Mathematical Analysis

Flatness of distributions vanishing on infinitely many hyperplanes

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Abstract

Let \( \{L_k\}_{k=1}^{\infty} \) be a family of hyperplanes in \( \mathbb{R}^n \) and let \( L_0 \) be a limiting hyperplane of \( \{L_k\} \). Let \( u \) be a distribution that satisfies a natural wave front condition and has vanishing restrictions to \( L_k \) for all \( k \geq 1 \). Then \( u \) must be flat at \( L_0 \).

Résumé

Platitude des distributions s’annulant sur une infinité d’hyperplans. Soit \( \{L_k\}_{k=1}^{\infty} \) une famille d’hyperplans dans \( \mathbb{R}^n \) et soit \( L_0 \) un hyperplan limite de \( \{L_k\} \). Si \( u \) est une distribution satisfaisant à une condition naturelle portant sur le front d’onde et qui s’annule sur \( L_k \) pour tout \( k \geq 1 \), alors \( u \) est plate sur \( L_0 \).

1. Introduction

Let \( \mathcal{L} \) be an infinite family of distinct hyperplanes \( L \) in \( \mathbb{R}^n \) with limit point (in the natural topology on the \( n \)-dimensional manifold of hyperplanes) \( L_0 \), and let \( U \) be an open set in \( \mathbb{R}^n \) intersecting \( L_0 \). Let \( u \) be an infinitely differentiable function vanishing on \( L \cap U \) for all \( L \in \mathcal{L} \). Then it is easy to see that \( u \) must be flat on \( L_0 \) in the sense that the derivatives of \( u \) of all orders vanish on \( L_0 \). To prove this, assume that some derivative of order \( m \) of \( u \) is different from zero at \( x_0 \in L_0 \) and that all derivatives of order \( < m \) vanish at \( x_0 \). We may assume that \( x_0 \) is the origin in \( \mathbb{R}^n \). Then \( u(x) = p(x)(1 + O(|x|)) \) as \( |x| \to 0 \), where \( p(x) \) is a non-zero homogeneous polynomial of degree \( m \).

Then the restriction of \( u \) to \( L \) must be non-identically zero for any hyperplane \( L \) with sufficiently small distance to the origin. This proves the statement.

The purpose of this note is to show that a similar statement is true for distributions \( u \), provided that

\[
WF(u) \cap N^*(L_0 \cap U) = \emptyset, \tag{1}
\]

a condition which is needed for the restriction of \( u \) to \( L_0 \) to be well defined [4, Corollary 8.2.7]. Here \( N^*(L_0) \) denotes the conormal manifold to \( L_0 \), i.e., the set of \( (x, \xi) \) where \( x \in L_0 \) and \( \xi \) is conormal to \( L_0 \) at \( x \), and \( WF(u) \) is the wave...
front set of $u$. By definition $(x^0, \xi^0) \notin WF(u)$ if there exists a function $\psi \in C^\infty_0$ such that $\psi(x^0) \neq 0$ and a conic neighborhood $\Gamma$ of $\xi^0$ such that $\hat{\psi}u(\xi)$ is rapidly decaying in $\Gamma$ in the sense that
\[
|\hat{\psi}u(\xi)| \leq C_m|\xi|^m, \quad \xi \in \Gamma, \ m = 1, 2, \ldots.
\]
(2)
Assume $L_0$ is the hyperplane $x_0 = 0$. If (1) holds and $x^0 \in L_0 \cap U$ we can choose $\psi \in C^\infty_0(U)$ with $\psi(x^0) \neq 0$ such that (2) holds with $\Gamma = \{\xi = (\xi', \xi_0); |\xi'| < \delta|\xi_0|\}$ for some $\delta > 0$. If $\varphi(\xi')$ is a test function in $C^\infty_c(\mathbb{R}^{n-1})$ defined on $L_0$, then the action of the restriction $\psi u|_{L_0}$ on the test function $\varphi$ can be defined by
\[
\langle \psi u|_{L_0}, \varphi \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\psi}u(\xi', \xi_0)\hat{\varphi}(-\xi') \, d\xi.
\]
Note that the integral must be absolutely convergent because of (2).

The space of (Schwartz) distributions on the open set $U$ is denoted $\mathcal{D}'(U)$.

**Theorem 1.** Let $\{L_k\}_{k=1}^\infty$ be an infinite family of (distinct) hyperplanes in $\mathbb{R}^n$, $n \geq 2$, and let $L_0 = \lim_{k \to \infty} L_k$ (in the topology of the manifold of hyperplanes). Let $U$ be a bounded open subset of $\mathbb{R}^n$. Assume $u \in \mathcal{D}'(U)$ satisfies (1) and the restriction $u|_{L_k \cap U}$ vanishes for all $k \geq 1$.

Then $u$ is flat in the set $L_0 \cap (U \setminus F)$, where $F$ is the (possibly empty) affine subspace of $L_0$ of codimension $\geq 1$.

\[
F = \bigcup_{m=1}^\infty \bigcap_{m=1}^\infty (L_k \cap L_0),
\]
that is, for every partial derivative $\partial^m$ of $u$ the restriction of $\partial^m u$ to $L_0 \cap (U \setminus F)$ vanishes.

Note that the wave front condition (1) is satisfied for all hyperplanes $L$ sufficiently close to $L_0$, since the wave front set $WF(u)$ is closed. Since $WF(\partial^m u) \subset WF(u)$ for any partial derivative and any distribution $u$, the same is true for all partial derivatives $\partial^m u$. Note also that the set $F_m = \bigcap_{k=m}^\infty (L_k \cap L_0)$ is a (possibly empty) affine subspace of $L_0$ of codimension $\geq 1$ for every $m$ (we may assume that $L_k \neq L_0$ for all $k$), and the sequence $F_m$ is increasing, so it is clear that $F$ is an affine subspace as stated in the theorem. $F$ can be described as the set of all points that are contained in all except finitely many of the sets $L_0 \cap L_k$.

The fact that the exceptional set $F$ may occur can be seen from the following example. Let $u$ be the distribution on $\mathbb{R}^2$ defined by $u(x_1, x_2) = x_2\delta_0(x_1)$. Then $WF(u)$ is equal to the conormal of the line $x_1 = 0$, that is, $WF(u) = \{(0, x_2; \xi_1, 0); x_2 \in \mathbb{R}, \xi_1 \neq 0\}$. The restriction of $u$ to any of the lines $L_k = \{x \in \mathbb{R}^2; x_2 = x_1/k\}$, $k = 1, 2, \ldots$, is well defined and vanishes, but the restriction of $\partial_{x_2} u = \delta_0(x_1)$ to $L_0 = \{x \in \mathbb{R}^2; x_2 = 0\}$ is $\delta_0(x_1)$, so $u$ is not flat on all of $L_0$ but only on $L_0 \setminus F$ where $F = \{(0, 0)\}$.

The assertion of Theorem 1 is in fact valid also if $n = 1$, because then condition (1) means that $u$ is $C^\infty$ in some neighborhood of the point $L_0$, and a smooth function vanishing at an infinite sequence of points must be flat at a limit point of that sequence; note that a hyperplane (affine submanifold of codimension 1) means a point in this case.

By our vanishing theorem for microlocally real analytic flat distributions [2], a distribution $u$ that satisfies the analytic wave front condition
\[
WF_A(u) \cap N^*(L_0 \cap U) = \emptyset
\]
and is flat on $L_0 \cap U$ must vanish in some neighborhood of $L_0 \cap U$. (For the definition of the analytic wave front set, $WF_A(u)$, see [4].) Combining this fact with Theorem 1 we obtain the following extension of the familiar fact that a real analytic function of one variable that vanishes at infinitely many points with a finite limit point must vanish identically:

**Corollary 2.** Let $L_k$, $k = 1, 2, \ldots$, be an infinite sequence of distinct hyperplanes in $\mathbb{R}^n$ and let $L_0 = \lim_{k \to \infty} L_k$ as in Theorem 1. Let $U$ be a bounded open subset of $\mathbb{R}^n$. Assume $u \in \mathcal{D}'(U)$ satisfies (5) and (3). Then $u = 0$ in some neighborhood of $L_0 \cap (U \setminus F)$, where $F$ is the set defined by (4).

In the recent article [3] we applied this corollary to give a new proof of a uniqueness result for a ray transform [1].
2. Proof of Theorem 1

Since $F$ is closed it is sufficient to prove that $u$ is flat on $L_0 \cap U_1$ for some open neighborhood $U_1$ of an arbitrary point of $U \setminus F$. Thus from now on we denote $U_1$ by $U$ and assume that $U \cap F = \emptyset$.

We denote the coordinates in $\mathbb{R}^n$ by $(x, y)$ where $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, and the dual coordinates by $(\xi, \eta)$. We may assume that $L_0$ is the plane $y = 0$. Let $(x^0, 0) \in U \cap L_0$ and choose a neighborhood $V \subset U$ of $(x^0, 0)$, a conic neighborhood $\Gamma = \{(\xi, \eta); |\xi| < \delta|\eta|\}$ of the conormal $(0, \pm 1)$ to $L_0$, and a function $\psi \in C^\infty_0(V)$ such that $\psi(x^0, 0) \neq 0$ and $\hat{\psi}(\xi, \eta)$ is rapidly decaying in the cone $\Gamma$. From now on we shall denote $\hat{\psi}(\xi, \eta)$ by $\hat{\psi}(\xi)\eta$. We may assume that $u$ is a real-valued distribution in the sense that $(u, \varphi)$ is real for all real-valued test functions $\varphi$. Denote by $u|_{L_{a,b}}$ the restriction of $u$ to the plane $y = a \cdot x + b$, where $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$. If $|a|$ and $|b|$ are sufficiently small this restriction is well defined, and its action on a test function $\varphi \in C^\infty_0(\mathbb{R}^{n-1})$ can be written as

$$
\langle u|_{L_{a,b}}, \varphi \rangle = \int \int \hat{u}(\xi, \eta)\hat{\varphi}(-\xi - \eta a)e^{ib\eta} d\xi d\eta.
$$

The fact that $\hat{u}$ has at most polynomial growth in the $\xi$-variable and is rapidly decaying in the cone $|\xi| < \delta|\eta|$ implies that the integral is absolutely convergent if $|a|$ is sufficiently small. For real-valued $\varphi \in C^\infty_0(\mathbb{R}^{n-1})$ we set

$$
\rho_{a,b}(s) = \rho_{a,b}(s) = \langle u|_{L_{a,b}}, \varphi \rangle = \int \int \hat{u}(\xi, \eta)\hat{\varphi}(-\xi - s\eta a)e^{ib\eta} d\xi d\eta, \quad s \in \mathbb{R}.
$$

Then $\rho_{a,b}(0) = \langle u|_{L_0}, \varphi \rangle$ and $\rho_{a,b}(1) = \langle u|_{L_{a,b}}, \varphi \rangle$. It is clear that $\rho_{a,b} \in C^\infty(-1, 1)$, if $|a| < \delta/2$. Differentiating $m$ times with respect to $s$ we obtain

$$
\rho_{a,b}^{(m)}(s) = \langle \partial_y^m u|_{L_0}, \varphi \rangle, \quad m \geq 0 \quad (P_m)
$$

where $\psi(x) = (a \cdot x + b)^m \varphi(x)$.

We shall prove this by induction over $m$. For $m = 0$ ($P_m$) is true with $s_0 = 1$ by the assumption that $u|_{L_{a,b}} = 0$ for all $k$ and $L_{a,b}$. Let $m$ be arbitrary and assume that ($P_m$) holds. Using the expression ($7$) for $\rho_{a,b}^{(m)}$ in ($P_m$), dividing by $(|a_k|^2 + b_k^2)^{1/2}$, passing to a subsequence such that $(a_k, b_k)/(|a_k|^2 + b_k^2)^{1/2}$ converges to a limit $(a_0, b_0)$, and letting $k$ tend to infinity we obtain

$$
0 = \int \hat{u}(\xi)\eta^m(-a \cdot \nabla + b_0)^m \hat{\varphi}(-\xi) d\xi d\eta = \langle \partial_y^m u|_{L_0}, (a_0 \cdot x + b_0)^m \varphi \rangle.
$$

Our assumption that $U \cap F = \emptyset$ implies that $a_0 \cdot x + b_0 \neq 0$ on $L_0 \cap U$. Since $\varphi$ is arbitrary it follows that $\partial_y^m u|_{L_0} = 0$ in $U$, and hence $\rho_{a,b}^{(m)}(0) = 0$ for every $\varphi$ and every $a$ and $b$. For an arbitrary $k$ we now use the induction assumption ($P_m$) once more together with Rolle’s theorem and obtain

$$
0 = \rho_{a_k,b_k}^{(m)}(s_0) - \rho_{a_k,b_k}^{(m)}(0) = \rho_{a_k,b_k}^{(m+1)}(s_1)
$$

for some $s_1 \in (0, s_0)$, which proves ($P_{m+1}$), and hence proves ($P_m$) for every $m$. By ($6$) we know that $\rho_{a,b}$ is real-valued, so the application of Rolle’s theorem is appropriate. Letting $k$ tend to infinity and using ($7$) now gives

$$
0 = \int \int \hat{u}(\xi, \eta)\eta^m \hat{\varphi}(-\xi) d\xi d\eta,
$$

that is, $\partial_y^m u|_{L_0} = 0$ for every $m$. 


It remains to show that an arbitrary mixed derivative $\partial_\alpha^x \partial_\beta^y u$, where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, must have vanishing restriction to $L_0$. By definition this means that

$$\int \int \hat{u}(\xi, \eta) \xi^\alpha \eta^m \hat{\varphi}(-\xi) \, d\xi \, d\eta = 0, \quad \varphi \in C_0^\infty (\mathbb{R}^{n-1}) .$$

(9)

Replacing $\varphi$ in (8) with $\partial_\alpha^x \varphi$ gives (9) and completes the proof.

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References