Symmetric Containers

Representing functors on groupoids

Håkon Robbestad Gylterud

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Context

- Based on the work by Thorsten Altenkirch and others on containers and differentiation of containers.
- Generalised notions of containers.

The groupoid interpretation of type theory

Martin Hofmann and Thomas Streicher (1996)

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- More generally depententendent types are modelled by families of groupoids.

A family of groupoids is a pair $(G, \mathcal{F})$, where $G : \mathit{Grpd}$ and $\mathcal{F} : G \to \mathit{Grpd}$ is a functor.

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  - The cyclic groups $\mathbb{Z}_n$ for $n \in \mathbb{N}_+$.
  - The permutation groups S_n for $n \in \mathbb{N}$.

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- We may take disjoint unions of groups. For instance
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  - $C : \mathbb{Z}_\bullet \to Grpd$, where each object $n : Ob(\mathbb{Z}_\bullet)$ is mapped to the set $Fin(n) = \{0, 1, \cdots, n-1\}$, and the arrows (i.e. the elements of the group $\mathbb{Z}_n$) act on $Fin(n)$ by additon.

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  - $P : S_\bullet \to Grpd$, where each object $n : Ob(S_\bullet)$ is mapped to the set $Fin(n) = \{0, 1, \cdots, n-1\}$, and the arrows (i.e. permutations) act on $Fin(n)$ by permutation.

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A 2-morphism of symmetric containers, $\epsilon : (f, \sigma) \to (g, \tau)$, is a natural transformation $\epsilon : f \Rightarrow g$ such that $\tau \circ Q\epsilon = \sigma$.

We assign the name $SCon$ to the 2-category of symmetric containers, their morphism and 2-morphisms.

Representing functors

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Let $(S \lhd P) : SCon$. Given a groupoid (or a set) X, we can construct a groupoid, $[\![S \lhd P]\!](X)$, of ways to select a shape $s \in S$, and fill the positions $P(s)$ with objects from X.

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$$Ob([\![S \lhd P]\!](X)) = \sum_{s \in S} X^{P(s)} \tag{1}$$

$$Mor_{[\![S \lhd P]\!](X)}((s, \phi), (t, \psi)) = \sum_{m : s \to t} (\phi \Rightarrow \psi P(m)) \tag{2}$$

Represenation theorem

Theorem

Let $Grpd^{Grpd}$ be the 2-category of 2-functors from Grpd to itself. There is a full and faithful strict 2-functor $[\![-]\!] : SCon \to Grpd^{Grpd}$.

Properties of symmetric containers

The 2-category of symmetric containers is closed under (and $[\![-]\!]$ respects):

- Finite products and co-products.
- Composition. That is, given $(S \lhd P)$ and $(T \lhd Q)$ there is a symmetric container $(S \lhd P) \circ (T \lhd Q)$ such that $[\![(S \lhd P) \circ (T \lhd Q)]\!] \cong [\![S \lhd P]\!] \circ [\![T \lhd Q]\!]$

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This structure allow to use techniques such as Taylor-series to analyse containers, and similar techniques can be used on symmetric containers.

Differentiation, definition

Definition

Given a symmetric container $(S \lhd P)$ where the sets of positions are all decidable, we may define the derivative symmetric container $\partial(S \lhd P) = (S' \lhd P')$ as follows:

$$Ob(S') = \sum_{s \in S} P(s)$$

$$Mor_{S'}((s, p), (t, q)) = \sum_{m : s \to t} P(m)(p) = q$$

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$$P(s,p) = P(s) - p$$
$$P(m) = P(m)|_p$$

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- $\partial(F \circ G) \cong (\partial F \circ G) \times \partial G$.

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- $\partial(F \times G) \cong \partial F \times G + F \times \partial G$.
- $\partial(F \circ G) \cong (\partial F \circ G) \times \partial G$.

Theorem

All containers with discrete set of shapes and finite, decidable sets of positions, have an anti-derivative. That is, if $F : SCon$ satisfy the hypothesis, then there is some G such that $\partial G \simeq F$.

Current project

- Formalise definitions in Agda using E-categories and E-bicategories.

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- Investigate fixed points of functors represented by symmetric containers.

Thank you all for listening!