A LOCAL UNIQUENESS THEOREM FOR WEIGHTED RADON TRANSFORMS

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Abstract. We consider a weighted Radon transform in the plane,
\[ R_m(\xi, \eta) = \int_{R} f(x, \xi x + \eta) m(x, \xi, \eta) \, dx, \]
where \( m(x, \xi, \eta) \) is a smooth, positive function. Using an extension of an argument of Strichartz we prove a local injectivity theorem for \( R_m \) for essentially the same class of \( m(x, \xi, \eta) \) that was considered by Gindikin in his article in this issue.

1. Introduction. In the note [4] in this issue Simon Gindikin gives a new inversion formula for a class of weighted plane Radon transforms similar to the attenuated Radon transform. Gindikin's proof is very short and uses no tools other than the fundamental theorem of calculus. Beginning with Novikov, [6], several authors have given similar results before using much more elaborate methods, in most cases based on complex analysis, [1], [5], [2], [3]. Inspired by Gindikin's paper I got the idea to try Strichartz' method in [8] for proving a local injectivity theorem (support theorem) for Gindikin's class of transforms. This led to Theorem 1 and Theorem 2 below.

Following Gindikin I shall consider a weighted Radon transform in the plane
\[ R_m(\xi, \eta) = \int_{R} f(x, \xi x + \eta) m(x, \xi, \eta) \, dx, \]
where \( m(x, \xi, \eta) \) is a smooth function.

Theorem 1. Assume that \( f(x, y) \) is a continuous function that vanishes for \( y < 0 \) and for \( |x| > A \) and that
\[ R_m f(\xi, \eta) = 0 \]
for \( (\xi, \eta) \) in some neighborhood of the origin. Assume that \( m(x, \xi, \eta) \) is strictly positive and satisfies the condition
\[ m'_x(x, \xi, \eta) - x m'_\eta(x, \xi, \eta) = (x a(\xi, \eta) + b(\xi, \eta)) m(x, \xi, \eta), \]
for some functions \( a(\xi, \eta) \) and \( b(\xi, \eta) \) that are independent of \( x \). Then \( f = 0 \) in some neighborhood of the \( x \)-axis.

If \( b(\xi, \eta) = 0 \), then (1.3) is condition (7) in [4]. However, (1.3) is only apparently more general than Gindikin's condition, because if \( m \) satisfies (1.3), then the normalized weight function \( m(x, \xi, \eta)/m(0, \xi, \eta) \) will satisfy (1.3) with \( b(\xi, \eta) = 0 \) (see Section 4). We shall see that condition (1.3) is invariant under affine coordinate transformations that preserve the direction of the \( y \)-axis. This leads to the more general injectivity result in Theorem 2. The relationship between condition (1.3)
and various conditions that have been introduced in previous studies of weighted Radon transforms will be discussed in Section 3. A comment on Gindikin’s inversion formula, based on the affine invariance of condition (1.3), is given in Section 4.

2. Proof of Theorem 1. The proof is an extension of Strichartz’ argument in the proof of the theorem on page 380 in [8]. For \( k = 0, 1, \ldots \) we set

\[
G_k(\xi, \eta) = \int_{\mathbb{R}} x^k f(x, \xi x + \eta) m(x, \xi, \eta) dx.
\]

Our assumption is that \( G_0(\xi, \eta) = 0 \) for all \((\xi, \eta)\) in some neighborhood \( V = \{(\xi, \eta); |\xi| < \varepsilon, |\eta| < \delta\}\) of the origin. I shall prove that \( G_k \) vanishes identically in \( V \) for every \( k \), if \( \varepsilon < \delta/2A \). Then in particular \( \int x^k f(x, \eta)m(x, 0, \eta) dx = 0 \) for all \( k \) and \( |\eta| < \delta \), which implies that \( f(x, \eta)m(x, 0, \eta) = 0 \), and since \( m \) is positive, this implies that \( f(x, \eta) = 0 \) for \( y < \delta \).

I shall argue by induction over \( k \). Assuming we know that \( G_k = 0 \) in \( V \) I shall prove that \( G_{k+1} = 0 \) in \( V \). For simplicity I will assume here that \( f \) is smooth, but the arguments given are actually valid for continuous \( f \) (and in fact for distributions), if all equations are interpreted in distribution sense. The induction assumption implies that \( \partial_\xi G_k = 0 \) in \( V \), that is,

\[
0 = \int x^{k+1} f'(x, \xi x + \eta)m(x, \xi, \eta) dx + \int x^k f(x, \xi x + \eta)m'(x, \xi, \eta) dx.
\]

On the other hand

\[
\partial_\eta G_{k+1}(\xi, \eta) = \int x^{k+1} f'(x, x\xi + \eta)m(x, \xi, \eta) dx
\]

\[
+ \int x^{k+1} f(x, x\xi + \eta)m'(x, \xi, \eta) dx.
\]

Replacing \( m'(x, \xi, \eta) \) in (2.2) by the expression obtained from (1.3) and then comparing with (2.3) we now obtain

\[
0 = \int x^{k+1} f'(x, \xi x + \eta)m(x, \xi, \eta) dx + \int x^{k+1} f(x, x\xi + \eta)m'(x, \xi, \eta) dx
\]

\[
+ a(\xi, \eta) \int x^{k+1} f(x, \xi x + \eta)m(x, \xi, \eta) dx + b(\xi, \eta) \int x^k f(x, \xi x + \eta)m(x, \xi, \eta) dx
\]

\[
= \partial_\eta G_k(\xi, \eta) + a(\xi, \eta)G_k(\xi, \eta) + b(\xi, \eta)G_k(\xi, \eta).
\]

By the induction assumption \( G_k = 0 \). Thus for every fixed \( \xi \) in \((-\varepsilon, \varepsilon)\) the function \( u(\eta) = G_{k+1}(\xi, \eta) \) satisfies the ordinary differential equation

\[
(2.4) \quad u''(\eta) + a(\xi, \eta)u(\eta) = 0
\]

in the interval \(-\delta < \eta < \delta\). Moreover, the function \( G_{k+1}(\xi, \eta) \) vanishes trivially whenever the line \( y = \xi x + \eta \) is contained in the region \((x, y); y < 0 \cup \{(x, y); |x| > A\}\). Since \( \varepsilon < \delta/2A \) this must happen for every \( (\xi, \eta) \) in the rectangle \((\xi, \eta); |\xi| < \varepsilon, -\delta < \eta < -\delta/2\). By the uniqueness theorem for the differential equation (2.4) the function \( u(\eta) = G_{k+1}(\xi, \eta) \) must then vanish in all of \( V \) (note that \( a(\xi, \eta) \) must be smooth since \( m \) is smooth). As already mentioned this implies that \( f(x, y) = 0 \) whenever \( y < \delta \), which completes the proof.

As a corollary we now obtain the following global injectivity theorem.

Corollary 1. Assume that \( f \) has compact support and satisfies \( R_m f(\xi, \eta) = 0 \) for \( \xi \) in some neighborhood of the origin and all \( \eta \), where \( m(x, \xi, \eta) \) is positive and satisfies (1.3). Then \( f = 0 \).
It is natural to ask if a neighborhood of the origin in the \( \xi \eta \)-plane in Theorem 1 and Corollary 1 can be replaced by an arbitrary open set. The answer is yes, and the reason is that the condition (1.3) is invariant under affine coordinate transformations in the \( xy \)-plane that preserve the direction of the \( y \)-axis. To prove this, consider an arbitrary such coordinate transformation
\[
x' = x - x_0, \quad y' = y - y_0 + c(x - x_0).
\]
The line \( y = \xi x + \eta \) takes in the \( x' y' \)-coordinates the form \( y' = \xi' x' + \eta' \), where
\[
\xi = \xi' - c, \quad \eta = \eta' + y_0 - \xi' x_0 + c x_0.
\]

We have to prove that
\[
m_1(x', \xi', \eta') = m(x' + x_0, \xi' - c, \eta' + y_0 - \xi' x_0 + c x_0)
\]
(2.5) satisfies (1.3) if \( m(x, \xi, \eta) \) does. Indeed, writing \( \eta_0 = y_0 + c x_0 \) and using first the chain rule and then the fact that \( m(x, \xi, \eta) \) satisfies (1.3) we obtain
\[
\begin{align*}
(\partial x' - x' \partial y') & \log m_1(x', \xi', \eta') \\
& = ((\partial x - x_0 \partial y - x' \partial y') \log m)(x' + x_0, \xi' - c, \eta' + y_0 - \xi' x_0) \\
& = (x' + x_0) a(\xi' - c, \eta' + \eta_0 - \xi' x_0) + b(\xi' - c, \eta' + \eta_0 - \xi' x_0) \\
& = x' a_1(\xi', \eta') + b_1(\xi', \eta'),
\end{align*}
\]
where
\[
\begin{align*}
a_1(\xi', \eta') &= a(\xi' - c, \eta' + \eta_0 - \xi' x_0), \\
b_1(\xi', \eta') &= x_0 a(\xi' - c, \eta' + \eta_0 - \xi' x_0) + b(\xi' - c, \eta' + \eta_0 - \xi' x_0).
\end{align*}
\]
This proves the claim.

We can now prove a stronger injectivity theorem as follows.

**Theorem 2.** Assume that \( f \) is continuous, compactly supported, and satisfies \( R_m f(\xi, \eta) = 0 \) for an open, connected and unbounded set \( E \) of lines \( y = \xi x + \eta \), and that \( m(x, \xi, \eta) \) is positive and satisfies (1.3). Then \( f = 0 \) on the union of all lines in \( E \).

**Proof.** Let \( L_0 \) be a line in \( E \) not meeting the convex hull of the support of \( f \) and let \( L_1 \) be an arbitrary line in \( E \). It is enough to prove that \( f = 0 \) on \( L_1 \). Since \( E \) is connected we can join \( L_0 \) and \( L_1 \) by a continuous curve \( t \to L_t, 0 < t \leq 1, \) in \( E \). If \( L_1 \) meets the support of \( f \), there must be a \( t_0 \leq 1 \) such that \( L_{t_0} \) meets the support of \( f \) and \( f = 0 \) on one side of \( L_{t_0} \). Because of the coordinate invariance that we just proved we may assume that \( L_{t_0} \) is the \( x \)-axis, and since \( E \) is open we know that \( R_m f(\xi, \eta) = 0 \) for \( (\xi, \eta) \) in some neighborhood of the origin. By Theorem 1 \( f \) must then vanish in some neighborhood of \( L_{t_0} \), which is a contradiction, and hence the theorem is proved.

The fact that condition (1.3) is affine invariant can be better understood from a geometric point of view as will be shown in the next section.

3. **Discussion of the condition (1.3).** The weighted Radon transform in the Euclidean plane can be written
\[
R_p f(\omega, p) = \int_{\omega \cdot x = p} f(x) \rho(x, \omega) ds,
\]
where $ds$ is arclength measure on the line $\omega \cdot x = \omega_1 x_1 + \omega_2 x_2 = p$, $\omega = (\omega_1, \omega_2) \in S^1$ is a unit vector, and $\rho(x, \omega)$ is a weight function defined on $\mathbb{R}^2 \times S^1$. Note that we have changed notation so that $(x, y)$ is now denoted $(x_1, x_2)$. In the special case when $\rho$ has the form

$$\rho_0(x, \omega) = \exp \left( - \int_0^\infty a(x + t\omega^\perp)dt \right),$$

for some smooth function $a(x)$ on $\mathbb{R}^2$ with compact support, $R_\rho$ is called the attenuated Radon transform; here $\omega^\perp$ denotes $(-\omega_2, \omega_1)$. As already mentioned, inversion formulas for the attenuated Radon transform were proved by several authors. The function (3.2) satisfies

$$\langle \omega^\perp, \partial_v \rangle \log \rho(x, \omega) = a(x),$$

and if $\rho$ satisfies (3.3), then $\rho/\rho_0$ is constant on every line $x \cdot \omega = p$, so $\rho$ is equal to $\rho_0$ up to a trivial factor. In [3] we proved an inversion formula under the condition that there exist functions $a(x), b_1(x), b_2(x)$ depending only on $x$ such that

$$\langle \omega^\perp, \partial_v \rangle \log \rho(x, \omega) = a(x) + \omega_1 b_1(x) + \omega_2 b_2(x).$$

If $b_1 = b_2 = 0$, then $\omega \mapsto \log \rho(x, \omega)$ is odd up to a term that is constant on lines, and if $a(x) = 0$ then $\omega \mapsto \rho(x, \omega)$ is even up to a factor that is constant on lines.

The condition (3.4) as formulated here as well as the definition of $R_\rho$ depend on the Euclidean structure in the plane. On the other hand, the fact that a local injectivity theorem holds for a weighted Radon transform is clearly preserved by an affine coordinate transformation in the plane. More generally, this property is also preserved by projective transformations. In fact projective transformations are what Palamodov calls factorable in his book [7, Section 3.1]. This means that a projective transformation taking points and lines (curves) $x$ and $L$ into $\tilde{x}$ and $\tilde{L}$, respectively, transforms surface measure $ds_L$ on $L$ into $c_1(x)c_2(\tilde{L})ds_{\tilde{L}}$ on $\tilde{L}$, that is, $ds_L/ds_{\tilde{L}}$ is equal to a product of a function depending only on $x$ and a function depending only on $L$. If we denote the weighted Radon transform in the new coordinates by $R_{\tilde{\rho}}$, this means — with obvious notation — that $(R_{\tilde{\rho}}f)(L) = c_2(L)(R_\rho(c_1f))(L)$. As a consequence local injectivity must hold for $R_{\tilde{\rho}}$ if it holds for $R_\rho$. Similarly, if we have an explicit inversion formula for $R_\rho$ we immediately get one for $R_{\tilde{\rho}}$.

Therefore it is interesting to study the invariance properties of the condition (3.4). Here we shall only be concerned with the case $a(x) = 0$. In this case the condition (3.4) can be formulated as follows: there exist $b_1(x)$ and $b_2(x)$ such that

$$\langle v, \partial_x \rangle \log \rho(x, \omega) = v_1 b_1(x) + v_2 b_2(x)$$

for all $\omega$ and all tangent vectors $v = (v_1, v_2)$ to $\mathbb{R}^2$ that are perpendicular to $\omega$. We next prove that this condition is invariant under projective transformations.

As a model of the real projective space $\mathbb{P}^2$ we take the set of lines through the origin in $\mathbb{R}^3$. Points of $\mathbb{R}^3$ are denoted $X = (X_0, X_1, X_2)$ and for the corresponding point of $\mathbb{P}^2 = \mathbb{R}^3/(\mathbb{R} \setminus \{0\})$ we use the standard notation $X = (X_0 : X_1 : X_2)$. The set of lines in $\mathbb{P}^2$ can be represented by the lines through the origin in another (dual) copy of $\mathbb{R}^3$, denoted $\tilde{\mathbb{R}}^3$, as follows: if $\Theta \in \tilde{\mathbb{R}}^3$ we denote by $L(\Theta)$ the set of all $(X_0 : X_1 : X_2) \in \mathbb{P}^2$ such that $(\Theta, X) = \Theta_1 X_1 + \Theta_2 X_2 + \Theta_3 X_3 = 0$. The plane $\mathbb{R}^2$ is imbedded in $\mathbb{P}^2$ in the usual way so that $(x_1, x_2)$ corresponds to $(1 : x_1 : x_2)$, and the line $x_1 \omega_1 + x_2 \omega_2 = p = 0$ corresponds to $(-p : \omega_1 : \omega_2)$. The weight function $\rho(X, \Theta)$ is defined on the set of pairs $(X, \Theta) \in \mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ satisfying $(X, \Theta) = 0$, and
it is even and homogeneous of degree zero in each variable. Consider the condition that there exist functions $b_0(X), b_1(X), b_2(X)$ such that

$$\langle V, \partial X \rangle \log \rho(X, \Theta) = V_0 b_0(X) + V_1 b_1(X) + V_2 b_2(X)$$

for all vectors $V = (V_0, V_1, V_2) \in T_r(\mathbb{R}^3) \approx \mathbb{R}^3$ that are perpendicular to $\Theta$. A projective transformation on $\mathbb{P}^2$ is given by a non-singular linear transformation $A$ on $\mathbb{R}^3$ taking $X$ into $AX$, hence taking the line $L(\Theta)$ into $L(A^{*-1}\Theta)$ and thereby transforming the weight function $\rho(X, \Theta)$ into $\tilde{\rho}(X, \Theta) = \rho(AX, A^{*-1}\Theta)$. It is straightforward to verify that the condition (3.6) is preserved by this transformation and that (3.6) takes the form (3.5) if $\rho$ is expressed in the inhomogeneous coordinates $x = (x_1, x_2)$ and $\omega = (\omega_1, \omega_2)$.

To define the weighted Radon transform (3.1) for even $\rho(x, \omega)$ without using the Euclidean structure we need to choose a Hahn measure $d\mu_\omega$ on every line $L$ through the origin in $\mathbb{R}^2$ and translate it to every line parallel to $L$. Denote the dual space to $\mathbb{R}^2$ by $\mathbb{R}^2$. Writing $d\mu_\omega = d\mu_L$ for an arbitrary non-zero element $\omega \in \mathbb{R}^2$ that is perpendicular to $L$ and extending $\rho(x, \omega)$ to $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{(0,0)\})$ as homogeneous of degree zero in $\omega$, we can define $R_\rho$ by

$$R_\rho f(\omega, p) = \int_{\omega \cdot x = p} f(x) \rho(x, \omega) d\mu_\omega, \quad (\omega, p) \in (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}.$$ 

The function $R_\rho f(\omega, p)$ is homogeneous of degree zero with respect to $(\omega, p)$. The correspondence between the operator $R_\rho$ and the function $\rho(x, \omega)$ is then only defined up to a multiplicative factor $c(\omega)$ depending on the choice of $d\mu_\omega$, but such a factor influences neither the properties of $R_\rho$ that we are interested in nor the validity of the condition (3.5).

As an example we can choose $d\mu_\omega = dx_1$ for all $\omega$ with $\omega_2 \neq 0$. If $\omega_2 \neq 0$, the line $\omega \cdot x = p$ can be written as $x_2 = \xi x_1 + \eta$ with $\xi = -\omega_1/\omega_2$, $\eta = p/\omega_2$, so $R_\rho f(\omega, p)$ can be expressed as a parametric Radon transform

$$R_\rho f(\omega, p) = \int_\mathbb{R} f(x_1, \xi x_1 + \eta) \rho(x_1, \xi x_1 + \eta, \omega(\xi)) dx_1, \quad \omega_2 \neq 0, \ p \in \mathbb{R},$$

with $\omega(\xi) = (\xi, -1)$. If we take $v = (1, \xi)$ in (3.5) that condition becomes

$$\nabla x_1 + \xi \nabla x_2 \log \rho(x_1, x_2, \omega(\xi)) = b_1(x) + \xi b_2(x)$$

for some functions $b_1(x)$ and $b_2(x)$ that are independent of $\xi$.

As pointed out by Gindikin, the condition (1.3) is very similar to the condition (3.8), although dual to it in the following sense. For fixed $(x_1, x_2)$ the equation $x_2 = \xi x_1 + \eta$ defines a line in $\xi \eta$-space with (co-)normal $(x_1, 1)$. The condition (1.3) means that the derivative of log $m$ in the direction perpendicular to $(x_1, 1)$ — that is, along the line — is a linear function of $v$. This means that condition (3.8) is the same as (1.3) applied to the adjoint transform. To see this we choose $d\xi d\eta$ as the measure on the manifold of lines in order to get an inner product in the space of functions on lines. Then the adjoint $R_\rho^*$ of the operator (1.1) can be written

$$R_\rho^* g(x_1, x_2) = \int_\mathbb{R} g(\xi, -\xi x_1 + y) m(x_1, \xi, -\xi x_1 + x_2) d\xi.$$ 

Replacing variables according to the scheme $(x_1, x_2, \xi) \rightarrow (\xi, \eta, x_1)$ and taking into account the minus signs in equation (3.9) we see that condition (3.8) means exactly the same as condition (1.3) applied to $R_\rho^*$. Accordingly, if we express $m(x_1, \xi, \eta)$ in
terms of homogeneous coordinates,
\[ \rho(X, \Theta) = m(X_1/X_0, -\Theta_1/\Theta_2, \Theta_0/\Theta_2), \]
then \( \rho \) satisfies the dual condition to (3.6)

\[ (\Psi, \partial_X)\rho(X, \Theta) = (\Psi, b(\Theta)) = \Psi_0 c_0(\Theta) + \Psi_1 c_1(\Theta) + \Psi_2 c_2(\Theta) \]

for some functions \( c_j(\Theta) \) and for all vectors \( \Psi = (\Psi_0, \Psi_1, \Psi_2) \in T_4(\mathbb{R}^3) \approx \mathbb{R}^3 \) that are perpendicular to \( X \). An argument given above shows that this condition is projectively invariant, hence condition (1.3) is projectively invariant, and in particular we see again that (1.3) is affine invariant in the sense verified in Section 2.

Let us finally compute the expression for (1.3) in terms of the function \( \rho(x, \omega) \) in the case when \( \omega \mapsto \rho(x, \omega) \) is even. With \( \omega(\xi) = (\xi, -1) \) we have \( m(x_1, \xi, \eta) = \rho(x_1, \xi x_1 + \eta, \omega(\xi)) \). By the chain rule we easily obtain
\[ (\partial_\xi - x_1 \partial_\eta) \log m(x_1, \xi, \eta) = \partial_\xi \log \rho(x_1, x_2, \omega(\xi)) \bigg|_{x_2 = \xi x_1 + \eta}, \]
It follows that (1.3) is equivalent to

\[ \partial_\xi \log \rho(x_1, x_2, \omega(\xi)) \bigg|_{x_2 = \xi x_1 + \eta} = x_1 a(\xi, \eta) + b(\xi, \eta) \]

for some functions \( a(\xi, \eta) \) and \( b(\xi, \eta) \). Expressed geometrically this means that the derivative of \( \log \rho(x_1, x_2, \omega(\xi)) \) with respect to \( \xi \) is an affine function of \( x_1 \) (or of \( x_2 \)) as long as the point \((x_1, x_2)\) stays on a fixed line \( L(\xi, \eta) \), with coefficients that depend on the line.

It follows that Gindikin’s result implies that there exists an inversion formula for \( R_\rho \) for even \( \rho(x, \omega) \) satisfying (3.11). Similarly, our Theorem 2 implies that local injectivity must hold for such \( R_\rho \).

4. A remark on Gindikin’s formula. Gindikin’s first and main step is to prove the formula (9) in [4], that is,

\[ cf(0, 0) = \int \int \left( \frac{\partial}{\partial \eta} R_m f(\xi, \eta) + a(\xi, \eta) R_m f(\xi, \eta) \right) d\xi \frac{d\eta}{\eta}, \]
under the condition that

\[ m(0, \xi, \eta) = 1 \text{ for all } \xi, \eta \]

and that \( m(x, \xi, \eta) \) satisfies (1.3) with \( b(\xi, \eta) = 0 \), which is condition (7) in [4]. The extension of this formula to arbitrary \((x_0, y_0)\) is not quite trivial, since those conditions on \( m(x, \xi, \eta) \) are not translation invariant. However, the fact that our condition (1.3) is affine invariant makes it easy to make this extension. Set \( f_1(x, y) = f(x_0 + x, y_0 + y) \) and observe that

\[ R_m f_1(\xi, \eta) = R_m f(\xi, \eta - \xi x_0 + y_0), \]

with \( m_1(x, \xi, \eta) = m(x + x_0, \xi, \eta - \xi x_0 + y_0) \). To be able to apply (4.1) we introduce the normalized weight function

\[ m_0(x, \xi, \eta) = \frac{m_1(x, \xi, \eta)}{m_1(0, \xi, \eta)} = \frac{m(x + x_0, \xi, \eta - \xi x_0 + y_0)}{m(x_0, \xi, \eta - \xi x_0 + y_0)}, \]

which satisfies (4.2). Next observe that a normalized weight function satisfying (1.3) must satisfy (1.3) with \( b(\xi, \eta) = 0 \); to see this choose \( x = 0 \) in (1.3). A simple computation shows that in fact

\[ (\partial_\xi - x \partial_\eta) \log m_0(x, \xi, \eta) = x a_1(\xi, \eta - \xi x_0 + y_0), \]
where
\begin{equation}
a_1(\xi, \eta) = a(\xi, \eta) + \frac{m'(x_0, \xi, \eta)}{m(x_0, \xi, \eta)},
\end{equation}
if \( m(x, \xi, \eta) \) satisfies (1.3). An application of (4.1) with \( m = m_0 \) together with (4.3) and the change of variable \( \eta \to \eta - y_0 \) now gives
\[ cf(x_0, y_0) = cf_1(0, 0) \]
\[ = \int \int \left( \frac{\partial}{\partial \eta} R_m f(\xi, \eta - \xi x_0) + a_1(\xi, \eta - \xi x_0) \frac{R_m f(\xi, \eta - \xi x_0)}{m(x_0, \xi, \eta - \xi x_0)} \right) d\xi \frac{d\eta}{\eta - y_0}. \]
Using the fact that
\[ \frac{\partial}{\partial \eta} \frac{R_m f}{m} = \frac{1}{m} \frac{\partial}{\partial \eta} R_m f - \frac{m'}{m} \frac{R_m f}{m} \]
together with (4.4) we can simplify the formula slightly to read
\[ cf(x_0, y_0) \]
\[ = \int \int \frac{\partial}{\partial \eta} R_m f(\xi, \eta - \xi x_0) + a(\xi, \eta - \xi x_0) \frac{R_m f(\xi, \eta - \xi x_0)}{m(x_0, \xi, \eta - \xi x_0)} d\xi \frac{d\eta}{\eta - y_0}. \]

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