Univalent multisets

V through the eyes of the identity type

Håkon Robbestad Gylterud

March 2014

Outline of the talk

1. Present common intuition about multisets

Outline of the talk

1. Present common intuition about multisets
2. Iterative multisets and Aczel's V

Outline of the talk

1. Present common intuition about multisets
2. Iterative multisets and Aczel's V
3. Give a model

Outline of the talk

1. Present common intuition about multisets
2. Iterative multisets and Aczel's V
3. Give a model
4. A simple result about W-types

Outline of the talk

1. Present common intuition about multisets
2. Iterative multisets and Aczel's V
3. Give a model
4. A simple result about W-types
5. Apply this result to the model

Outline of the talk

1. Present common intuition about multisets
2. Iterative multisets and Aczel's V
3. Give a model
4. A simple result about W-types
5. Apply this result to the model

Mathematical context

In this talk...

- We work in Martin-Löf type theory.

Mathematical context

In this talk...

- We work in Martin-Löf type theory.
- The notion of "set" is that of a type in type theory

Mathematical context

In this talk...

- We work in Martin-Löf type theory.
- The notion of "set" is that of a type in type theory (or rather element in the type *Set* in the logical framework).

Mathematical context

In this talk...

- We work in Martin-Löf type theory.
- The notion of "set" is that of a type in type theory (or rather element in the type *Set* in the logical framework).
- We will use the term "iterative set" to refer to the notion of set which is studied in Set Theory.

Mathematical context

In this talk...

- We work in Martin-Löf type theory.
- The notion of "set" is that of a type in type theory (or rather element in the type *Set* in the logical framework).
- We will use the term "iterative set" to refer to the notion of set which is studied in Set Theory.
- Juxtaposition denotes (left associative) function application. That is, $f\ x$ denotes f applied to x, and $f\ x\ y := (f\ x)\ y$

Mathematical context

In this talk...

- We work in Martin-Löf type theory.
- The notion of "set" is that of a type in type theory (or rather element in the type *Set* in the logical framework).
- We will use the term "iterative set" to refer to the notion of set which is studied in Set Theory.
- Juxtaposition denotes (left associative) function application. That is, $f\ x$ denotes f applied to x, and $f\ x\ y := (f\ x)\ y$
- The technical parts are formalized in Agda.

What are multisets?

What are multisets?

Our intuition is that multisets...

- Consists of elements.

What are multisets?

Our intuition is that multisets...

- Consists of elements.
- Elements are considered to be unordered.

What are multisets?

Our intuition is that multisets...

- Consists of elements.
- Elements are considered to be unordered.
- For each element the number of occurences is relevant.

What are multisets?

Our intuition is that multisets...

- Consists of elements.
- Elements are considered to be unordered.
- For each element the number of occurences is relevant.

The first two points are applies to sets as well. The third point distinguishes the two notions.

Examples

- The roots of a polynomial is a multiset if we respect multiplicity. $x^3 - 2x^2 + x$ has roots $\{0, 1, 1\}$.

Examples

- The roots of a polynomial is a multiset if we respect multiplicity. $x^3 - 2x^2 + x$ has roots $\{0, 1, 1\}$.
- Sequent calculus. $A, A \vdash \phi$

Examples

- The roots of a polynomial is a multiset if we respect multiplicity. $x^3 - 2x^2 + x$ has roots $\{0, 1, 1\}$.
- Sequent calculus. $A, A \vdash \phi$
- Bags in computer science.

Classical vs Constructive

Classically, a multiset is modelled as a set X, called the domain, and a function, $e : X \to \mathbb{N}$. Or if extended into the infinite, a function $e : X \to \mathrm{Card}$.

Classical vs Constructive

Classically, a multiset is modelled as a set X, called the domain, and a function, $e : X \to \mathbb{N}$. Or if extended into the infinite, a function $e : X \to \text{Card}$.

Constructively, there might not be interestion functions into $\mathbb{N}$, and the notion of cardinals is problematic.

Classical vs Constructive

Classically, a multiset is modelled as a set X, called the domain, and a function, $e : X \to \mathbb{N}$. Or if extended into the infinite, a function $e : X \to \text{Card}$.

Constructively, there might not be interestion functions into $\mathbb{N}$, and the notion of cardinals is problematic.

The general solution is to define

Definition

A multiset with domain $X : \mathit{Set}$ is a family $M : X \to \mathit{Set}$.

Iterative multisets

Is it possible to parallell the construction of iterative sets?

Iterative multisets

Is it possible to parallell the construction of iterative sets?
For iterative sets, we consider the totality V, consisting of sets where all elements of the sets, them selves are sets.

Iterative multisets

Is it possible to parallell the construction of iterative sets?
For iterative sets, we consider the totality V, consisting of sets where all elements of the sets, them selves are sets.
One may then wish for a totality M, consistsing of multisets of multisets, all with with domain M it self.

Trees

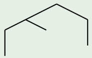
It is well know that (wellfounded) trees can serve as models of (wellfounded) iterative sets.

Trees

It is well know that (wellfounded) trees can serve as models of (wellfounded) iterative sets.

Example

The iterative set $\{\{\{\emptyset\},\emptyset\},\{\emptyset\}\}$ is represented by

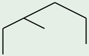


Trees

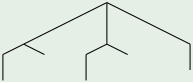
It is well know that (wellfounded) trees can serve as models of (wellfounded) iterative sets.

Example

The iterative set $\{\{\{\emptyset\},\emptyset\},\{\emptyset\}\}$ is represented by



but also by

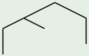


Trees

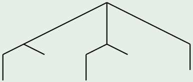
It is well know that (wellfounded) trees can serve as models of (wellfounded) iterative sets.

Example

The iterative set $\{\{\{\emptyset\}, \emptyset\}, \{\emptyset\}\}$ is represented by



but also by



For iterative multisets, we want to keep these two distinct.

Aczel's model of iterative sets in type theory

Aczel's model of iterative sets in type theory

Definition

(Aczel) Given en a universe $U : Set$ with decoding familty $T : U \to Set$, define a setoid (V, E_V) by

$$
\begin{aligned}
V &: Set \\
V &:= W_{a:U}\, Ta \\
E_V &: V \to V \to Set \\
E_V(\sup a\, f)(\sup b\, g) &:= \prod_{x:Ta} \sum_{y:Tb} E_V(f\, x)(g\, y) \wedge \prod_{y:Ta} \sum_{x:Tb} E_V(f\, x)(g\, y)
\end{aligned}
$$

Aczel's model of iterative sets in type theory

Lemma

E_V is equivalent to

$$E'_V : V \to V \to Set$$
$$E'_V(\sup a\, f)(\sup b\, g) :=$$
$$\sum_{\alpha : Ta \to Tb} \prod_{x : Ta} E_V(f\, x)(g\, (\alpha\, x)) \wedge \sum_{\beta : Tb \to Ta} \prod_{y : Tb} E_V(f\, (\beta y))(g\, y)$$

Aczel's model of iterative sets in type theory

Lemma

E_V is equivalent to

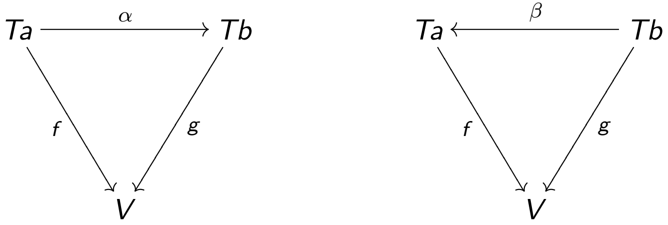
$$E'_V : V \to V \to Set$$
$$E'_V(\sup a\, f)(\sup b\, g) :=$$
$$\sum_{\alpha : Ta \to Tb} \prod_{x : Ta} E_V(f\, x)(g\,(\alpha\, x)) \wedge \sum_{\beta : Tb \to Ta} \prod_{y : Tb} E_V(f\,(\beta y))(g\, y)$$

Proof.

W-induction on V and apply the axiom of choice twice. □

Aczel's model of iterative sets in type theory

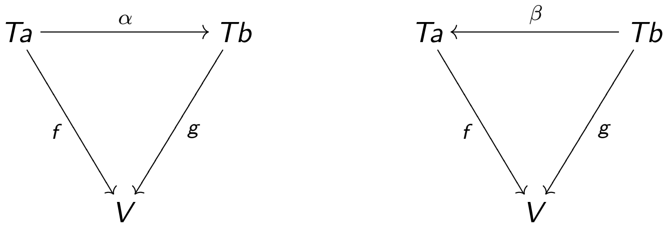
Diagramatically, $(\sup a\, f)$ is equal, according to E_V, to $(\sup b\, g)$ if the diagrams



commutes up to E_V.

Aczel's model of iterative sets in type theory

Diagramatically, $(\sup a\, f)$ is equal, according to E_V, to $(\sup b\, g)$ if the diagrams



commutes up to E_V.
The natural change to make is to require that α and β form an equivalence of types.

The model

Definition

$$M : Set$$

$$M := W_{a:U}\, Ta$$

$$E_M : M \to M \to Set$$

$$E_M(\sup a\, f)(\sup b\, g) := \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} E_M(f\, x)(g\,(\alpha\, x))$$

The model

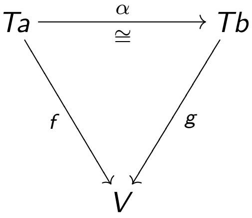
Definition

$$M : Set$$

$$M := W_{a:U}\,Ta$$

$$E_M : M \to M \to Set$$

$$E_M(\sup a\, f)(\sup b\, g) := \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} E_M(f\, x)(g\,(\alpha\, x))$$



The identity type and Equivalence

In Martin-Löf type theory, every $A : Set$ is equipped with a type $=_A : A \to A \to Set$, which is inductively generated by

- If $a : A$ then $(\mathit{refl}\ a) : a =_A a$.

The identity type and Equivalence

In Martin-Löf type theory, every $A : Set$ is equipped with a type $=_A : A \to A \to Set$, which is inductively generated by

- If $a : A$ then $(\mathit{refl}\ a) : a =_A a$.

This induces a notion of extensional equality on functions, and a notion of equivalence between types, which are essential in Homotopy Type Theory.

The identity type and Equivalence

In Martin-Löf type theory, every $A : Set$ is equipped with a type $=_A : A \to A \to Set$, which is inductively generated by

- If $a : A$ then $(refl\ a) : a =_A a$.

This induces a notion of extensional equality on functions, and a notion of equivalence between types, which are essential in Homotopy Type Theory. If $A, B : Set$ we denote by $A \cong B$the type of equivalences from A to B. And if $f, g : A \to B$, we denote by $f \simeq g$ the type of extensional equalities (homotopies) from f to g.

A result on the identity type of W types

Lemma

Let $a, a' : A$ and $f : Ba \to W_A B$ and $g : Ba' \to W_A B$.
If $\alpha : a = a'$ is such that $f = (g \cdot B\alpha)$ then $(\sup a\, f) = (\sup a'\, g)$.

A result on the identity type of W types

Lemma

Let $a, a' : A$ and $f : Ba \to W_A B$ and $g : Ba' \to W_A B$.
If $\alpha : a = a'$ is such that $f = (g \cdot B\alpha)$ then $(\mathrm{sup}\, a\, f) = (\mathrm{sup}\, a'\, g)$.

Proof.

By induction on α it is sufficient to show the claim for the case $\alpha \equiv \mathit{refl}\ a$.

A result on the identity type of W types

Lemma

Let $a, a' : A$ and $f : Ba \to W_A B$ and $g : Ba' \to W_A B$.
If $\alpha : a = a'$ is such that $f = (g \cdot B\alpha)$ then $(\sup a\, f) = (\sup a'\, g)$.

Proof.

By induction on α it is sufficient to show the claim for the case $\alpha \equiv \mathit{refl}\ a$.
In this case, $B\alpha \equiv id_{Ba}$, and so the second hypothesis becomes $f = g \cdot id$.

A result on the identity type of W types

Lemma

Let $a, a' : A$ and $f : Ba \to W_A B$ and $g : Ba' \to W_A B$.
If $\alpha : a = a'$ is such that $f = (g \cdot B\alpha)$ then $(\sup a\, f) = (\sup a'\, g)$.

Proof.

By induction on α it is sufficient to show the claim for the case $\alpha \equiv \mathit{refl}\ a$. In this case, $B\alpha \equiv id_{Ba}$, and so the second hypothesis becomes $f = g \cdot id$. Now $g \cdot id \equiv \lambda x.g\, x$. We have $g = \lambda x.g\, x$ (but not $g \equiv \lambda x.g\, x$). So we conclude $f = g$.

A result on the identity type of W types

Lemma

Let $a, a' : A$ and $f : Ba \to W_A B$ and $g : Ba' \to W_A B$.
If $\alpha : a = a'$ is such that $f = (g \cdot B\alpha)$ then $(\sup a\, f) = (\sup a'\, g)$.

Proof.

By induction on α it is sufficient to show the claim for the case $\alpha \equiv \mathit{refl}\ a$.
In this case, $B\alpha \equiv id_{Ba}$, and so the second hypothesis becomes $f = g \cdot id$.
Now $g \cdot id \equiv \lambda x.g\,x$. We have $g = \lambda x.g\,x$ (but not $g \equiv \lambda x.g\,x$). So we conclude $f = g$.
By induction, we can proove that if $f = g$, then $(\sup a\, f) = (\sup a'\, g)$.
Apply this to the above, and we get our desired conclusion. □

The univalence axiom

Definition

The axiom of extensionality states that for each $f, g : A \to B$, the obvious function

$$f = g \to f \simeq g$$

is an equivalence of types.

Definition

The axiom of univalence for a universe $U : Set$ with decoding family $T : U \to Set$, states that for each $a, b : U$, the obvious function

$$Ta = Tb \to Ta \cong Tb$$

is an equivalence of types.

$Id \leftrightarrow E_M$

Theorem

The univalence axiom implies that for any $m, m' : M$ we have that

$$m = m' \leftrightarrow E_M m m'$$

Proof.

($\rightarrow$) follows from reflexivity of E_M.
($\leftarrow$) By W-induction. Assume $a, b : U$ and $f : Ta \rightarrow M$ and $g : Tb \rightarrow M$. Then

$$
\begin{aligned}
E_M(\sup a\, f)(\sup b\, g) &:= \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} E_M(fx)(g(\alpha x)) && (1)\\
\text{Inducion hypotheis} \quad &\leftrightarrow \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} fx = g(\alpha x)) && (2)\\
\text{Definition of } \simeq \quad &\equiv \sum_{\alpha : Ta \cong Tb} f \simeq g \cdot \alpha) && (3)\\
\text{Extensionality} \quad &\cong \sum_{\alpha : Ta \cong Tb} f = g \cdot \alpha) && (4)\\
\text{Univalence} \quad &\cong \sum_{\alpha : a = b} f = g \cdot T\alpha && (5)\\
\text{Previous lemma} \quad &\leftrightarrow (\sup a\, f) = (\sup b\, g) && (6)
\end{aligned}
$$

Axiomatisation of multiset theory

Axiomatisation of multiset theory

Extensionality

$$\forall xy \; x = y \leftrightarrow \forall z \; (z \in x \cong z \in y)$$

Axiomatisation of multiset theory

Extensionality

$$\forall xy \; x = y \leftrightarrow \forall z \; (z \in x \cong z \in y)$$

Pairing

$$\forall xy \exists u \forall z \; z \in u \cong (z = x \vee z = y))$$

Axiomatisation of multiset theory

Extensionality

$$\forall xy\ x = y \leftrightarrow \forall z\ (z \in x \cong z \in y)$$

Pairing

$$\forall xy \exists u \forall z\ z \in u \cong (z = x \vee z = y))$$

Restricted separation

$$\forall x \exists u \forall z\ z \in u \cong (z \in x \wedge \phi(z))$$

Conclusion

This is work in progress, but the result on the identity type of M indicates that it is a good model of multisets in type theory. The current project is to give this more substance to this claim by giving an axiomatisation of iterative multiset theory.