Univalent multisets

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A multiset, like a set, consists of elements. But the notion of a multiset (see (Blizard 1988) for a survey of the litterature) is distinguished from that of a set by the fact that the number of occurences of an element in the multiset is relevant. The aim of this work is investigate the notion of iterative multisets, where multisets are iteratively built up from the empty multiset, in the context of Martin-Löf Type Theory with Voevodsky's Univalence Axiom (see for example (Awodey, Pelayo and Warren 2013) on the Univalence Axiom).

In type theory there is a model of iterative sets, constructed by (Aczel 1978). It is a setoid, where the underlying type is the W-type $V := W_{a:U} Ta$, where $U : Set$ is a universe with decoding family $T : U \to Set$. The equality is given by the U-small relation $Eq_V : V \to V \to U$, defined inductively by

$$Eq_V(\sup x\, f)(\sup y\, g) := \left(\prod_{a:Tx} \sum_{b:Ty} Eq_V(f\, a)(g\, b) \right) \wedge \left(\prod_{b:Ty} \sum_{a:Tx} Eq_V(f\, a)(g\, b) \right).$$

When building a model of iterative multisets in type theory, we consider the same underlying type, but to indicate our differing intension, we give it a new name $M := W_{a:U} Ta$. And we change the equality to the relation $Eq_M : M \to M \to U$, defined inductively by

$$Eq_M(\sup x\, f)(\sup y\, g) := \sum_{\alpha:Tx \cong Ty} \prod_{a:Tx} Eq_M(f\, a)(g\, (\alpha a))$$

where $\cong$ denotes equivalence of types. From the point of view of homotopy type theory, we show that this is the natural way to look at M in the way that Eq_M is equivalent to the identity type on M.

The elementhood relation $\in : M \to M \to U$ is then defined similarily to how it is done in Aczel's model, $x \in (\sup y\, g) := \sum_{a:Ty} Eq_M(g\, a)x$, but the proposition $x \in y$ is now considered to be the *set of occurences of x in y*. We then explore axiomatisations of multiset theory, and the interpretation of the axioms in the model.

References

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