors R.Newton and K.Chadan for helpful suggestions. It is a pleasure for him to thank professors J.Boman, B.S.Pavlov and S.N Naboko for continuous interest to the work. He is grateful to professor S.Albeverio for fruitful and encouraging discussions. Financial support by Alexander von Humboldt foundation is gratefully acknowledged.
References.