Univalent multisets

V through the eyes of the identity type

Håkon Robbestad Gylterud

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Outline of the talk

1. Present common intuition about multisets

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3. A result about W-types
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5. Outline of current and future work

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- The technical parts are formalized in Agda.

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The first two points are applies to sets as well. The third point distinguishes the two notions.

Examples

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- Sequent calculus. $A, A \vdash \phi$
- Bags in computer science.

Related work

Blizzard (1989), develops a classical, two sorted, first order theory of multisets which, when restricted to sets, is equivalent to ZFC.

Elementhood in multisets

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- $\big(2 \in \mathrm{Roots}(x^3 - 2x^2 + x)\big) \cong \emptyset$.
- $(3 \in \{3, 3, 3, \cdots\}) \cong \mathbb{N}$.

Exensionality

In set theory

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The principle of extensionality for multisets

Two multisets x and y are considered equal iff for any z, the number of occurences of z in x and the number of occurences of z in y are in bijective correspondence (in our symbolism: $(z \in x) \cong (z \in y)$).

Classical vs Constructive

Classically, one can model a multiset as a set X, called the domain, and a function, $e : X \to \mathbb{N}$. Or if extended into the infinite, a function $e : X \to \mathrm{Card}$.

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A solution is to consider a multiset as a family. $m : X \to Set$, or $m : I \to X$.

Iterative multisets

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For iterative sets, we consider the totality V, consisting of sets where all elements of the sets, them selves are sets.
One may then wish for a totality M, consistsing of multisets of multisets, all with with domain M it self.

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It is well known that (wellfounded) trees can serve as models of (wellfounded) iterative sets.

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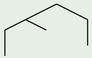
The iterative set $\{\{\{\emptyset\},\emptyset\},\{\emptyset\}\}$ is represented by



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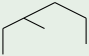

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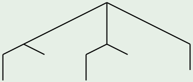

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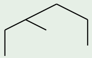

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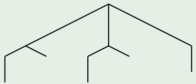

For iterative multisets, we want to keep these two distinct.

The W-type

Definition

Given a family $A : Set$, $B : A \to Set$, the set of all well founded trees with branchings in this family, denoted $W_{a:A} Ba$ is inductively generated by the rule:

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- For each $a : A$ and $f : Ba \to W_{a:A} Ba$, there is a unique element $(\sup a\, f) : W_{a:A} Ba$.

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$$V : Set$$
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$$=_V : V \to V \to Set$$
$$(\sup\ a\ f) =_V (\sup\ b\ g) :=$$
$$\prod_{i:Ta} \sum_{j:Tb} (f\ i) =_V (g\ j) \land \prod_{j:Tb} \sum_{i:Ta} (f\ i) =_V (g\ j)$$

Aczel's model of iterative sets in type theory

Lemma

$=_V$ *is equivalent to*

$$=_V' : V \to V \to Set$$
$$(\sup a\, f) =_V' (\sup b\, g) :=$$
$$\sum_{\alpha : Ta \to Tb} \prod_{x : Ta} (f\, x) =_V (g\, (\alpha\, x)) \wedge \sum_{\beta : Tb \to Ta} \prod_{y : Tb} (f\, (\beta y)) =_V' (g\, y)$$

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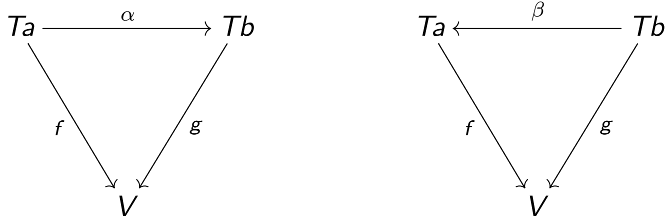
Proof.

W-induction on V and apply the (type theoretical) axiom of choice twice. □

Diagramatically, $(\sup a\, f)$ is equal, according to $=_V$, to $(\sup b\, g)$ if the diagrams



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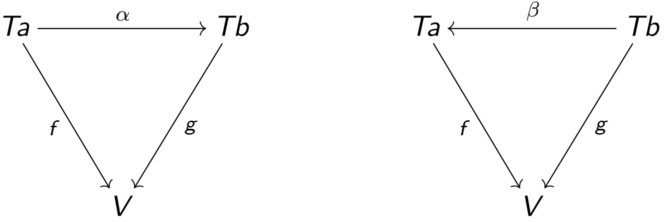

commutes up to $=_V$.
The natural change to make is to require that α and β form an equivalence of types.

The model

Definition

$$M : Set$$

$$M := W_{a:U}\, Ta$$

$$=_M : M \to M \to Set$$

$$(\sup a\, f) =_M (\sup b\, g) := \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} (f\, x) =_M (g\, (\alpha\, x))$$

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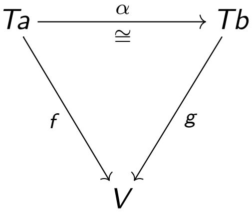

The model

Definition

Elementhood between multisets is defined by

$$\in : M \to M \to Set$$

$$x \in (\sup a\, f) := \sum_{i : Ta} (f\, a =_M x)$$

The identity type and Equivalence

In Martin-Löf type theory, every $A : Set$ is equipped with a type $Id_A : A \to A \to Set$, which is inductively generated by

- If $a : A$ then $(refl\ a) : Id_A\ a\ a$.

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This induces a notion of extensional equality on functions, and a notion of equivalence between types, which are essential in Homotopy Type Theory. If $A, B : Set$ we denote by $A \cong B$ the type of equivalences from A to B. And if $f, g : A \to B$, we denote by $f \simeq g$ the type of extensional equalities (homotopies) from f to g.

A result on the identity type of W types

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Lemma

For any $A : Set$ *and* $B : A \to Set$, *and all* $(\text{sup}\ a\, f), (\text{sup}\ b\, g) : W_A B$, *there is an equivalence*

$$Id_{W_A B}\,(\text{sup}\ a\, f)\,(\text{sup}\ b\, g) \cong \sum_{\alpha : Id_A\ a\ b} Id\ f\ (B\alpha \cdot g)$$

A result on the identity type of W types

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Proof.

There is a map going from left to right by induction on $Id_{W_A B}$. That is, for each $(\sup\ a\, f)$ the element $(refl_a, refl_f)$ works. Call this map ϕ.

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So assume that $p : \sum_{\alpha : Id_A\ a\ b} Id\ f\ (B\alpha \cdot g)$.
By induction (on the Σ-type and the two Id-types), it is enough to consider the case where $p \equiv (refl_a, refl_f)$.

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So assume that $p : \sum_{\alpha : Id_A\, a\, b} Id\ f\ (B\alpha \cdot g)$.
By induction (on the Σ-type and the two Id-types), it is enough to consider the case where $p \equiv (refl_a, refl_f)$.
We check that $\phi\, refl_{(\sup\ a\, f)} \equiv p$, by the above definiton. And by induction on Id, we can show that every element in the inverse image of p is equal to $refl_{(\sup\ a\, f)}$.

□

The univalence axiom

Definition

The axiom of extensionality states that for each $f, g : A \to B$, the obvious function

$$Id\, f\, g \to f \simeq g$$

is an equivalence of types.

Definition

The axiom of univalence for a universe $U : Set$ with decoding family $T : U \to Set$, states that for each $a, b : U$, the obvious function

$$Id\, a\, b \to Ta \cong Tb$$

is an equivalence of types.

Id is equivalent to $=_M$

Theorem

The univalence axiom implies that for any $m, m' : M$ *we have that*

$$Id\ m\ m' \cong m =_M m'$$

Proof.

By W-induction. Assume $a, b : U$ and $f : Ta \to M$ and $g : Tb \to M$. Then

$$
\begin{aligned}
(\sup a\, f) =_M (\sup b\, g) &\equiv \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} (fx) =_M (g(\alpha x)) \\
\text{Inducion hypotheis} \quad &\cong \sum_{\alpha : Ta \cong Tb} \prod_{x : Ta} Id\,(f\,x)\,(g(\alpha x)) \\
\text{Definition of } \simeq \quad &\equiv \sum_{\alpha : Ta \cong Tb} f \simeq g \cdot \alpha \\
\text{Extensionality} \quad &\cong \sum_{\alpha : Ta \cong Tb} Id\, f\, (g \cdot \alpha) \\
\text{Univalence} \quad &\cong \sum_{\alpha : a = b} Id\, f\, (g \cdot T\alpha) \\
\text{Previous lemma} \quad &\cong Id\,(\sup a\, f)\,(\sup b\, g)
\end{aligned}
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□

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Pairing

$$\forall xy \exists u \forall z\ z \in u \cong (z = x \lor z = y))$$

Restricted separation

$$\forall x \exists u \forall z\ z \in u \cong (z \in x \land \phi(z))$$

Conclusion

This is work in progress, but the result on the identity type of M indicates that it is a good model of multisets in type theory. The current project is to give this more substance to this claim by giving an axiomatisation of iterative multiset theory.