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Introduction to Sectorial Operators

HT 2017

1. problem sheet

Problem 1
Discuss the details of Example 1.4 from the lecture: show that in $X = (C([0,1]), \|\cdot\|_\infty)$ the operator

$$Tf = f', \quad \operatorname{dom} T = C^1([0,1]).$$

is closed but not bounded.

Problem 2
Provide an example which shows that the sum of two closed operators in a Banach (or Hilbert) space is not closed in general.

Problem 3
Prove Lemma 1.20 from the lecture: for a densely defined linear operator S in a Hilbert space $\mathcal{H}$, the following are equivalent:

(i) S is symmetric;

(ii) $(Sf, g) = (f, Sg)$ for all $f, g \in \operatorname{dom} S$.

If $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$ then (i) and (ii) are equivalent to

(iii) $(Sf, f) \in \mathbb{R}$ for all $f \in \operatorname{dom} S$.

Hint: For the difficult implication, look at $(S(f + \lambda g), f + \lambda g)$ for appropriate values of λ.

Problem 4
Let S, T be densely defined operators in a Hilbert space such that ST is again densely defined. Show that $T^* S^* \subset (ST)^*$ and that in general the converse inclusion fails. Find additional conditions under which the converse inclusion is true.

Problem 5
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let T be the operator in $L^2(\mathbb{R})$ given by

$$Tg := fg, \quad \operatorname{dom} T := \left\{ g \in L^2(\mathbb{R}) : fg \in L^2(\mathbb{R}) \right\};$$

cf. Example 1.23 from the lecture. Show that the point spectrum of T is given by

$$\sigma_{\mathrm{p}}(T) = \left\{ \mu \in \mathbb{R} : |f^{-1}(\{\mu\})| > 0 \right\},$$

where $|\cdot|$ denotes the Lebesgue measure.