Stockholms universitet Jonathan Rohleder

# Introduction to Sectorial Operators HT 2017

2. problem sheet

**Problem 1**
Let $\mathfrak{t}$ be a sectorial form with vertex $\gamma$ and semi-angle $\theta$ and let $\mathfrak{h} = \operatorname{Re}\mathfrak{t}$ and $\mathfrak{g} = \operatorname{Im}\mathfrak{t}$. Show the estimate

$$|\mathfrak{g}[u, v]| \leq (\tan\theta)\sqrt{(\mathfrak{h}-\gamma)[u]}\sqrt{(\mathfrak{h}-\gamma)[v]}, \quad u, v \in \operatorname{dom}\mathfrak{t}$$

(cf. Lemma 2.8 in the lecture notes).

**Problem 2**
Prove Proposition 2.14 from the lecture: If $\mathfrak{t}$ is sectorial then $\mathfrak{t}$ is closable iff

$$\left.\begin{array}{c}(u_n)_n \subset \operatorname{dom}\mathfrak{t} \\ u_n \to 0 \text{ in } \mathcal{H} \text{ and} \\ \mathfrak{t}[u_n - u_m] \to 0\end{array}\right\} \Longrightarrow \mathfrak{t}[u_n] \to 0.$$

In this case,

$$\begin{aligned}\bar{\mathfrak{t}}[u] &= \lim_{n\to\infty} \mathfrak{t}[u_n], \\ \operatorname{dom}\bar{\mathfrak{t}} &= \{u \in \mathcal{H} : \exists (u_n)_n \subset \operatorname{dom}\mathfrak{t} \text{ s.t. } u_n \to u \text{ in } \mathcal{H}, \mathfrak{t}[u_n - u_m] \to 0\}.\end{aligned}$$

**Problem 3**
Show that, in general, the numerical range of a closed operator $T$ is not closed and the inclusion $\sigma_c(T) \subset \Theta(T)$ does not hold.

**Problem 4**
Which of the following operators in $\ell^2(\mathbb{N})$ are sectorial and how does a sector containing the numerical range look like? Which of these operators are even m-sectorial?

(i) $R(x_n)_{n\in\mathbb{N}} := (inx_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} R = \{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : (nx_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\}$;

(ii) $S(x_n)_{n\in\mathbb{N}} := ((n+in)x_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} S = \{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : ((n+in)x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\}$;

(iii) $T(x_n)_{n\in\mathbb{N}} := (a_n^{\phi}x_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} T = \{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : (a_n^{\phi}x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\}$, where for a fixed $\phi \in (-\pi, \pi]$, $(a_n^{\phi})_{n\in\mathbb{N}}$ is a sequence of complex numbers being dense in $\{re^{i\phi} : r \in \mathbb{R}\}$.

**Problem 5**
How do the spectra of the operators in Problem 4 look like? Check in each case whether they fill out all of the (closure of the) numerical range of the operator.