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Introduction to Sectorial Operators HT 2017

3. problem sheet

Problem 1
Use the first representation theorem to verify that the operator T_0 in $L^2(0,\infty)$,

$$T_0 f = -f'', \quad \operatorname{dom} T_0 = \left\{ f \in H^1(0,\infty) : f' \text{ AC}, f'' \in L^2(0,\infty), f'(0) = 0 \right\},$$

is m-sectorial, and specify a sector that contains the numerical range of T_0. (Cf. Section 4 of the lecture.)

Problem 2
Provide counterexamples which show that the following statements are **not** true.

(i) If A, V are linear operators in a Hilbert space, A is self-adjoint and V is A-bounded with A-bound one then $A + V$ is self-adjoint.

(ii) If $\mathfrak{t}, \mathfrak{t}'$ are sectorial forms on a Hilbert space, $\mathfrak{t}$ is closed and $\mathfrak{t}'$ is $\mathfrak{t}$-bounded with $\mathfrak{t}$-bound one then $\mathfrak{t} + \mathfrak{t}'$ is closed.

Problem 3
Prove the following statement which is used in the lecture: If U, V are finite-dimensional subspaces of a Hilbert space then

$$\dim U - \dim V = \dim \left(U \cap V^\perp \right) - \dim \left(V \cap U^\perp \right)$$

holds.

Problem 4
Let T, V be linear operators in a Hilbert space such that T is closed and V is T-compact. Show that V is T-bounded with T-bound zero.

Problem 5
Let $T \in \mathcal{C}(\mathcal{H})$ and $\xi \in \rho(T)$. Show that for all $\lambda \neq \xi$ the following hold.[1]

(i) $\lambda \in \sigma(T)$ iff $\frac{1}{\lambda - \xi} \in \sigma((T - \xi)^{-1})$.

(ii) $\lambda \in \sigma_{\mathrm{ess,k}}(T)$ iff $\frac{1}{\lambda - \xi} \in \sigma_{\mathrm{ess,k}}((T - \xi)^{-1})$, $k = 1, 2, 3, 4$.

Here is a "bonus problem" suggested by Tobias:

Problem 6
Let T be a self-adjoint operator such that T^2 is T-bounded. Does it follow that T is bounded?

[1] Statements of this type are usually called spectral mapping theorems.