SIGN-CHANGING AND SYMMETRY-BREAKING SOLUTIONS TO SINGULAR PROBLEMS

ANDRZEJ SZULKIN AND SHOYEB WALIULLAH

Abstract. We consider the degenerate elliptic equation

\[-\text{div}(|x|^{-a} |\nabla u|^{p-2} \nabla u) - \lambda |x|^{-p(a+1)} |u|^{p-2} u = |x|^{-b} |u|^{p-2} u \quad \text{in } \mathbb{R}^N\]

related to the Caffarelli-Kohn-Nirenberg inequality. We show that it possesses infinitely many solutions which are sign-changing and nonradial. The solutions are obtained by constrained minimization on subspaces consisting of functions which have certain prescribed symmetry properties. We also extend these results to higher order equations.

1. Introduction

The degenerate elliptic equation

\[(1.1) \quad -\text{div}(|x|^{-a} |\nabla u|^{p-2} \nabla u) - \lambda |x|^{-p(a+1)} |u|^{p-2} u = |x|^{-b} |u|^{p-2} u \quad \text{in } \mathbb{R}^N\]

has recently received considerable amount of attention, see e.g. [2, 3, 8, 9, 10, 11, 23, 26]. In these papers the authors have studied the existence and the nonexistence of ground state solutions and also conditions under which these solutions respectively are and are not radially symmetric. A somewhat different but related problem has been considered in [7].

Our aim here is to prove that there are infinitely many distinct solutions to (1.1). The solutions that we find are sign-changing and are not radially symmetric though they have certain prescribed symmetry properties. Note that existence of multiple positive solutions which are symmetric but nonradial have been shown in [9].

Let

\[(1.2) \quad S^1_\lambda (a, b) := \inf_{u \in D^1_\mu (\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-a} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{-p(a+1)} |u|^p \, dx}{(\int_{\mathbb{R}^N} |x|^{-b} |u|^q \, dx)^{p/q}},\]

where \(D^1_\mu (\mathbb{R}^N)\) is the closure of \(C^\infty_0 (\mathbb{R}^N)\) under the norm

\[\| |x|^{-a} \nabla u \|_p := \left( \int_{\mathbb{R}^N} |x|^{-a} |\nabla u|^p \, dx \right)^{1/p}.\]

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When
\[
1 < p < N, \quad 0 \leq a < \frac{N - p}{p}, \quad 0 \leq a \leq b \leq a + 1 \quad \text{and}
\]
(1.3)
\[
q = q(a, b, p) := \frac{Np}{N + p(b - a - 1)},
\]
the Caffarelli-Kohn-Nirenberg inequality
\[
\|x|^{-b}u\|_q \leq C\|x|^{-a}\nabla u\|_p
\]
for all \(u \in C_0^\infty(\mathbb{R}^N)\) (see [6]) implies that \(S_{1, p}^\lambda(a, b) > 0\) provided \(\lambda < S_{0, p}^1(a, a + 1)\). To show this, we observe that
\[
\int_{\mathbb{R}^N} |x|^{-a}\nabla u|_p^p dx - \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)}u|_p^p dx
\]
\[
= \frac{\int_{\mathbb{R}^N} |x|^{-a}\nabla u|_p^p dx}{(\int_{\mathbb{R}^N} |x|^{-b}u|_q^q dx)^{p/q}} \left( 1 - \frac{\lambda \int_{\mathbb{R}^N} |x|^{-(a+1)}u|_p^p dx}{\int_{\mathbb{R}^N} |x|^{-a}\nabla u|_p^p dx} \right).
\]
Since \(S_{0, p}^1(a, b) > 0\), we obtain
\[
S_{1, p}^\lambda(a, b) \geq S_{0, p}^1(a, b) \left( 1 - \frac{\lambda}{S_{0, p}^1(a, a + 1)} \right) > 0
\]
as claimed.

If the infimum in (1.2) is attained at some \(\bar{u}\), then \(\bar{u}\) satisfies (1.1) with the right-hand side replaced by \(\mu|x|^{-b}u|_q^{q-2}u\), where \(\mu\) is the Lagrange multiplier. Hence \(u = \alpha \bar{u}\) is a solution of (1.1) for a suitable \(\alpha > 0\).

It is known that when conditions (1.3) are satisfied, then ground state solutions exist if
\[
1) \quad a < b < a + 1 \quad \text{and} \quad \lambda < S_{0, p}^1(a, a + 1), \quad \text{or}
\]
\[
2) \quad a = b > 0 \quad \text{and} \quad 0 < \lambda < S_{0, p}^1(a, a + 1),
\]
see [2, 23, 26].

Note that we do not treat the case \(b = a + 1\). In fact it has been shown in [10] that \(S_{1, p}^1(a, a + 1)\) is never achieved. Our arguments below rely on Lemma 2.3 (the concentration-compactness principle) whose last conclusion may fail if \(b = a + 1\). Note also that the cases \(a < b\) and \(a = b > 0\) are quite different. In the first of them we have \(q < p^*\) while in the second one \(q = p^*\), where \(p^* := Np/(N - p)\) is the critical Sobolev exponent. Therefore the Rellich-Kondrachov theorem can be applied to functions having compact support if \(a < b\) but not if \(a = b\).

The paper is divided into three sections. In Section 2 we prove the existence of symmetry-breaking sign-changing solutions to (1.1) which have a prescribed number of nodal domains. By a nodal domain we mean here a largest open connected set on which a function \(u\) does not change sign (i.e., \(u \geq 0 \text{ or } u \leq 0\)). The crucial part of the existence proof relies on an extended version of the concentration-compactness principle due to the
Section 3 extends the results of Section 2 to higher order equations in the case $a < b$. We are not able to treat the case $a = b > 0$ in full generality; however, in Section 4 we consider the equation

\[(1.5) \quad \Delta(|x|^{-2a} \Delta u) - \lambda |x|^{-2(a+2)} u = |x|^{-aq} |u|^{q-2} u \quad \text{in} \quad \mathbb{R}^N,\]

where $q = 2^* := 2N/(N - 4)$ is the corresponding critical Sobolev exponent. Finally, in Appendix A we prove a version of the concentration-compactness lemma (Lemma 3.4). Although the argument is similar to that which may be found in [25], we include it for the reader’s convenience.

The authors would like to thank Thomas Bartsch for showing a simple proof of Lemma 2.1, and Carlos Kenig for suggesting an argument in the proof of Lemma 3.1.

2. Second order equations

Recall that a subset $U$ of a Banach space $E$ is called invariant with respect to an action of a group $G$ (or $G$-invariant) if $gU \subset U$ for all $g \in G$, and a functional $I : U \to \mathbb{R}$ is invariant (or $G$-invariant) if $I(gu) = I(u)$ for all $g \in G$, $u \in U$. The subspace $E^G := \{ u \in E : gu = u \text{ for all } g \in G \}$ is called the fixed point space of this action.

In this section we assume that conditions (1.3) are satisfied. Let $x = (y, z) = (y_1, y_2, z_1, \ldots, z_{N-2}) \in \mathbb{R}^N$ and let $O(2)$ be the group of orthogonal transformations acting on $\mathbb{R}^2$ by $(g, y) \mapsto gy$. For any positive integer $m$ we define $G_m$ to be the finite subgroup of $O(2)$ generated by the two elements $\alpha$ and $\beta$ in $O(2)$, where $\alpha$ is the rotation in the $y$-plane by the angle $2\pi/m$ and $\beta$ is the reflection in the line $y_1 = 0$ if $m = 2$, and in the line $y_2 = \tan(\pi/m)y_1$ for other $m$ (so in complex notation $w = y_1 + iy_2$, $\alpha w = we^{2\pi i/m}$ and $\beta w = \overline{w}e^{2\pi i/m}$). Since $\alpha^m = \beta^2 = e$ and $\alpha \beta \alpha = \beta$, $G_m = \{ \alpha^k \beta^l : k = 0, 1, \ldots, m-1, l = 0, 1 \}$. The group $G_m$ acts on $\mathbb{R}^N$ by $gx := (gy, z)$. Let $G_{m,x} := \{ gx : g \in G_m \}$ be the orbit of $x$ and denote the number of elements in $G_{m,x}$ by $|G_{m,x}|$. One easily sees that $|G_{m,x}| = 1$ if $w = 0$, $|G_{m,x}| = m$ if $w \neq 0$ and $w = |w|e^{\pi ji/m}$, $0 \leq j \leq 2m - 1$, and $|G_{m,x}| = 2m$ otherwise. Finally, let

\[(2.1) \quad V_m := \{ u \in D^{1,p}_a(\mathbb{R}^N) : u(gx) = \det(g)u(x) \text{ for all } g \in G_m \},\]

where $\det(g)$ is the determinant of $g$. Define the action of $G_m$ on $D^{1,p}_a(\mathbb{R}^N)$ by setting

\[(gu)(x) := \det(g)u(g^{-1}x).\]
Then $V_m = [D_{a}^{1,p}(\mathbb{R}^N)]^{G_m}$ (because $\det(g) = \det(g^{-1})$). Hence the principle of symmetric criticality [19] implies that the minimizers of

$$
S^{1,p}_{\lambda,m}(a,b) := \inf_{u \in V_m \setminus \{0\}} \frac{\int_{\mathbb{R}^N}||x|^{-a}\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N}||x|^{-(a+1)}u|^p \, dx}{(\int_{\mathbb{R}^N}||x|^{-b}|v|^q \, dx)^{p/q}}
$$

correspond to solutions of (1.1). Note that $\det(\alpha) = 1$, $\det(\beta) = -1$ and $\alpha x = \beta x$ if $x = (y,z)$ and $y = (y_1,0)$. It follows that each $u \in V_m$ is sign-changing and has at least $2m$ nodal domains (sign changes must occur at $x$ with $|G_{m,n}| = m$). The idea of introducing spaces like $V_m$ goes back to [13], where bounded domains with suitable symmetries were considered and $V_m$ was defined in terms of Fourier expansions rather than group actions. To unify notation, it will be convenient to set $V_0 := D_{a}^{1,p}(\mathbb{R}^N)$ and denote $S^{1,p}_{\lambda}(a,b)$ by $S^{1,p}_{\lambda,0}(a,b)$. In this way (1.2) is a special case of (2.2) corresponding to the action of the trivial subgroup $G_0$ consisting of the identity element in $O(2)$.

For the reader’s convenience we include here a simple proof of the principle of symmetric criticality which we owe to Thomas Bartsch. A simple proof for Hilbert spaces may be found in [27].

**Lemma 2.1.** Let $E$ be a Banach space and $G$ a compact Lie group acting on $E$ by isometries. If $U$ is an open $G$-invariant subset of $E$, $I \in C^1(U, \mathbb{R})$ is a $G$-invariant functional and $u \in E^G \cap U$ is a critical point of $I|_{E^G \cap U}$, then $u$ is a critical point of $I$.

**Proof.** Let $u$ be a critical point of $I|_{E^G \cap U}$. Since $I(u + tv) = I(gu + tgv) = I(u + tgv)$ ($t$ small and positive), one easily sees that $I'(u)v = I'(u)(gv)$ for each $g \in G$, $v \in E$. Observe that $\bar{v} := \int_G gv \in E^G$, hence $0 = I'(u)\bar{v} = \int_G I'(u)(gv) = I'(u)v$ for each $v \in E$. So $I'(u) = 0$. \qed

The next result, in a sense an extended version of the Rellich-Kondrachov theorem, has been proven in [26] for $p = 2$ and in [23] for general $p > 1$. In the next section we will prove a more general result.

**Lemma 2.2.** If $0 \leq a < (N-p)/p$ and $u_n \rightharpoonup u$ in $D_{a}^{1,p}(\mathbb{R}^N)$, then $|x|^{-a}u_n \rightharpoonup |x|^{-a}u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$.

We now state a generalization of the concentration-compactness lemma which is originally due to Lions [16, 17] and has been subsequently extended in [4, 5], see also [27]. The version below is essentially a special case of a result contained in [25]. It will be extended in Lemma 3.4 in the next section.

Denote by $\mathcal{M}(\mathbb{R}^N)$ the space of bounded measures on $\mathbb{R}^N$.

**Lemma 2.3** (Concentration-compactness). Assume that the conditions in (1.3) are satisfied and $\lambda < S^{1,p}_{0,0}(a,a+1)$. Let $\{u_n\}_{n=1}^{\infty} \subset V_m$ with $m \geq 0$ be
a sequence such that
\[ u_n \to u \quad \text{in } V_m \]
\[ ||x|^{-a}\nabla(u_n - u)|^p - \lambda||x|^{-a+1}(u_n - u)|^p \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{R}^N) \]
\[ ||x|^{-b}(u_n - u)|^q \rightharpoonup \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N) \]
\[ u_n \to u \quad \text{a.e. on } \mathbb{R}^N \]
and define
\[ \mu_\infty := \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-a}\nabla u_n|^p - \lambda||x|^{-a+1}u_n|^p \, dx, \tag{2.3} \]
\[ \nu_\infty := \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-b}u_n|^q \, dx. \]
Then
\[ \nu = \sum_{j \in J} \nu_j \delta_{x_j}, \tag{2.4} \]
\[ ||\nu||^{p/q} \leq S_{\lambda,m}^{1,p}(a,b)^{-1}||\mu_s||, \tag{2.5} \]
\[ \nu_{\mu_\infty}^{p/q} \leq S_{\lambda,m}^{1,p}(a,b)^{-1} \mu_\infty, \tag{2.6} \]
\[ \lim_{n \to \infty} |||x|^{-a}\nabla u_n||_p^p - \lambda|||x|^{-a+1}u_n||_p^p \geq |||x|^{-a}\nabla u||_p^p - \lambda|||x|^{-a+1}u||_p^p + ||\mu_s|| + \mu_\infty, \tag{2.7} \]
\[ \lim_{n \to \infty} |||x|^{-b}u_n||_q^q = |||x|^{-b}u||_q^q + ||\nu|| + \nu_\infty, \tag{2.8} \]
where \( \mu_s \) is the atomic part of \( \mu \). Further, if \( u = 0 \), then (2.7), with \( \mu_s \) replaced by \( \mu \), is an equality. If in addition \( b < a + 1 \) and \( ||\nu||^{p/q} = S_{\lambda,m}^{1,p}(a,b)^{-1}||\mu|| \), then \( \nu \) and \( \mu \) concentrate on a single orbit of \( G_m \) or are zero.

Before we present the main theorem of this section we make some simple observations. According to [23, 26], \( \lambda_{0,0}^{1,p}(a,a) < \lambda_{0,0}^{1,p}(0,0) \) if \( a > 0 \). Since \( p^* := Np/(N - p) = q(0,0) = q(a,a) \) is the critical Sobolev exponent and the Sobolev constant \( \lambda = \lambda_{0,0}^{1,p}(0,0) \) is attained (for \( U(x) = (1 + |x|^{p/(p-1)})^{-N-p}/p \)), it follows that there exists a non-negative radial function \( u \in C_0^\infty(\mathbb{R}^N) \) such that
\[ \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^p \, dx}{(\int_{\mathbb{R}^N} ||x|^{-a}u|^{p^*} \, dx)^{p/p^*}} < \lambda_{0,0}^{1,p}(0,0). \tag{2.9} \]
Let \( m \geq 1 \), denote the polar angle of \( y \) by \( \theta \) and choose \( \tilde{\epsilon} = (\tilde{y}, \tilde{z}) \in \mathbb{R}^N \) such that \( \text{supp } u(\cdot - \tilde{\epsilon}) \subset \{ x = (y, z) \in \mathbb{R}^N : 0 < \theta < \pi/\tilde{m} \} \). Then
\[ \nu(x) := \sum_{g \in G_m} \det(g)u(gx - \tilde{\epsilon}) \tag{2.10} \]
is in \( V_m \) and
\[ \lambda_{0,m}^{1,p}(a,a) \leq \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^p \, dx}{(\int_{\mathbb{R}^N} ||x|^{-a}u|^{p^*} \, dx)^{p/p^*}} \leq \frac{2m \int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^p \, dx}{(2m \int_{\mathbb{R}^N} ||x|^{-a}u|^{p^*} \, dx)^{p/p^*}}. \]
Hence by (2.9),

\[(2.11) \quad S_{\lambda,m}^{1,p}(a,a) \leq S_{0,m}^{1,p}(a,a) < (2m)^{1-p/p^*} S_{0,0}^{1,p}(0,0) \quad \text{if } \lambda \geq 0. \]

Suppose \(\{u_n\}_{n=1}^{\infty} \subset V_m\) is a minimizing sequence for (2.2) such that

\[(2.12) \quad \|x|^{-b}u_n\|_q = 1 \quad \text{and} \quad \|x|^{-a}\nabla u_n\|_p - \lambda \|x|^{-(a+1)}u_n\|_p \to S_{\lambda,m}^{1,p}(a,b)\]

as \(n \to \infty\). Going if necessary to a subsequence, still denoted by \(u_n\), we may assume that \(u_n \rightharpoonup u\) in \(V_m\), and so

\[
\|x|^{-a}\nabla u\|_p - \lambda \|x|^{-(a+1)}u\|_p \leq \lim_{n \to \infty} \left( \|x|^{-a}\nabla u_n\|_p - \lambda \|x|^{-(a+1)}u_n\|_p \right) = S_{\lambda,m}^{1,p}(a,b).
\]

Hence \(u\) is a minimizer provided \(\|x|^{-b}u\|_q = 1\). Denote

\[u'(x) := \frac{t^{N/q-b}}{u(tx)}.\]

**Theorem 2.4.** Assume that \(m \geq 1\), and (1.3) and one of the following conditions hold:

1. \(a < b < a + 1\) and \(\lambda < S_{0,0}^{1,p}(a,a + 1)\),
2. \(a = b > 0\) and \(0 < \lambda < S_{0,0}^{1,p}(a,a + 1)\).

Let \(\{u_n\} \subset V_m\) be a sequence satisfying (2.12). Then there exists a sequence \(\{t_n\} \subset (0,\infty)\) such that \(\{u_n^t\}\) contains a convergent subsequence. In particular, there exists a minimizer for \(S_{\lambda,m}^{1,p}(a,b)\).

**Proof.** For every \(n\) there exists \(t_n\) such that

\[\int_{B(0,t_n)} \|x|^{-b}u_n\|^q dx = \frac{1}{2}.\]

If \(v_n(x) := u_n^t(x)\), then \(\|x|^{-b}v_n\|_q = 1\), \(\|x|^{-a}\nabla v_n\|_p - \lambda \|x|^{-(a+1)}v_n\|_p \to S_{\lambda,m}^{1,p}(a,b)\) and

\[(2.13) \quad \int_{B(0,1)} \|x|^{-b}v_n\|^q dx = \frac{1}{2}.\]

Since \(\{v_n\}\) is bounded in \(V_m\), going if necessary to a subsequence, we may assume that

\[
v_n \rightharpoonup v \quad \text{in } V_m\]
\[
\|x|^{-a}\nabla (v_n - v)\|_p - \lambda \|x|^{-(a+1)}(v_n - v)\|_p \rightharpoonup \mu \quad \text{in } M(\mathbb{R}^N)\]
\[
\|x|^{-b}(v_n - v)\|_q \rightharpoonup \nu \quad \text{in } M(\mathbb{R}^N)\]
\[
v_n \to v \quad \text{a.e. on } \mathbb{R}^N.
\]

Lemma 2.3 asserts that

\[(2.14) \quad S_{\lambda,m}^{1,p}(a,b) \geq \|x|^{-a}\nabla v\|_p - \lambda \|x|^{-(a+1)}v\|_p + \|\mu\| + \mu_\infty\]

and

\[(2.15) \quad 1 = \|x|^{-b}v\|_q + \|\nu\| + \nu_\infty.\]
We obtain from (2.5), (2.6), (2.14) and the definition of $S^{1,p}_{\lambda,m}(a, b)$ that

$$S^{1,p}_{\lambda,m}(a, b) \geq S^{1,p}_{\lambda,m}(a, b) \left( \left\| |x|^{-b}v\right\|^q + \left\| v\right\|^p + \nu^{p/q}_\infty \right).$$

This together with (2.15) implies that only one of the numbers $\left\| |x|^{-b}v\right\|^q$, $\left\| v\right\|^p$ and $\nu^{p/q}_\infty$ is equal to 1 and the other two are zero. But by (2.13), $\nu_\infty \leq 1/2$, and so $\nu_\infty = 0$.

If $v = 0$, then $S^{1,p}_{\lambda,m}(a, b)\left|\nu\right|^p/q = S^{1,p}_{\lambda,m}(a, b) \geq \left\| \mu_x\right\| = \left\| \mu\right\|$ by (2.14). This and (2.5) imply that $\left|\nu\right|^p/q = S^{1,p}_{\lambda,m}(a, b)\left|\mu\right|$ and so $\nu, \mu$ concentrate on a single orbit, say $G_{m, \tilde{x}}$. By (2.13), $\tilde{x} \neq 0$. Since $G_{m, \tilde{x}}$ is finite, for each $r > 0$ we can construct a function $\xi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that the support of $\xi$ is a subset of $\bigcup_{p \in G_m} B(g\tilde{x}, r)$, $\xi$ is invariant with respect to the action of $G_m$ and $\xi(x) = 1$ for $x$ in a neighbourhood of $\tilde{x}$. We choose $r < \left|\tilde{x}\right|$ such that the sets $B(g\tilde{x}, r)$ and $B(h\tilde{x}, r)$ are disjoint if $g\tilde{x} \neq h\tilde{x}$.

Case $a < b$. Since $q < p^*$ and $0 \notin \text{supp} \xi$, the Rellich-Kondrachov theorem asserts that $\left| |x|^{-b}v\right|\xi \nu \to 0$ in $L^p(\mathbb{R}^N)$. Hence $\nu$ does not concentrate at $G_{m, \tilde{x}}$, so $\nu = 0$ and $\left\| |x|^{-b}v\right\|^p/q = 1$.

Case $a = b > 0$. Since $q = p^*$ here, the argument above does not apply. For any $\epsilon > 0$ we can choose $r$ so that $\left| |x|^{-a} - |g\tilde{x}|^{-a}\right| < \epsilon$ and $\left| |x|^{-a}r - |g\tilde{x}|^{-a}r\right| < \epsilon$ whenever $x \in B(g\tilde{x}, r)$. Since $|g\tilde{x}| = |\tilde{x}|$, $\left| |x|^{-(a+1)}\xi v\right| 0 \in L^p(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \left| |x|^{-a}\xi v\right|^p dx \to 1$, we obtain

\begin{equation}
S^{1,p}_{0,0}(0, 0)(2m)^{1-p/p^*} \left| \tilde{x}\right|^{-ap} - \epsilon \frac{\left| \tilde{x}\right|^{-ap} - \epsilon}{(\left| \tilde{x}\right|^{-ap} + \epsilon)p/p^*}
\leq \lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} \nabla (\xi v_n)^p dx}{(\int_{\mathbb{R}^N} |\xi v_n|^p dx)^{p/p^*}} \left( \frac{\left| \tilde{x}\right|^{-ap} - \epsilon}{(\left| \tilde{x}\right|^{-ap} + \epsilon)p/p^*}\right)
\leq \lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} \left| |x|^{-a}\nabla (\xi v_n)^p dx - \lambda \int_{\mathbb{R}^N} \left| |x|^{-(a+1)}\xi v_n\right|^p dx}{(\int_{\mathbb{R}^N} |\xi v_n|^p dx)^{p/p^*}} = S^{1,p}_{\lambda,m}(a, a).
\end{equation}

The first inequality above is clear for $\tilde{x}$ such that $|G_{m, \tilde{x}}| = 2m$. For other $\tilde{x}$, let $D_k := \bigcup_{g \in G_m} B(g\tilde{x}, r) \cap \{x = (y, z) : y \neq 0, (k - 1)\pi/m < \theta < k\pi/m\}$, $1 \leq k \leq 2m$ (as before, $\theta$ is the polar angle of $y$). Since $\xi v_n \in V_m$, $\xi v_n = 0$ on the boundary of $D_k$ (in the sense of traces). Hence

$$\frac{\int_{D_k} \nabla (\xi v_n)^p dx}{(\int_{D_k} |\xi v_n|^p dx)^{p/p^*}} \geq S \equiv S^{1,p}_{0,0}(0, 0),$$

so the first inequality in (2.16) is valid also in this case. Equality in the last line of (2.16) holds because

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |x|^{-a}\nabla (\xi v_n)^p dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \xi^p \left| |x|^{-a}\nabla v_n\right|^p dx.$$

By letting $\epsilon \to 0$ we conclude that $S^{1,p}_{0,0}(0, 0)(2m)^{1-p/p^*} \leq S^{1,p}_{\lambda,m}(a, a)$. This contradicts (2.11), so we deduce that $\left\| |x|^{-a}v\right\|^{p^*/p}_p = 1$. □
Remark 2.5. (i) Since $S_{1,0}^{1,p}(a,a) < S_{0,0}^{1,p}(0,0)$ if $a > 0$ and $\lambda \geq 0$, the argument above also applies to the case $m = 0$ (it becomes in fact simpler). Hence we recover some of the main results in [26] ($p = 2$) and [2, 23] ($p > 1$).

(ii) Since $S_{1,m}^{1,p}(a,a) < (2m)^{1-p/p^*} S_{0,0}^{1,p}(0,0)$ if $\lambda$ is negative and sufficiently close to 0, the conclusion of Theorem 2.4 remains valid for such $\lambda$. This is consistent with the results in [2, 23, 26].

Corollary 2.6. Assume that $m \geq 1$ and $a = b = 0$. Then the conclusion of Theorem 2.4 remains valid for $0 < \lambda < \left( \frac{N-p}{p} \right)^p$.

Proof. By the argument of case $a = b > 0$ in Theorem 2.4 we obtain

$$S_{1,0}^{1,p}(0,0)/(2m)^{1-p/p^*} \leq S_{\lambda,m}^{1,p}(0,0).$$

We claim that this is impossible. Indeed, $S = S_{0,0}^{1,p}(0,0)$ is attained, hence $S_{1,0}^{1,p}(0,0) < S_{0,0}^{1,p}(0,0)$. So there exists a positive radial function $u \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} |x|^{-1} u|^p dx < S_{0,0}^{1,p}(0,0).$$

Then, defining $v$ in the same way as in (2.10), we see as before that

$$S_{\lambda,m}^{1,p}(0,0) \leq \frac{(\int_{\mathbb{R}^N} |\nabla v|^p dx - \lambda \int_{\mathbb{R}^N} |x|^{-1} v|^p dx)}{(2m \int_{\mathbb{R}^N} |v|^{p^*} dx)^{p/p^*}} < (2m)^{1-p/p^*} S_{0,0}^{1,p}(0,0),$$

a contradiction to (2.17). \qed

We remark that the minimizers for $S_{1,0}^{1,2}(0,0)$ are explicitly known [14, 20].

Corollary 2.7. In all cases considered above there exists a minimizer for $S_{1,m}^{1,p}(a,b)$ ($m \geq 1$), and hence a solution to (1.1), which has exactly $2m$ nodal domains.

Proof. Let $u$ be a minimizer obtained in Theorem 2.4 or Corollary 2.6 and let $v(x) = |u(x)|$ for $x = (y, z)$ such that $0 < \theta < \pi/m$. Extend $v$ to $\mathbb{R}^N$ by using the action of $G_m$. Then the integrals in (2.2) are the same for $u$ and $v$. Hence $v$ is a minimizer for $S_{\lambda,m}^{1,p}(a,b)$ and obviously, it has exactly $2m$ nodal domains. \qed

Remark 2.8. Let $\Omega_0 := \{ x = (y, z) : y \neq 0 \text{ and } 0 < \theta < \pi/m \}$ and consider the space $D_{a,0}^{1,p}(\Omega_0)$ (the closure of $C_0^\infty(\Omega_0)$ under the norm $||x|^{-a} \nabla u||_p$).

A simple modification of our arguments shows that the quotient in (1.2), with $\lambda$ in an appropriate range and $\mathbb{R}^N$ replaced by $\Omega_0$, has a minimizer $v \in D_{a,0}^{1,p}(\Omega_0)$. Hence using the action of $G_m$, we obtain a solution $u \in D_{a,0}^{1,p}(\Omega)$, where $\Omega = \{ (y, z) : y \neq 0 \}$. Suppose $a = b = 0$. The subspace $\mathbb{R}^N \setminus \Omega$ has $p$-capacity 0 if and only if $1 < p \leq 2$, hence $D_{0,0}^{1,p}(\Omega) = D^{1,p}(\mathbb{R}^N)$ for $1 < p \leq 2$ (and $D_{0,0}^{1,p}(\Omega) \neq D^{1,p}(\mathbb{R}^N)$ for $p > 2$), see [12, Theorems 4.7.3.
and 4.7.4] and [18, Theorem 9.2.3]. So $u$ is a solution of (1.1) in all of $\mathbb{R}^N$ if $a = b = 0$ and $1 < p \leq 2$. This gives an alternative proof of Corollary 2.6 for such $a, b, p$. We do not know whether in general this $u$ satisfies (1.1) also on the set $y = 0$.

3. Higher order problems

Here we shall generalize the results of the previous section to higher order equations. For this purpose we need an inequality which generalizes that of Caffarelli, Kohn and Nirenberg and is due to Lin [15].

Let $k$ be a positive integer,

$$D^\alpha u := \frac{\partial|\alpha|u}{\partial x_1 \cdots \partial x_N}$$

and $k = |\alpha| = \sum_{i=1}^N \alpha_i$.

Similarly as in (1.3), let

$$1 < p < N/k, \quad 0 < a < \frac{N - kp}{p}, \quad 0 \leq a \leq b \leq a + k \quad \text{and}$$

$$q = q(a, b, p) := \frac{Np}{N + p(b - a - k)}.$$  

Then a special case of Lin’s inequality states that

$$\| |x|^{-b}u\|_q \leq C\| |x|^{-a}D^k u\|_p$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Here

$$\| |x|^{-a}D^k u\|_p := \max_{|\alpha|=k} \| |x|^{-a}D^\alpha u\|_p.$$  

Denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the above norm by $D_a^{k,p}(\mathbb{R}^N)$. Let

$$\nabla^k u := \begin{cases} (-\Delta)^{k/2} & \text{if } k \text{ is even} \\ \nabla(-\Delta)^{(k-1)/2} & \text{if } k \text{ is odd} \end{cases}$$

(cf. [17]).

**Lemma 3.1.** $\| |x|^{-a}\nabla^k u\|_p$ is an equivalent norm in $D_a^{k,p}(\mathbb{R}^N)$.

**Proof.** It is clear that $\| |x|^{-a}\nabla^k u\|_p \leq C_1\| |x|^{-a}D^k u\|_p$ for some $C_1 > 0$ and $\| |x|^{-a} \frac{\partial u}{\partial x_j}\|_p \leq \| |x|^{-a}\nabla u\|_p$. So the result will easily follow if we prove that $\| |x|^{-a} \frac{\partial^2 u}{\partial x_j \partial x_k}\|_p \leq C_2\| |x|^{-a}\Delta u\|_p$. Let $u \in C_0^\infty(\mathbb{R}^N)$ and let $R_j$, $1 \leq j \leq N$, denote the Riesz transforms, i.e., $R_j(u) := u * K_j$, where $K_j(x) = c_N x_j/|x|^{N+1}$, see e.g. §I.6.3 in [21]. Then $\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_j R_k(\Delta u)$ (since $\hat{K}_j(\xi) = -\xi_j/|\xi|$, this can be seen by taking Fourier transforms of both sides). As $|x|^{-ap}$ is in the class of $A_p$-weights if $a$ is as in (3.1), applying Corollary in §V.4.2 of [21], we obtain

$$\| |x|^{-a} \frac{\partial^2 u}{\partial x_j \partial x_k}\|_p = \| |x|^{-a}R_j R_k(\Delta u)\|_p \leq C_2\| |x|^{-a}\Delta u\|_p.$$
We consider the problem

\[ S_{\lambda,m}^{k,p}(a, b) := \inf_{u \in V_m} \frac{\int_{\mathbb{R}^N} |x|^{-a}\nabla^k u|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{-(a+k)} u|^p \, dx}{(\int_{\mathbb{R}^N} |x|^{-b} u|^q \, dx)^{p/q}}, \]

where \( V_m \) is as in (2.1), but with \( D_{a}^{1,p}(\mathbb{R}^N) \) replaced by \( D_{a}^{k,p}(\mathbb{R}^N) \). If \( \lambda < S_{0,0}^{k,p}(a, a + k) \), then \( S_{\lambda,m}^{k,p}(a, b) > 0 \) and once again, by the principle of symmetric criticality, the minimizers of (3.4) correspond to sign-changing solutions for the same equation as the minimizers on the whole space \( D_{a}^{k,p}(\mathbb{R}^N) \).

**Remark 3.2.** In (3.4) we have used the norm \( \| |x|^{-a}\nabla^k u|_p \). Although Theorem 3.5 below remains valid for any other equivalent differentiable norm satisfying \( \| u \| = \| u^t \| \) for all \( t > 0 \), where \( u^t(x) = t^{N/a-k}u(tx) \), the numerical value of the constant on the left-hand side of (3.4) and the equation that the minimizers satisfy do depend on the particular choice of such norm.

The following result will be needed in the proof of Lemma 3.4.

**Lemma 3.3.** If \( 0 \leq a < (N - kp)/p, |\alpha| < k \) and \( u_n \rightharpoonup u \in D_{a}^{k,p}(\mathbb{R}^N) \), then \( |x|^{-a} D^\alpha u_n \to |x|^{-a} D^\alpha u \) in \( L_{loc}^p(\mathbb{R}^N) \).

**Proof.** We may assume that \( u = 0 \). An application of (3.2) with \( k \) replaced by \( k - |\alpha| \), \( u \) replaced by \( D^\alpha u \) and \( b = a + k - |\alpha| \) gives

\[
\int_{B(0,r)} |x|^{-a} D^\alpha u_n|^p \, dx \leq s^{(k-|\alpha|)p} \int_{B(0,r)} |x|^{-(a+k-|\alpha|)} D^\alpha u_n|^p \, dx
\leq s^{(k-|\alpha|)p} C_1 \int_{\mathbb{R}^N} |x|^{-a} D^k u_n|^p \, dx \leq C_2 s^{(k-|\alpha|)p},
\]

where the constants \( C_1, C_2 \) are independent of \( n \). Hence for any \( \epsilon > 0 \) there exists \( r > 0 \) such that

\[
\int_{B(0,r)} |x|^{-a} D^\alpha u_n|^p \, dx < \epsilon.
\]

Since \( |x|^{-a} D^\alpha u_n \to 0 \) in \( L_{loc}^p(\mathbb{R}^N \setminus B(0, r)) \) by the Rellich-Kondrachov theorem, the conclusion follows. \( \square \)

**Lemma 3.4 (Concentration-compactness).** Assume that the conditions in (3.1) are satisfied and \( \lambda < S_{0,0}^{k,p}(a, a + k) \). Let \( \{u_n\}_{n=1}^\infty \subset V_m \) with \( m \geq 0 \) be a sequence such that

\[
\begin{align*}
&u_n \rightharpoonup u \quad \text{in} \ V_m, \\
&|x|^{-a}\nabla^k (u_n - u)|^p - \lambda |x|^{-(a+k)}(u_n - u)|^p \rightharpoonup \mu \quad \text{in} \ \mathcal{M}(\mathbb{R}^N), \\
&|x|^{-b}(u_n - u)|^q \rightharpoonup \nu \quad \text{in} \ \mathcal{M}(\mathbb{R}^N), \\
&u_n \to u \quad \text{a.e. on} \ \mathbb{R}^N.
\end{align*}
\]
and define
\[
\mu_\infty := \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-a} \nabla^k u_n|^p + \lambda ||x|^{-(a+k)} u_n|^p \, dx
\]
(3.5)
\[
\nu_\infty := \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-b} u_n|^q \, dx.
\]
Then
\[
\nu = \sum_{j \in J} \nu_j \delta_{x_j},
\]
(3.6)
\[
||\nu||^{p/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1} ||\mu_s||,
\]
(3.7)
\[
\nu^{p/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1} \mu_\infty,
\]
(3.8)
\[
\lim_{n \to \infty} |||x|^{-a} \nabla^k u_n||^p_p - \lambda |||x|^{-(a+k)} u_n||^p_p \geq \|\| |x|^{-a} \nabla^k u||^p_p - \lambda \|\| |x|^{-(a+k)} u||^p_p + ||\mu_s|| + \mu_\infty,
\]
(3.9)
\[
\lim_{n \to \infty} \|\| |x|^{-b} u_n||^q_q = \|\| |x|^{-b} u||^q_q + \|\nu\| + \nu_\infty,
\]
(3.10)
where \(\mu_s\) is the atomic part of \(\mu\). Further, if \(u = 0\), then (3.9), with \(\mu_s\) replaced by \(\mu\), is an equality. If in addition \(b < a + k\) and \(\|\nu\|^{p/q} = S_{\lambda,m}^{k,p}(a,b)^{-1} ||\mu||\), then \(\nu\) and \(\mu\) concentrate on a single orbit of \(G_m\) or are zero.

As we have already mentioned in the introduction, the proof of the above lemma is similar to that in [25], but we include it in Appendix A for the reader’s convenience. We also remark that the result remains valid for any action of a closed subgroup \(G\) of \(O(N)\), with \(V_m\) replaced by \([D_{a,p}^k(\mathbb{R}^N)]^G\) and \(S_{\lambda,m}^{k,p}(a,b)\) replaced by the infimum of the quotient in (3.4), taken over \([D_{a,p}^k(\mathbb{R}^N)]^G \setminus \{0\}\).

As before, let \(u'(x) = t^{N/q-b}u(tx)\).

**Theorem 3.5.** Assume that \(k \geq 2\), \(m \geq 0\) and (3.1) holds. Assume also that \(a < b < a + k\) and \(\lambda < S_{0,0}^{k,p}(a,a+k)\). Let \(\{u_n\} \subset V_m\) be a minimizing sequence for \(S_{\lambda,m}^{k,p}(a,b)\) such that \(\|\| |x|^{-b} u_n||^q_q = 1\). Then there exists a sequence \(\{t_n\} \subset (0,\infty)\) such that \(\{u_{n,t_n}\}\) contains a convergent subsequence. In particular, there exists a minimizer for \(S_{\lambda,m}^{k,p}(a,b)\).

The proof is similar to that of Theorem 2.4, case \(a < b\). If \(m = 0\), cf. Remark 2.5.

If \(a = b > 0\), then we are not able to show that an inequality corresponding to (2.11) holds for general \(k \geq 2\) and \(p > 1\). However, we can do this in the particular case \(k = p = 2\), see the next section.

**Remark 3.6.** Since for \(k \geq 2\) we cannot replace \(u\) by \(|u|\) as we did in the proof of Corollary 2.7, we can only infer here that the minimizers in \(V_m\) \((m \geq 1)\) have at least \(2m\) nodal domains.
4. A FOURTH ORDER EQUATION

As we have mentioned, the results of the preceding section do not take into account the case when \( a = b > 0 \) and \( \lambda \geq 0 \). Here we consider the problem of finding minimizers for \( a = b > 0 \) and \( k = p = 2 \). Note that (3.1) now becomes

\[
(4.1) \quad N > 4, \quad 0 < a = b < \frac{N - 4}{2}, \quad q = 2^* := \frac{2N}{N - 4}.
\]

Let

\[
(4.2) \quad S^{2,2}_{\lambda,m}(a, a) := \inf_{u \in V_m, u \neq 0} \frac{\int_{\mathbb{R}^N} |x|^{-a} \Delta u^2 \, dx - \lambda \int_{\mathbb{R}^N} |x|^{-2(a+2)} u^2 \, dx}{(\int_{\mathbb{R}^N} |x|^{-a} u^2 \, dx)^{2/2^*}}.
\]

The minimizers satisfy the following fourth order equation:

\[
(4.3) \quad \Delta(|x|^{-2a} \Delta u) - \lambda |x|^{-2(a+2)} u = |x|^{-2a} |u|^{2^*-2} u \quad \text{in} \ \mathbb{R}^N.
\]

We shall need an inequality corresponding to (2.11). The essential step is given in the following lemma.

**Lemma 4.1.** Under the conditions in (4.1), \( S^{2,2}_{0,0}(a, a) < S^{2,2}_{0,0}(0, 0) \).

**Proof.** We use the same general ideas as in [26]. It is known [22] that \( u(x) = (1 + |x|^2)^{-N/2} \) is a minimizer for \( S = S^{2,2}_{0,0}(0, 0) \). We will prove the lemma by showing that the function

\[
g(a) := \frac{\int_{\mathbb{R}^N} |x|^{-a} \Delta u^2 \, dx}{(\int_{\mathbb{R}^N} |x|^{-a} u^2 \, dx)^{2/2^*}}, \quad 0 \leq a < \frac{N - 4}{2},
\]

is strictly decreasing. The beta and gamma functions satisfy the following relations, see [1]:

\[
(4.4) \quad \int_0^\infty (1 + t)^{-(y+z)} t^{y-1} \, dt = B(y, z) = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)} \quad \text{and} \quad \Gamma(y+1) = y\Gamma(y).
\]

We note that

\[
g(a) = \frac{\int_{\mathbb{R}^N} (N - 4)^2 |x|^{-2a} (1 + |x|^2)^{-N} (N^2 + 4N|x|^2 + 4|x|^4) \, dx}{(\int_{\mathbb{R}^N} |x|^{-a} u^2 \, dx)^{2/2^*}}.
\]

Going over to polar coordinates gives

\[
g(a) = (N - 4)^2 \omega^{1-2/2^*} \alpha(a) \beta(a) \gamma(a),
\]

where \( \omega \) is the volume of \( S^{N-1} \),

\[
\alpha(a) := N^2 \frac{\int_0^\infty (1 + r^2)^{-N} r^{-2a+N-1} \, dr}{(\int_0^\infty (1 + r^2)^{-N} r^{-2a+N-1} \, dr)^{2/2^*}},
\]

\[
\beta(a) := 4N \frac{\int_0^\infty (1 + y^2)^{-N} y^{-2a+1+N} \, dy}{(\int_0^\infty (1 + r^2)^{-N} r^{-2a+N-1} \, dr)^{2/2^*}}.
\]
and

\[ \gamma(a) := 4 \frac{\int_0^\infty (1 + r^2)^{-Nr-2a+3+N} \, dr}{\left( \int_0^\infty (1 + r^2)^{-Nr-2a+N-1} \, dr \right)^{2/2^*}}. \]

Let

\[ s(a) := \Gamma\left( \frac{N}{2} + a \right) \Gamma\left( \frac{N}{2} + \frac{a^*}{2} \right)^{-2/2^*}, \]

\[ t(a) := \Gamma\left( \frac{N}{2} - a \right) \Gamma\left( \frac{N}{2} - \frac{a^*}{2} \right)^{-2/2^*}. \]

Substituting \( t = r^2 \) in (4.4), we see that

\[ \int_0^\infty (1 + r^2)^{-Nr-2a+N-1} \, dr = \frac{\Gamma\left( \frac{N}{2} + a \right) \Gamma\left( \frac{N}{2} - a \right)}{2\Gamma(N)} \]

and

\[ \int_0^\infty (1 + r^2)^{-Nr-2^*a+N-1} \, dr = \frac{\Gamma\left( \frac{N}{2} + 2^*a \right) \Gamma\left( \frac{N}{2} - 2^*a \right)}{2\Gamma(N)}. \]

Hence

\[ \alpha(a) = \frac{N^2}{(2\Gamma(N))^{1-2/2^*}} s(a)t(a) \]

and by the second formula in (4.4),

\[ \Gamma\left( \frac{N}{2} + a - 1 \right) \Gamma\left( \frac{N}{2} - a + 1 \right) = \frac{N - 2a}{N + 2a - 2} \Gamma\left( \frac{N}{2} + a \right) \Gamma\left( \frac{N}{2} - a \right). \]

So

\[ \beta(a) = \frac{4N}{(2\Gamma(N))^{1-2/2^*}} \frac{N - 2a}{N + 2a - 2} s(a)t(a), \]

and similarly,

\[ \gamma(a) = \frac{4}{(2\Gamma(N))^{1-2/2^*}} \frac{N - 2a + 2}{N + 2a - 4} \frac{N - 2a}{N + 2a - 2} s(a)t(a). \]

Since the functions \( \frac{N-2a+2}{N+2a-4} \) and \( \frac{N-2a}{N+2a-2} \) are strictly decreasing, the proof will be complete if we can show that \( s'(a) < 0 \) and \( t'(a) < 0 \). A simple calculation gives

\[ s'(a) = s(a) \begin{bmatrix} \Gamma'(\frac{N}{2} + a) & -\frac{\Gamma'(\frac{N}{2} + a^*)}{\Gamma(\frac{N}{2} + a^*)} \\ \Gamma(\frac{N}{2} + a) & \Gamma(\frac{N}{2} + a^*) \end{bmatrix} \]

\[ t'(a) = t(a) \begin{bmatrix} \Gamma'(\frac{N}{2} - a^*) & \frac{\Gamma'(\frac{N}{2} - a)}{\Gamma(\frac{N}{2} - a^*)} \\ \Gamma(\frac{N}{2} - a) & \Gamma(\frac{N}{2} - a^*) \end{bmatrix}. \]

Since \( \log \Gamma(z) \) is strictly convex in \((0, \infty)\) (see [1]), we obtain the conclusion.
Theorem 4.2. Assume that \( m \geq 0 \), the conditions (4.1) are satisfied, \( a = b > 0 \) and \( 0 \leq \lambda < S_{a,a+2}^2 \). Let \( \{u_n\} \subset V_m \) be a minimizing sequence for \( S_{\lambda,m}^{a,a} \) such that \( \|\cdot - a\|_{a+2} = 1 \). Then there exists a sequence \( \{t_n\} \subset (0,\infty) \) such that \( \{u_{t_n}^n\} \) contains a convergent subsequence. In particular, there exists a minimizer for \( S_{\lambda,m}^{a,a} \).

Proof. Since \( S_{a,0}^{2,2}(a,0) \) is attained, by the same argument as before (cf. (2.10) and (2.11)), we obtain
\[
S_{\lambda,m}^{a,a} \leq S_{a,0}^{2,2}(a,a) < 2^{1/2} S_{a,0}^{2,2}(0,0) \quad \text{if} \quad \lambda \geq 0 \quad \text{and} \quad m \geq 1.
\]
Hence we may follow the arguments in the proof of case \( a = b > 0 \) of Theorem 2.4 keeping in mind that \( \xi v_n \in H^2(D_k) \cap H^1_0(D_k) \) and, by [24], the best Sobolev constant in this space is \( S_{a,a} \). We also note as in Remark 2.5 that since \( S_{\lambda,0}^{2,2}(a,a) < S_{a,0}^{2,2}(0,0) \), the conclusion remains valid for \( m = 0 \). \( \square \)

Appendix A

In this appendix we prove Lemma 3.4. Let \( G \) be a closed subgroup of the group \( O(N) \) of orthogonal transformations of \( \mathbb{R}^N \) and \( C_0^\infty_0(\mathbb{R}^N) \) the set of functions in \( C_0^\infty(\mathbb{R}^N) \) which are invariant with respect to an action of \( G \). We note that for every \( \epsilon > 0 \) there exists a constant \( C(\epsilon,p) > 0 \) such that
\[
|A + B|^p - |A|^p \leq \epsilon |A|^p + C(\epsilon,p)|B|^p \quad \forall A, B \in \mathbb{R}^d
\]
(we will use this inequality with \( d = 1 \) and \( N \)).

We shall need the following lemma, due to Lions (see [17]), which is a type of reverse Hölder’s inequality. The important fact here is that the measure \( \nu \) below is purely atomic. A slightly more concise proof of this result can be found in [25].

Lemma A.1. Let \( \mu, \nu \) be two bounded nonnegative measures on \( \mathbb{R}^N \) satisfying for some constant \( C \geq 0 \)
\[
(\int_{\mathbb{R}^N} |\phi|^q d\nu)^{1/q} \leq C \left( \int_{\mathbb{R}^N} |\phi|^p d\mu \right)^{1/p} \quad \forall \phi \in C_0^\infty(\mathbb{R}^N),
\]
where \( 1 \leq p < q < \infty \), and let \( \mu_s \) be the atomic part of \( \mu \). Then there exists an at most countable set \( (x_j)_{j \in J} \) of distinct points in \( \mathbb{R}^N \) and a set of numbers \( (\nu_j)_{j \in J} \) in \( (0,\infty) \) such that
\[
\nu = \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu_s \geq C^{-\rho} \sum_{j \in J} \nu_j^{\rho/q} \delta_{x_j}.
\]

Proof of Lemma 3.4. Although our argument works for any action of a closed subgroup \( G \subset O(N) \) (with the changes indicated earlier), we assume here that \( G = G_m \).
1) Assume that \( u = 0 \) and let \( \xi \in C_0^\infty(\mathbb{R}^N) \). (3.4) with \( m = 0 \) implies
\[
\left( \int_{\mathbb{R}^N} |x|^{-b} \xi u_n|^q \, dx \right)^{p/q} \leq S_{\lambda,0}^{k,p}(a,b)^{-1} \int_{\mathbb{R}^N} |x|^{-a} \nabla^k(\xi u_n)|^p - \lambda |x|^{-(a+k)} \xi u_n|^p \, dx.
\]
An application of Lemma 3.3 while taking the limits gives
\[
(A.3) \quad \left( \int_{\mathbb{R}^N} |\xi|^q \, d\nu \right)^{p/q} \leq S_{\lambda,0}^{k,p}(a,b)^{-1} \int_{\mathbb{R}^N} |\xi|^p \, d\mu,
\]
where \( \mu, \nu \geq 0 \). We can now assert (3.6) via Lemma A.1. To obtain (3.7) we repeat the argument above with \( \xi \in C_0^\infty(\mathbb{R}^N) \) and \( S_{\lambda,0}^{k,p}(a,b) \) replaced by \( S_{\lambda,m}^{k,p}(a,b) \). In order to obtain \( \mu_s \) and not \( \mu \) on the right-hand side of (3.7) we note that by (3.6) it suffices to take \( \xi \) having support of arbitrarily small measure.

2) For \( R > 1 \), let \( \psi_R \in C^\infty(\mathbb{R}^N) \) be a radially symmetric function such that \( \psi_R(x) = 1 \) for \( |x| > R + 1 \), \( \psi_R(x) = 0 \) for \( |x| < R \) and \( 0 \leq \psi_R \leq 1 \) on \( \mathbb{R}^N \). We then obtain
\[
\left( \int_{\mathbb{R}^N} |x|^{-b} \psi_R u_n|^q \, dx \right)^{p/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1} \int_{\mathbb{R}^N} |x|^{-a} \nabla^k(\psi_R u_n)|^p - \lambda |x|^{-(a+k)} \psi_R u_n|^p \, dx.
\]
Lemma 3.3 implies that
\[
(A.4) \quad \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |x|^{-b} \psi_R u_n|^q \, dx \right)^{p/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1} \lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-a} \psi_R \nabla^k u_n|^p - \lambda |x|^{-(a+k)} \psi_R u_n|^p \, dx.
\]
Since
\[
\int_{|x| > R+1} |x|^{-a} \nabla^k u_n|^p \, dx \leq \int_{\mathbb{R}^N} |x|^{-a} \psi_R \nabla^k u_n|^p \, dx 
\leq \int_{|x| > R} |x|^{-a} \nabla^k u_n|^p \, dx,
\]
\[
\int_{|x| > R+1} |x|^{-(a+k)} u_n|^p \, dx \leq \int_{\mathbb{R}^N} |x|^{-(a+k)} \psi_R u_n|^p \, dx
\leq \int_{|x| > R} |x|^{-(a+k)} u_n|^p \, dx
\]
and \( \int_{|x| < R+1} |x|^{-(a+k)} u_n|^p \, dx \to 0 \), it follows that
\[
\mu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-a} \psi_R \nabla^k u_n|^p - \lambda |x|^{-(a+k)} \psi_R u_n|^p \, dx.
\]
Similarly,
\[
\int_{|x| > R+1} |x|^{-b} u_n|^q \, dx \leq \int_{\mathbb{R}^N} |x|^{-b} \psi_R u_n|^q \, dx \leq \int_{|x| > R} |x|^{-b} u_n|^q \, dx
\]
implies
\[
\nu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-b} \psi_R u_n|^q \, dx.
\]
Inequality (3.8) now follows from (A.4).

3) Further assume that \(\|\nu\|^{p/q} = S_{\lambda,m}^{k,p}(a,b)^{-1}\|\mu\| \neq 0\) and \(b < a + k\). Since \(p < q\), from Hölder’s inequality we have, for \(\xi \in C^\infty_{0,G_m}(\mathbb{R}^N)\),

\[
\left(\int |\xi|^p d\mu\right)^{1/p} \leq \|\mu\|^{1/p-1/q} \left(\int |\xi|^q d\mu\right)^{1/q}.
\]

Combining this with (A.5)

\[
\left(\int_{\mathbb{R}^N} |\xi|^q d\nu\right)^{p/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1} \int_{\mathbb{R}^N} |\xi|^p d\mu
\]

gives

\[
\left(\int |\xi|^q d\nu\right)^{1/q} \leq S_{\lambda,m}^{k,p}(a,b)^{-1}\|\mu\|^{1/p-1/q} \left(\int |\xi|^q d\mu\right)^{1/q}.
\]

The above inequality implies \(\nu \leq S_{\lambda,m}^{k,p}(a,b)^{-q/p}\|\mu\|^{q/p-1}\mu\). This and the equality \(\|\nu\|^{p/q} = S_{\lambda,m}^{k,p}(a,b)^{-1}\|\mu\|\) give

\[
\nu = S_{\lambda,m}^{k,p}(a,b)^{-q/p}\|\mu\|^{q/p-1}\mu\quad\text{and}\quad \mu = S_{\lambda,m}^{k,p}(a,b)\|\nu\|^{p/q-1}\nu.
\]

So for \(\xi \in C^\infty_{0,G_m}(\mathbb{R}^N)\) we have from (A.5)

\[
\left(\int |\xi|^q d\nu\right)^{p/q} \leq \int |\xi|^p \|\nu\|^{p/q-1} d\nu.
\]

Hence for each open \(G_m\)-invariant set \(\Omega \subset \mathbb{R}^N\),

\[
\left(\frac{\nu(\Omega)}{\nu(\mathbb{R}^N)}\right)^{p/q} \leq \frac{\nu(\Omega)}{\nu(\mathbb{R}^N)}.
\]

Since \(p < q\), it follows that either \(\nu(\Omega) = 0\) or \(\nu(\mathbb{R}^N) = \nu(\Omega)\). Therefore \(\nu\) and \(\mu\) are concentrated on a single orbit.

4) In the general case we set \(v_n := u - u_n\). Since \(v_n \to 0\) in \(D^k_{a,p}(\mathbb{R}^N)\), equation (3.7) follows from part 1) of the proof.

5) For any \(\epsilon > 0\), set \(A = |x|^{-a}\nabla^k u_n\) and \(B = -|x|^{-a}\nabla^k u\) in inequality (A.1) to obtain

\[
|||x|^{-a}\nabla^k u_n|^p - |||x|^{-a}\nabla^k u_n|^p| \leq \epsilon |||x|^{-a}\nabla^k u_n|^p + C(\epsilon,p)|||x|^{-a}\nabla^k u|^p|.
\]

It follows that

\[
\int_{|x| > R} \left(|||x|^{-a}\nabla^k v_n|^p - |||x|^{-a}\nabla^k u_n|^p|\right) dx \\
\leq \epsilon \int_{|x| > R} |||x|^{-a}\nabla^k u_n|^p| dx + C(\epsilon,p) \int_{|x| > R} |||x|^{-a}\nabla^k u|^p| dx.
\]

Similarly,

\[
\int_{|x| > R} \left(|||x|^{-(a+k)} v_n|^p - |||x|^{-(a+k)} u_n|^p|\right) dx \\
\leq \epsilon \int_{|x| > R} |||x|^{-(a+k)} u_n|^p| dx + C(\epsilon,p) \int_{|x| > R} |||x|^{-(a+k)} u|^p| dx.
\]
Since $\epsilon$ is arbitrary, by letting $n \to \infty$ and $R \to \infty$, we conclude that
\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p \, dx = \mu_\infty.
\]

Since the Brézis-Lieb lemma [27] gives
\[
\lim_{n \to \infty} \int_{|x| > R} ||x|^{-b} v_n|^q \, dx = \lim_{n \to \infty} \int_{|x| > R} ||x|^{-b} u_n|^q \, dx - \int_{|x| > R} ||x|^{-b} u|^q \, dx,
\]
we conclude that
\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} ||x|^{-b} v_n|^q = \nu_\infty.
\]

Now inequality (3.8) follows from part 2) of the proof.

6) There exists a finite measure $\bar{\mu}$ such that passing to a subsequence,
\[
||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p \overset{\ast}{\rightharpoonup} \bar{\mu}
\]
in $\mathcal{M}(\mathbb{R}^N)$. Let $\phi_\eta \in C_0^\infty(B(x_j, \eta))$, $0 \leq \phi_\eta \leq 1$ and $\phi_\eta(x_j) = 1$, where $x_j$ is an atom of $\mu$. An application of inequality (A.1) shows that
\[
|\mu(\phi_\eta) - \bar{\mu}(\phi_\eta)|
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} \phi_\eta \left( ||x|^{-a} \nabla^k v_n|^p - \lambda ||x|^{-(a+k)} v_n|^p \right. \\
- \left. (||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p) \right) \, dx
\leq \epsilon \lim_{n \to \infty} \int_{\mathbb{R}^N} \phi_\eta (||x|^{-a} \nabla^k u_n|^p + |\lambda| ||x|^{-(a+k)} u_n|^p) \, dx
+ C \int_{\mathbb{R}^N} \phi_\eta (||x|^{-a} \nabla^k u|^p + |\lambda| ||x|^{-(a+k)} u|^p) \, dx.
\]

Letting $\eta \to 0$ we have
\[
|\bar{\mu}(\{x_j\}) - \mu_s(\{x_j\})| \leq \epsilon C_1.
\]

From the fact that $\epsilon$ is arbitrary and $C_1$ is independent of $\epsilon$, we see that the atomic part of $\bar{\mu}$ is equal to $\mu_s$. Since $|x|^{-(a+k)} u_n \to |x|^{-(a+k)} u$ in $L^p(\mathbb{R}^N)$ and $\xi |x|^{-a} \nabla^k u_n \to \xi |x|^{-a} \nabla^k u$ in $L^p(\mathbb{R}^N)$ for all positive $\xi \in C_0^\infty(\mathbb{R}^N)$, we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \xi (||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p) \, dx
\geq \int_{\mathbb{R}^N} \xi (||x|^{-a} \nabla^k u|^p - \lambda ||x|^{-(a+k)} u|^p) \, dx.
\]

Now, $||x|^{-a} \nabla^k u|^p - \lambda ||x|^{-(a+k)} u|^p$ seen as a measure is relatively singular to the Dirac measures $\delta_{x_j}$, and it follows that
\[
(A.6) \quad ||\bar{\mu}|| \geq |||x|^{-a} \nabla^k u||^p - \lambda ||x|^{-(a+k)} u||^p + ||\mu_s||.
\]

For $R > 1$ we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} ||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p \, dx
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \psi_R (||x|^{-a} \nabla^k u_n|^p - \lambda ||x|^{-(a+k)} u_n|^p) \, dx + \int_{\mathbb{R}^N} (1 - \psi_R) \, d\bar{\mu}.
\]
As $R \to \infty$, by Lebesgue’s dominated convergence theorem and (A.6), we obtain

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-a} \nabla^k u_n|^p - \lambda |x|^{-(a+k)} u_n|^p \, dx = \mu_\infty + \|\tilde{\mu}\| \geq \mu_\infty + \|x|^{-a} \nabla^k u\|_p^p - \lambda \|x|^{-(a+k)} u\|_p^p + \|\mu_s\|.
$$

This gives (3.9). By a similar (but simpler) argument we obtain (3.10).

If $u = 0$, then by definition, $\tilde{\mu} = \mu$, hence

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-a} \nabla^k u_n|^p - \lambda |x|^{-(a+k)} u_n|^p \, dx = \mu_\infty + \|\mu\|
$$

and (3.9) with $\mu_s$ replaced by $\mu$ is an equality.

\[\square\]

**References**


DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, 106 91 STOCKHOLM, SWEDEN

E-mail address: andrzejs@math.su.se

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, 106 91 STOCKHOLM, SWEDEN

E-mail address: shoyeb@math.su.se