The method of Nehari manifold

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Abstract. We present a unified approach to the method of Nehari manifold for functionals which have a local minimum at 0 and we give several examples where this method is applied to the problem of finding ground states and multiple solutions for nonlinear elliptic boundary value problems. We also consider a recent generalization of this method to problems where 0 is a saddle point of the functional.

Contents

Chapter 1. Introduction 1

Chapter 2. Preliminaries 5
- 2.1. Critical point theory 5
- 2.2. Differential equations and boundary value problems 6

Chapter 3. Nehari manifold 9
- 3.1. Abstract setting 9
- 3.2. Elliptic equations in bounded domains 14
- 3.3. Elliptic problems in $\mathbb{R}^N$ 17
- 3.4. A multiplicity result for a singularly perturbed problem with domain topology 21
- 3.5. Newtonian systems 23

Chapter 4. Generalized Nehari manifold 25
- 4.1. Abstract setting 25
- 4.2. Elliptic equations – the indefinite case 28

Bibliography 35

CHAPTER 1

Introduction

In the two papers [**33**] and [**34**] Nehari has introduced a method which turned out to be very useful in critical point theory and eventually came to bear his name. He considered a boundary value problem for a certain nonlinear second order ordinary differential equation in an interval (a, b) and showed that it has a nontrivial solution which may be obtained by constrained minimization. In [**34**] he also considered existence of solutions with a prescribed number of nodes in (a, b).

To describe Nehari's method in an abstract setting, let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ a functional. The Fréchet derivative of Φ at u, $\Phi'(u)$, is an element of the dual space E^*, and we shall denote $\Phi'(u)$ evaluated at $v \in E$ by $\Phi'(u)v$. Suppose $u \neq 0$ is a critical point of Φ, i.e., $\Phi'(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} := \{u \in E \setminus \{0\} : \Phi'(u)u = 0\}. \tag{1}$$

So $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial (i.e., $\neq 0$) critical points of Φ. $\mathcal{N}$ is called the *Nehari manifold* though in general it may not be a manifold. Set

$$c := \inf_{u \in \mathcal{N}} \Phi(u). \tag{2}$$

Under appropriate conditions on Φ one hopes that c is attained at some $u_0 \in \mathcal{N}$ and that u_0 is a critical point.

Assume without loss of generality that $\Phi(0) = 0$. Assume that for each $w \in S_1(0) := \{w \in E : \|w\| = 1\}$ the function $\alpha_w(s) := \Phi(sw)$ attains a unique maximum s_w in $(0, \infty)$ such that $\alpha_w'(s) > 0$ whenever $0 < s < s_w$, $\alpha_w'(s) < 0$ whenever $s > s_w$ and $s_w \geq \delta$ for some $\delta > 0$ independent of $w \in S_1(0)$. Then $\alpha_w'(s_w) = \Phi'(s_w w)w = 0$. Hence $s_w w$ is the unique point on the ray $s \mapsto sw$, $s > 0$, which intersects $\mathcal{N}$. Moreover, $\mathcal{N}$ is bounded away from 0. It is easy to see that $\mathcal{N}$ is closed in E and there exists a radial bijection between $\mathcal{N}$ and $S_1(0)$. In Section 3.1 we shall see that if s_w is bounded on compact subsets of $S_1(0)$, then this bijection is in fact a homeomorphism. Clearly, c in (2), if attained, is positive. We shall also show in Section 3.1 that $u_0 \in \mathcal{N}$ is a critical point whenever $\Phi(u_0) = c$. Note that since $s \mapsto \alpha_w(s)$ is increasing for all $w \in S_1(0)$ and $0 < s < \delta$, 0 is a local minimum and hence a critical point of Φ. Since u_0 is a solution to the equation $\Phi'(u) = 0$ which has minimal "energy" Φ in the set of all nontrivial solutions, we shall call it a *ground state*.

Suppose in addition to the assumptions already made that E is a Hilbert space and $\Phi \in C^2(E, \mathbb{R})$. Then

$$\alpha_w''(s_w) = \Phi''(s_w w)(w, w) = s_w^{-2}\Phi''(u)(u, u) \leq 0, \text{ where } u = s_w w \in \mathcal{N}.$$

If $\Phi''(u)(u,u) < 0$ for all $u \in \mathcal{N}$, then, setting $G(u) := \Phi'(u)u$, we see that

$$G'(u)u = \Phi''(u)(u,u) + \Phi'(u)u = \Phi''(u)(u,u) < 0, \quad u \in \mathcal{N}.$$

Since $\mathcal{N} = \{u \in E \setminus \{0\} : G(u) = 0\}$, it follows from the implicit function theorem that $\mathcal{N}$ is a C^1-manifold of codimension 1 and $E = T_u(\mathcal{N}) \oplus \mathbb{R}u$ for each $u \in \mathcal{N}$. Hence in this case it is easily seen that any $u \in \mathcal{N}$ with $\Phi(u) = c$ (i.e., any minimizer of $\Phi|_{\mathcal{N}}$) satisfies $\Phi'(u) = 0$. More generally, a point $u \in E$ is a nonzero critical point of Φ if and only if $u \in \mathcal{N}$ and u is critical for the restriction of Φ to $\mathcal{N}$. In view of this property, one may apply critical point theory on the manifold $\mathcal{N}$ in order to find critical points of Φ.

Our goal in this survey is to present a unified approach to the method of Nehari manifold and to illustrate it with a number of examples where it can be applied in order to show the existence of solutions to nonlinear boundary value problems. Our approach is a little different from the usual and is taken from [**44**]. In particular, we do not need to make customary assumptions which imply that $\Phi \in C^2(E, \mathbb{R})$ and $\Phi''(u)(u,u) < 0$ on $\mathcal{N}$. Therefore our results will be somewhat more general than those which may be found in the literature. We shall also treat so-called indefinite problems in which 0 is a saddle point rather than a local minimum of Φ. Then $\mathcal{N}$ need neither be closed (its closure may contain the origin) nor does it need to intersect all rays $s \mapsto sw$, $s > 0$, $w \in S_1(0)$. In the applications we shall consider it turns out that in this case $c = 0$ and is not attained. In order to circumvent this difficulty we shall replace $\mathcal{N}$ by the generalized Nehari manifold $\mathcal{M}$ which has been introduced by Pankov in [**35**]. Ground states will be obtained by minimizing Φ over $\mathcal{M}$. Also here we follow the approach in [**44**].

The literature on the method of Nehari manifold is rather extensive. It would be impossible to cover all different aspects of this method or to provide a reasonably complete bibliography. Therefore we focus on a few topics which we think are representative and are in line with our interests and research experience. A particular topic which we leave out in order to keep this survey reasonably short is the so-called fibering method which has been introduced by Pohozaev. See [**12, 17, 24**] and the references there.

Although we assume the reader is somewhat familiar with critical point theory and its applications to nonlinear boundary value problems, we summarize some pertinent results in Chapter 2. More material may be found e.g. in [**3, 23, 32, 37, 42, 45, 46**].

The central part of this survey is Chapter 3 where we consider elliptic boundary value problems in bounded domains and in $\mathbb{R}^N$. We make assumptions which imply that the functionals Φ corresponding to these problems have a local minimum at 0 and show that $\inf_{\mathcal{N}} \Phi$ is attained and hence there exists a ground state. We also discuss the existence of sign-changing solutions, and of infinitely many solutions for even Φ.

In Chapter 4 we consider indefinite elliptic problems and obtain solutions by minimizing Φ over $\mathcal{M}$. More precisely, we have an orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$ of a Hilbert space E and the functionals which correspond to our problems are of the form $\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u)$, where $u^\pm \in E^\pm$. Then the generalized Nehari manifold is defined by

$$\mathcal{M} := \{u \in E \setminus (E^0 \oplus E^-) : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^0 \oplus E^-\}.$$

So all critical points $u \notin E^0 \oplus E^-$ must be in $\mathcal{M}$ and one wants to show that $c := \inf_{\mathcal{M}} \Phi$ is positive, is attained and if $u_0 \in \mathcal{M}$, $\Phi(u_0) = c$, then u_0 is a critical point of Φ. Under some additional assumptions there are no nontrivial critical points in $E^0 \oplus E^-$, hence the minimizers on $\mathcal{M}$ are ground states.

We would like to mention that we have included a few results which seem to be new: Theorem 19 where existence of solutions for an equation involving the p-Laplacian is shown and Theorems 41, 42 where existence of a ground state is shown for an indefinite system of elliptic equations. The novelty is that in both cases we admit a nonlinearity satisfying a weak superlinearity condition at infinity instead of more common conditions of Ambrosetti-Rabinowitz type.

Notation. $B_\rho(p)$ and $S_\rho(p)$ will respectively denote the open ball and the sphere centered at p and having radius ρ. We also write S as a shorthand for $S_1(0)$. The sublevel set of c will be denoted by Φ^c, i.e.,

$$\Phi^c := \{u \in E : \Phi(u) \leq c\}.$$

The symbol "$\rightharpoonup$" will stand for weak convergence in E.

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CHAPTER 2

Preliminaries

Below we summarize some facts from critical point theory and briefly describe a variational framework for problems we shall consider.

2.1. Critical point theory

Let E be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ a functional. A point $u \in E$ is called *critical* if $\Phi'(u) = 0$. The corresponding value $c = \Phi(u)$ is a *critical value* or a *critical level*. A sequence $(u_n) \subset E$ is called a *Palais-Smale sequence* if $(\Phi(u_n))$ is bounded and $\Phi'(u_n) \to 0$. If $\Phi(u_n) \to c \in \mathbb{R}$ and $\Phi'(u_n) \to 0$, then (u_n) is a $(PS)_c$*-sequence.* The functional Φ is said to satisfy the *Palais-Smale condition* (or $(PS)_c$*-condition*) if each Palais-Smale sequence (or $(PS)_c$-sequence) has a convergent subsequence. It is clear that if a (subsequence of) a Palais-Smale sequence converges to u, then u is a critical point. The above definitions can be easily carried over to C^1-submanifolds of E (or more generally, to Finsler manifolds). The only difference is that Φ' must be considered as an element of the cotangent bundle of the manifold M, i.e., $\Phi'(u) \in T_u(M)^*$, the dual of the tangent space to M at u.

In order to formulate our results we shall need the notion of genus. Let A be a closed subset of $E \setminus \{0\}$ such that $A = -A$. The *genus* of A, denoted $\gamma(A)$, is the smallest integer k such that there exists an odd mapping $h \in C(A, \mathbb{R}^k \setminus \{0\})$. We also set $\gamma(\emptyset) := 0$ and $\gamma(A) := \infty$ if no h exists for any finite k. It can be shown that if A is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd homeomorphism, then $\gamma(A) = k$. This implies that the unit sphere S in an infinite-dimensional Banach space E contains compact sets of any genus $k \geq 1$ and since $\gamma(A) \leq \gamma(B)$ whenever $A \subset B$, it follows that $\gamma(S) = \infty$. Properties of genus may be found e.g. in [**3, 37, 42**].

Let E be a Banach space such that the unit sphere S in E is a submanifold of class (at least) C^1 and let $\Phi \in C^1(S, \mathbb{R})$.

THEOREM 1. *If Φ is bounded below and satisfies the Palais-Smale condition, then $c := \inf_S \Phi$ is attained and is a critical value of Φ.*

PROOF. Let $(u_n) \subset S$ be a minimizing sequence for Φ. By Ekeland's variational principle [**25, 32, 45**] we may assume $\Phi'(u_n) \to 0$. So (u_n) is a Palais-Smale sequence and therefore $u_n \to u$ after passing to a subsequence. Hence $\Phi(u) = c$ and $\Phi'(u) = 0$. $\square$

Let

$$\Gamma_j := \{A \subset S : A = -A,\ A \text{ is compact and } \gamma(A) \geq j\}$$

and

$$c_j := \inf_{A \in \Gamma_j} \sup_{u \in A} \Phi(u), \quad j = 1, 2, \ldots .$$

If E is infinite-dimensional, then $\gamma(S) = \infty$ and $\Gamma_j \neq \emptyset$ for any j. If Φ is bounded below, then we have $c_1 \le c_2 \le \dots$ and $c_j < \infty$ for all j because the sets $A \in \Gamma_j$ are compact. The following holds:

THEOREM 2. *If E is infinite-dimensional, $\Phi \in C^1(S, \mathbb{R})$ is bounded below and satisfies the Palais-Smale condition, then Φ has infinitely many pairs of critical points.*

This result may be found in [**37, 42**] for S respectively in a Hilbert space and in a Banach space such that S is of class $C^{1,1}$, and in [**43**] for S in a Banach space such that $S \in C^1$. (For $S \in C^1$ Theorem 2 can also be easily deduced from [**19**]. One can in fact remove the compactness requirement in the definition of Γ_j; this requirement was essential for the argument in [**43**].)

We do not include the proof here but only point out that one shows all c_j are critical levels and if $c_j = \dots = c_{j+p}$ for some $p \ge 0$, then $\gamma(K_{c_j}) \ge p+1$, where

$$K_{c_j} := \{u \in S : \Phi(u) = c_j \text{ and } \Phi'(u) = 0\}.$$

Hence the number of critical points is infinite regardless of whether the number of distinct c_j's is finite or not.

2.2. Differential equations and boundary value problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Here we assume that λ is a real constant and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies the growth restriction

$$|f(x,u)| \le a(1 + |u|^{q-1}) \quad \text{for some } a > 0 \text{ and } 2 < q < 2^*, \tag{4}$$

where $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := \infty$ if $N = 1$ or 2. Recall that if $N \ge 3$, then 2^* is the critical exponent with respect to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$. Let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|$ given by

$$\|u\|^2 := \int_\Omega |\nabla u|^2 \, dx.$$

By Poincaré's inequality, on $H_0^1(\Omega)$ $\|\,.\,\|$ is equivalent to the usual $H^1(\Omega)$-norm. Let

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2}\int_\Omega \lambda u^2 \, dx - \int_\Omega F(x,u) \, dx \equiv \frac{1}{2}\|u\|^2 - I_1(u) - I(u),$$

where

$$F(x,u) := \int_0^u f(x,s) \, ds.$$

By a solution of (3) we shall always mean a weak solution $u \in H_0^1(\Omega)$.

THEOREM 3. *Suppose that f is continuous and satisfies (4). Then:*
(i) $\Phi \in C^1(H_0^1(\Omega), \mathbb{R})$ and $\Phi'(u) = 0$ if and only if $u \in H_0^1(\Omega)$ is a solution of (3).
(ii) The functionals I and I_1 are weakly continuous, i.e., if $u_n \rightharpoonup u$, then $I(u_n) \to I(u)$ and $I_1(u_n) \to I_1(u)$.
(iii) The operators I' and I_1' are completely continuous (or weak-to-strong continuous), i.e., if $u_n \rightharpoonup u$, then $I'(u_n) \to I'(u)$ and $I_1'(u_n) \to I_1'(u)$.
(iv) If $f(x,u) = o(u)$ uniformly in x as $u \to 0$, then $I'(u) = o(\|u\|)$ and $I(u) =$

$o(\|u\|^2)$ as $u \to 0$ in $H_0^1(\Omega)$.
(v) If $N = 1$, then f need not satisfy any growth restriction.

The proofs of (different parts of) the above result may be found in [**3, 23, 32, 37, 42, 45, 46**]. Here we only point out that weak continuity of I, I_1 and complete continuity of I', I_1' follow from the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \le r < 2^*$ (the Rellich-Kondrachov theorem) and the fact that the operator $u \mapsto f(x,u)$ is continuous from $L^q(\Omega)$ to $L^{q/(q-1)}(\Omega)$ (Krasnoselskii's theorem). (iv) is a consequence of the fact that $f(x,u) = o(u)$ implies that for each $\varepsilon > 0$ there is C_ε such that

$$|f(x,u)| \le \varepsilon |u| + C_\varepsilon |u|^{q-1}$$

and (v) follows by the compactness of the embedding $H_0^1(\Omega) \hookrightarrow C(\overline{\Omega})$.

Remark 4. Consider the Newtonian system of differential equations

$$-\ddot{q} + q = W_q(q,t), \quad q \in \mathbb{R}^N, \ t \in \mathbb{R},$$

where V given by $V(q,t) := W(q,t) - \frac{1}{2}|q|^2$ is the Newtonian potential. If W and W_q are continuous and 2π-periodic in t, then the conclusions of Theorem 3 remain valid in the space $H^1(S^1, \mathbb{R}^N)$ consisting of 2π-periodic functions with the norm given by

$$\|q\|^2 := \int_0^{2\pi} (|\dot{q}|^2 + |q|^2)\, dt,$$

see e.g. [**32**]. In particular, no growth restriction on W is necessary. The functional here is

$$\Phi(q) := \frac{1}{2}\|q\|^2 - I(q), \quad \text{where} \quad I(q) := \int_0^{2\pi} W(q,t)\, dt.$$

Let $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p \in (1,\infty)$. Δ_p is called the p-Laplacian and the problem corresponding to (3) is

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Now the function f needs to satisfy the growth restriction (4) with 2^* replaced by $p^* := Np/(N-p)$ if $N > p$ and $p^* := \infty$ otherwise. Let

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{1}{p}\int_\Omega \lambda |u|^p\, dx - \int_\Omega F(x,u)\, dx \equiv \frac{1}{p}\|u\|^p - I_1(u) - I(u),$$

$u \in W_0^{1,p}(\Omega)$.

Theorem 5. *Suppose that f is continuous and satisfies (4) with 2^* replaced by p^* and 2 by p. Then:*
(i) $\Phi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and $\Phi'(u) = 0$ if and only if $u \in W_0^{1,p}(\Omega)$ is a solution of (5).
(ii) The functionals I and I_1 are weakly continuous.
(iii) The operators I' and I_1' are completely continuous.
(iv) If $f(x,u) = o(|u|^{p-1})$ uniformly in x as $u \to 0$, then $I'(u) = o(\|u\|^{p-1})$ and $I(u) = o(\|u\|^p)$ as $u \to 0$ in $W_0^{1,p}(\Omega)$.
(v) If $N < p$, then f need not satisfy any growth restriction.

The proofs of the above statements are very similar to the corresponding ones in Theorem 3 and are outlined in [**23**]. A comprehensive treatment of problems involving the p-Laplacian in a bounded domain may be found in [**22**].

Finally we consider the problem

$$\left\{ \begin{array}{ll} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0, & |x| \to \infty, \end{array} \right. \tag{6}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies the growth restriction

$$|f(x,u)| \le a(|u| + |u|^{q-1}) \quad \text{for some } a > 0 \text{ and } 2 < q < 2^*. \tag{7}$$

This time we work in $H^1(\mathbb{R}^N)$, with the usual norm given by

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx,$$

and the functional is

$$\Phi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx \equiv \langle Lu, u\rangle - I(u).$$

Here $L : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is bounded linear and $\langle .\,, .\rangle$ is the inner product in $H^1(\mathbb{R}^N)$.

THEOREM 6. *Suppose that V, f are continuous, V is bounded and f satisfies (7). Then:*
(i) $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $\Phi'(u) = 0$ if and only if $u \in H^1(\mathbb{R}^N)$ is a solution of (6).
(ii) If $f(x,u) = o(u)$ uniformly in x as $u \to 0$, then $I'(u) = o(\|u\|)$ and $I(u) = o(\|u\|^2)$ as $u \to 0$ in $H^1(\mathbb{R}^N)$.

A proof that Φ is of class C^1 and $\Phi'(u) = 0$ if and only if $u \in H^1(\mathbb{R}^N)$ is a solution of (6) may be found in [**21**] or [**45**]. The fact that such solution must necessarily tend to 0 as $|x| \to \infty$ is discussed more in detail in [**18**] and [**36**] (in particular, if 0 is not in the essential spectrum of $-\Delta + V$, then $u(x) \to 0$ exponentially, see [**36**]). We note that in unbounded domains it is necessary to replace (4) by the stronger condition (7); however, the assumption that $f(x,u) = o(u)$ uniformly in x as $u \to 0$ and (4) imply (7). The proof of (ii) is exactly the same as that of (iv) in Theorem 3. Note that now one cannot expect I to be weakly continuous or I' to be completely continuous because the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is not compact for any r.

Finally, we recall a result concerning the spectrum of $-\Delta + V$ for periodic V.

PROPOSITION 7. *Supose that $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic with respect to $x_1, \dots, x_N$. Then the spectrum $\sigma(-\Delta + V)$ in $L^2(\mathbb{R}^N)$ is absolutely continuous, bounded below but not above and consists of disjoint closed intervals.*

A proof of this well-known fact may be found e.g. in [**27, 40**]. Note that we do not exclude the possibility that the spectrum consists of only one interval, i.e., $\sigma(-\Delta + V) = [a, \infty)$ for some $a \in \mathbb{R}$. Intervals (α, β) such that $\alpha, \beta \in \sigma(-\Delta + V)$ and $(\alpha, \beta) \cap \sigma(-\Delta + V) = \emptyset$ are called *spectral gaps.*

CHAPTER 3

Nehari manifold

3.1. Abstract setting

In what follows we always assume that E is a uniformly convex real Banach space, $\Phi \in C^1(E,\mathbb{R})$ and $\Phi(0) = 0$. Recall that $S := S_1(0)$. A function $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ is said to be a *normalization function* if $\varphi(0) = 0$, φ is strictly increasing and $\varphi(t) \to \infty$ as $t \to \infty$. We shall need the following further assumptions:

(A_1) There exists a normalization function φ such that
$$u \mapsto \psi(u) := \int_0^{\|u\|} \varphi(t)\,dt \in C^1(E \setminus \{0\},\mathbb{R}),$$
$J := \psi'$ is bounded on bounded sets and $J(w)w = 1$ for all $w \in S$.

(A_2) For each $w \in E \setminus \{0\}$ there exists s_w such that if $\alpha_w(s) := \Phi(sw)$, then $\alpha_w'(s) > 0$ for $0 < s < s_w$ and $\alpha_w'(s) < 0$ for $s > s_w$.

(A_3) There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and for each compact subset $\mathcal{W} \subset S$ there exists a constant $C_{\mathcal{W}}$ such that $s_w \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

J in (A_1) is called the duality mapping corresponding to φ, see [**22**] for a detailed discussion of this notion. An obvious sufficient condition for (A_1) to hold is that $\|\cdot\| \in C^1(E \setminus \{0\},\mathbb{R})$ (then one can take $\varphi(t) = t$). Here we will be mainly interested in two cases: E a Hilbert space and E the Sobolev space $W_0^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^N$ bounded and $p > 1$. In the first case we take $\varphi(t) := t$ - then J is the usual duality mapping between E and E^*. In the second case we put $\varphi(t) := t^{p-1}$. The associated functional ψ is given by $\psi(u) = \frac{1}{p}\|u\|^p$ and the duality mapping

$$J = \psi' : E \to E^*, \qquad J(w)v = \int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v\,dx \tag{8}$$

is continuous and bounded on bounded sets, see [**22**] or Section 7.5A in [**23**]. It follows from (A_1) that S is a C^1-submanifold of E and the tangent space of S at w is
$$T_w(S) = \{z \in E : J(w)z = 0\}.$$

Recall that
$$\mathcal{N} := \{u \in E \setminus \{0\} : \Phi'(u)u = 0\}.$$
As has been shown in the introduction, (A_2) implies that $sw \in \mathcal{N}$ if and only if $s = s_w$. Moreover, by the first part of (A_3), $\mathcal{N}$ is closed in E and bounded away from 0. Define the mappings $\widehat{m} : E \setminus \{0\} \to \mathcal{N}$ and $m : S \to \mathcal{N}$ by setting
$$\widehat{m}(w) := s_w w \quad \text{and} \quad m := \widehat{m}|_S. \tag{9}$$

Proposition 8. *Suppose Φ satisfies (A_2) and (A_3). Then:*
(a) The mapping $\widehat{m}$ is continuous.

(b) The mapping m is a homeomorphism between S and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = u/\|u\|$.

Proof. (a) Suppose $w_n \to w \neq 0$. Since $\widehat{m}(tw) = \widehat{m}(w)$ for each $t > 0$, we may assume $w_n \in S$ for all n and it suffices to show that $\widehat{m}(w_n) \to \widehat{m}(w)$ after passing to a subsequence. Write $\widehat{m}(w_n) = s_n w_n$. By (A_2) and (A_3), (s_n) is bounded and bounded away from 0, hence, taking a subsequence, $s_n \to \overline{s} > 0$. Since $\mathcal{N}$ is closed and $\widehat{m}(w_n) \to \overline{s}w$, $\overline{s}w \in \mathcal{N}$. Hence $\overline{s}w = s_w w = \widehat{m}(w)$.

(b) This is an immediate consequence of (a). □

We shall consider the functionals $\widehat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ defined by

$$\widehat{\Psi}(w) := \Phi(\widehat{m}(w)) \quad \text{and} \quad \Psi := \widehat{\Psi}|_S. \tag{10}$$

Although we do not claim that $\mathcal{N}$ is a C^1-manifold, we shall show that $\widehat{\Psi}$ is of class C^1 and there is a one-to-one correspondence between critical points of Ψ and nontrivial critical points of Φ. In the context of a standard saddle point reduction with respect to subspaces, a similar observation has been known for a long time, see e.g. [**2, 14**]. Somewhat surprisingly, the variant below has not been used – up to our knowledge – before [**44**]. However, see [**6, 7**] where the reduction to a C^1-functional $\Psi : S \to \mathbb{R}$ has been made under stronger smoothness assumptions.

Proposition 9. *Suppose E is a Banach space satisfying (A_1). If Φ satisfies (A_2) and (A_3), then $\widehat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ and*

$$\widehat{\Psi}'(w)z = \frac{\|\widehat{m}(w)\|}{\|w\|}\Phi'(\widehat{m}(w))z \quad \textit{for all } w, z \in E,\ w \neq 0.$$

Proof. Let $w \in E \setminus \{0\}$ and $z \in E$. Using the maximality property of $s_w w$ and the mean value theorem, we obtain

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &= \Phi(s_{w+tz}(w+tz)) - \Phi(s_w w) \\ &\leq \Phi(s_{w+tz}(w+tz)) - \Phi(s_{w+tz}w) \\ &= \Phi'(s_{w+tz}(w+\tau_t tz))s_{w+tz}tz,\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &\geq \Phi(s_w(w+tz)) - \Phi(s_w w) \\ &= \Phi'(s_w(w+\eta_t tz))s_w tz,\end{aligned}$$

where $\eta_t \in (0,1)$. Since the mapping $w \mapsto s_w$ is continuous according to Proposition 8, we see combining these two inequalities that

$$\lim_{t\to 0}\frac{\widehat{\Psi}(w+tz) - \widehat{\Psi}(w)}{t} = s_w\Phi'(s_w w)z = \frac{\|\widehat{m}(w)\|}{\|w\|}\Phi'(\widehat{m}(w))z.$$

Hence the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\widehat{\Psi}$ is of class C^1, see e.g. [**23, 45**]. □

Corollary 10. *Suppose E is a Banach space satisfying (A_1). If Φ satisfies (A_2) and (A_3), then:*
(a) $\Psi \in C^1(S, \mathbb{R})$ and

$$\Psi'(w)z = \|m(w)\|\Phi'(m(w))z \quad \textit{for all } z \in T_w(S).$$

(b) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{N}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.
(c) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding values of Ψ and Φ coincide and $\inf_S \Psi = \inf_{\mathcal{N}} \Phi$.
(d) If Φ is even, then so is Ψ.

PROOF. (a) follows from Proposition 9. Note only that since $w \in S$, we have $m(w) = \widehat{m}(w)$.

(b) We first note that by (A1) we have $E = T_w(S) \oplus \mathbb{R}w$ for every $w \in S$, and the projection $E \to T_w(S)$, $z + tw \mapsto z$ has uniformly bounded norm with respect to $w \in S$. Indeed, J is bounded on bounded sets and $J(w)(z + tw) = t$, so if $\|z + tw\| = 1$, then $|t| \le C$ for some constant $C > 0$ and therefore

$$\|z\| \le |t| + \|z + tw\| \le (C+1)\|z + tw\| \qquad \text{for all } w \in S,\ z \in T_w S \text{ and } t \in \mathbb{R}.$$

Moreover, by (a) we have

$$\|\Psi'(w)\| = \sup_{\substack{z \in T_w(S) \\ \|z\|=1}} \Psi'(w)z = \|u\| \sup_{\substack{z \in T_w(S) \\ \|z\|=1}} \Phi'(u)z \tag{11}$$

with $u = m(w)$, and since $\Phi'(u)w = \Phi'(u)\frac{u}{\|u\|} = 0$, we conclude using (a) again that

$$\begin{aligned}\|\Psi'(w)\| \le \|u\|\|\Phi'(u)\| &= \|u\| \sup_{\substack{z \in T_w(S),\, t \in \mathbb{R} \\ z+tw \neq 0}} \frac{\Phi'(u)(z + tw)}{\|z + tw\|} \\ &\le (C+1)\|u\| \sup_{z \in T_w(S)\setminus\{0\}} \frac{\Phi'(u)z}{\|z\|} = (C+1)\|\Psi'(w)\|.\end{aligned}$$

Since $u \in \mathcal{N}$ and $\mathcal{N}$ is bounded away from 0, this two-sided estimate, together with the fact that $\Phi(u) = \Psi(w)$, easily yields the assertion.

(c) By (11), $\Psi'(w) = 0$ if and only if $\Phi'(m(w)) = 0$. The other part is clear.

(d) If Φ is even, then $s_w = s_{-w}$. Hence $\widehat{m}(-w) = -\widehat{m}(w)$ and the conclusion follows from the definition of Ψ. $\square$

REMARK 11. We note that the infimum of Φ over $\mathcal{N}$ has the following minimax characterization:

$$c = \inf_{u \in \mathcal{N}} \Phi(u) = \inf_{w \in E\setminus\{0\}} \max_{s>0} \Phi(sw) = \inf_{w \in S} \max_{s>0} \Phi(sw).$$

Our next results give sufficient conditions for the existence of a ground state and of infinitely many critical points of Φ.

THEOREM 12. *Let E be a Hilbert space and suppose that $\Phi(u) = \frac{1}{2}\|u\|^2 - I(u)$, where*
(i) $I'(u) = o(\|u\|)$ as $u \to 0$,
(ii) $s \mapsto I'(su)u/s$ is strictly increasing for all $u \neq 0$ and $s > 0$,
(iii) $I(su)/s^2 \to \infty$ uniformly for u on weakly compact subsets of $E\setminus\{0\}$ as $s \to \infty$,
(iv) I' is completely continuous.
Then equation $\Phi'(u) = 0$ has a ground state solution. Moreover, if I is even, then this equation has infinitely many pairs of solutions.

In the following we say that Φ *satisfies the Palais-Smale condition on* $\mathcal{N}$ if every Palais-Smale sequence (u_n) for Φ with $u_n \in \mathcal{N}$ for all n contains a convergent subsequence.

PROOF. First we verify that (A_2) and (A_3) are satisfied. By (i) and (iii), $\alpha_w(s) > 0$ for $s > 0$ small and $\alpha_w(s) < 0$ for s large. Since

$$\alpha_w'(s) = \frac{d}{ds}\Phi(sw) = s(\|w\|^2 - s^{-1}I'(sw)w),$$

it follows from (ii) that there exists a unique s_w with $\alpha_w'(s_w) = 0$. Clearly, $s_w \geq \delta$ for some $\delta > 0$ and all $w \in S$ according to (i), and $s_w \leq C_{\mathcal{W}}$ for all w in a compact subset $\mathcal{W} \subset S$ according to (iii).

We shall demonstrate in Proposition 14 below that Φ satisfies the Palais-Smale condition on $\mathcal{N}$. Assuming this, let (w_n) be a Palais-Smale sequence for Ψ and set $u_n := m(w_n) \in \mathcal{N}$. Then (u_n) is a Palais-Smale sequence for Φ according to Corollary 10, hence $u_n \to u$ after passing to a subsequence and $w_n \to m^{-1}(u)$. It follows that Ψ satisfies the Palais-Smale condition.

Let (w_n) be a minimizing sequence for Ψ. By Ekeland's variational principle [**25, 32, 45**] we may assume $\Psi'(w_n) \to 0$, and by the Palais-Smale condition, $w_n \to w$ after passing to a subsequence. Hence w is a minimizer for Ψ and u is a ground state solution for the equation $\Phi'(u) = 0$, cf. Theorem 1 and Corollary 10.

Assume now that Φ is even. Then so is Ψ. Since $\inf_S \Psi > 0$, Ψ is bounded from below and the second conclusion follows from Corollary 10 and Theorem 2. $\square$

In one of the problems considered in the next section we shall need an extension of the above result to Banach spaces. Let $I_0 \in C^1(E, \mathbb{R})$ be even, positively homogeneous of degree $p > 1$ (i.e., $I_0(su) = s^p I_0(u)$, $s > 0$) and such that

$$c_0\|u\|^p \leq I_0(u) \leq c_0^{-1}\|u\|^p \tag{12}$$

for some constant $c_0 > 0$.

THEOREM 13. *Let E be a uniformly convex Banach space satisfying (A_1) and suppose that $\Phi(u) = I_0(u) - I(u)$, where*
(i) $I'(u) = o(\|u\|^{p-1})$ as $u \to 0$,
(ii) $s \mapsto I'(su)u/s^{p-1}$ is strictly increasing for all $u \neq 0$ and $s > 0$,
(iii) $I(su)/s^p \to \infty$ uniformly for u on weakly compact subsets of $E\setminus\{0\}$ as $s \to \infty$,
(iv) I' is completely continuous,
(v) I_0 is weakly lower semicontinuous, positively homogeneous of degree p, satisfies (12) and

$$(I_0'(v) - I_0'(w))(v - w) \geq c_1(\|v\|^{p-1} - \|w\|^{p-1})(\|v\| - \|w\|)$$

for some $c_1 > 0$ and all $v, w \in E$.
Then equation $\Phi'(u) = 0$ has a ground state solution. Moreover, if I is even, then this equation has infinitely many pairs of solutions.

PROOF. Using (12) and the homogeneity of I_0 it is easy to see that the argument of Theorem 12 goes through with obvious changes. $\square$

Note that while in Theorem 12 the sphere S is a C^∞-manifold, in Theorem 13 S may be of class C^1 only. Therefore it is important to have a version of Theorem 2 which holds for such S. In the next section we shall encounter a problem where $S \in C^1$ but $\notin C^{1,1}$.

To complete the proofs of Theorems 12 and 13 we still need to show that Φ satisfies the Palais-Smale condition on $\mathcal{N}$. Since (v) of Theorem 13 is automatically satisfied if E is a Hilbert space and $I_0(u) = \frac{1}{2}\|u\|^2$, it suffices to show this for Theorem 13. It was first noted in [**31**] that (in the context of semilinear boundary value problems) the Palais-Smale condition for Φ on $\mathcal{N}$ is satisfied under weaker growth assumptions than needed when no restriction to $\mathcal{N}$ is made.

Proposition 14. *The following holds under the assumptions of Theorem 13:*
(a) If $(u_n) \subset \mathcal{N}$ is a sequence such that $\sup_{n\in\mathbb{N}} \Phi(u_n) < \infty$, then – passing to a subsequence – we have $u_n \rightharpoonup u \neq 0$ as $n \to \infty$, and there is $s_u > 0$ such $s_u u \in \mathcal{N}$ and $\Phi(s_u u) \le \liminf_{n\to\infty} \Phi(u_n)$.
(b) $\Phi|_{\mathcal{N}}$ is coercive, i.e., $\Phi(u_n) \to \infty$ as $u_n \in \mathcal{N}$, $\|u_n\| \to \infty$.
(c) Φ satisfies the Palais-Smale condition on $\mathcal{N}$.

The proof below is inspired by [**31**].

Proof. (a) Let $(u_n) \subset \mathcal{N}$ be a sequence such that $\Phi(u_n) \le d$ for all n. We first claim that (u_n) is bounded. Otherwise $\|u_n\| \to \infty$ and $v_n := u_n/\|u_n\| \rightharpoonup v$ in E after passing to a subsequence. Suppose $v = 0$. Since $u_n \in \mathcal{N}$ and $v_n \rightharpoonup 0$, for each $s > 0$ we have

$$d \ge \Phi(u_n) = \Phi(s_{v_n} v_n) \ge \Phi(s v_n) \ge c_0 s^p - I(s v_n) \to c_0 s^p \tag{13}$$

because assumption (iv) of Theorem 13 implies that I is weakly continuous. This yields a contradiction upon choosing $s > (d/c_0)^{1/p}$. So $v \neq 0$ and hence

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^p} \le \frac{1}{c_0} - \frac{I(\|u_n\| v_n)}{\|u_n\|^p} \to -\infty \tag{14}$$

as $n \to \infty$ by (iii) of Theorem 13, a contradiction again. It follows that (u_n) is bounded, so $u_n \rightharpoonup u$ after passing to a subsequence. If $u = 0$, we see as in (13) that

$$d \ge \Phi(u_n) \ge \Phi(s u_n) \ge d_0 s^p - I(s u_n) \to d_0 s^p \tag{15}$$

for all $s > 0$, where $d_0 = c_0 \inf_{\mathcal{N}} \|u\| > 0$, a contradiction. Hence $u \neq 0$. Moreover, as $I(s_u u_n) \to I(s_u u)$, again by the weak continuity of I, we have

$$\Phi(s_u u) \le \liminf_{n\to\infty} \Phi(s_u u_n) \le \liminf_{n\to\infty} \Phi(u_n)$$

since $u_n \in \mathcal{N}$.

(b) This follows directly from (a).

(c) Let $(u_n) \subset \mathcal{N}$ be a Palais-Smale sequence. By (a), $u_n \rightharpoonup u$ after passing to a subsequence. Moreover,

$$\Phi'(u_n) = I_0'(u_n) - I'(u_n) \to 0 \quad \text{and} \quad I'(u_n) \to I'(u) \qquad \text{as } n \to \infty,$$

hence by (v) of Theorem 13,

$$o(1) = [I_0'(u_n) - I_0'(u)](u_n - u) \ge c_1(\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

If follows that $\|u_n\| \to \|u\|$ and therefore $u_n \to u$ by the uniform convexity of E. So Φ satisfies the Palais-Smale condition on $\mathcal{N}$. □

Remark 15. It is easily seen that the proof of Proposition 14 still goes through if the condition that I_0 satisfies (v) of Theorem 13 is replaced by the more general condition that I_0 can be written as a sum of a functional satisfying (v) and a functional with completely continuous derivative. It will be convenient to use this observation in the next section.

3.2. Elliptic equations in bounded domains

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{16}$$

Here $\lambda < \lambda_1$, where λ_1 denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies the growth restriction

$$|f(x,u)| \le a(1+|u|^{q-1}) \quad \text{for some } a > 0 \text{ and } 2 < q < 2^*. \tag{17}$$

Recall from Section 2.2 that $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := \infty$ otherwise. Since $\lambda < \lambda_1$, we may introduce an equivalent norm in $E \equiv H_0^1(\Omega)$ by setting

$$\|u\|^2 := \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx.$$

Let

$$F(x,u) := \int_0^u f(x,s)\, ds$$

and

$$\Phi(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \int_\Omega F(x,u)\, dx \equiv \frac{1}{2}\|u\|^2 - I(u).$$

By Theorem 3, $\Phi \in C^1(E, \mathbb{R})$, I' is completely continuous and critical points of Φ are solutions of (16). The following result on the existence and multiplicity of solutions of (16) is essentially contained in [**31**]. We note however that in [**31**] stronger assumptions on f were imposed for the multiplicity result in order to apply critical point theory on $\mathcal{N}$.

Theorem 16. *Suppose that $\lambda < \lambda_1$, f satisfies (17) and*
(i) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0,\infty)$,
(iii) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.
Then equation (16) has a ground state solution. Moreover, if f is odd in u, then (16) has infinitely many pairs of solutions.

Proof. We want to use Theorem 12. It follows from (i) and Theorem 3 that $I'(u) = o(\|u\|)$ as $u \to 0$, and it is easy to see from (ii) that $s \mapsto I'(su)u/s$ is strictly increasing if $u \ne 0$ and $s > 0$. Next we verify (iii) of Theorem 12. Let $\mathcal{W} \subset E \setminus \{0\}$ be weakly compact and let $(u_n) \subset \mathcal{W}$. It suffices to show that if $s_n \to \infty$ as $n \to \infty$, then so does a subsequence of $I(s_n u_n)/s_n^2$. Passing to a subsequence, $u_n \rightharpoonup u \in E \setminus \{0\}$ and $u_n(x) \to u(x)$ a.e. Since $|s_n u_n(x)| \to \infty$ if $u(x) \ne 0$, an application of Fatou's lemma yields

$$\frac{I(s_n u_n)}{s_n^2} = \int_\Omega \frac{F(x, s_n u_n)}{(s_n u_n)^2} u_n^2\, dx \to \infty \quad \text{as } n \to \infty$$

(that $F \ge 0$ is an easy consequence of (i) and (ii) above). Finally, (iv) of Theorem 12 is ensured by Theorem 3.

Now Theorem 12 yields that equation (16) has a ground state solution and if f is odd in u, then (16) has infinitely many pairs of solutions. □

Remark 17. One can show that each ground state solution u_0 is positive or negative. Indeed, if u is a sign-changing solution, then we let $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ denote the positive and the negative part of u, respectively.

Multiplying (16) by $u^\pm$ and integrating, we see that $u^\pm \in \mathcal{N}$. Hence $\Phi(u) = \Phi(u^+) + \Phi(u^-) \geq 2c$. It follows that $u_0 \geq 0$ or $u_0 \leq 0$ and by Harnack's inequality [**26**], $u_0 > 0$ or $u_0 < 0$ in Ω.

Suppose f satisfies (17), (i) of Theorem 16 and the following Ambrosetti-Rabinowitz condition [**4**]: There exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x,u) \leq f(x,u)u \quad \text{for all } |u| \geq R. \tag{18}$$

It is well known and may be found e.g. in [**3, 4, 37, 42, 45**] that under these assumptions (16) has a positive solution, and if in addition f is odd in u, then there are infinitely many pairs of solutions. A typical example of f satisfying both (18) and the assumptions of Theorem 16 is $f(x,u) = |u|^{q-2}u$, $2 < q < 2^*$. It is easy to show by integration that (18) implies $|F(x,u)| \geq a_1|u|^\mu - a_2$ for some $a_1, a_2 > 0$. On the other hand, there are functions which increase slower than that and yet satisfy (i)-(iii) of Theorem 16, e.g. $f(x,u) = ug(|u|)$, where $g(0) = 0$, g is strictly increasing and $g(s) \to \infty$, $g(s)/s^a \to 0$ for any $a > 0$ as $s \to \infty$. Of course, there also exist functions satisfying (18) but not (ii).

Next we add a result on the existence of least energy sign-changing solutions. For this we put

$$\mathcal{N}_{sc} := \{u \in E : u^\pm \in \mathcal{N}\} \qquad \text{and} \qquad c_{sc} := \inf_{u \in \mathcal{N}_{sc}} \Phi(u),$$

where as in Remark 17, u^+ and u^- are the positive and negative part of u. Since $u^\pm \in \mathcal{N}$, we see that every sign-changing solution of (16) lies in $\mathcal{N}_{sc}$.

Theorem 18. *Under the assumptions of Theorem 16,* (16) *admits a least energy sign-changing solution, i.e. a solution $u \in \mathcal{N}_{sc}$ with $\Phi(u) = c_{sc} \geq 2c$.*

This result is due to Liu and Wang [**31**]. The existence of least energy solution was first asserted in [**41**] under somewhat stronger assumptions. However, the proof in [**41**] is not fully correct since, as in many other papers, it has been overlooked that the functionals

$$E \to \mathbb{R}, \qquad u \mapsto \int_\Omega |\nabla u^\pm|^2\, dx$$

are *not* of class C^1, see [**9**, Section 3] for a discussion of this regularity problem. The first correct proof – under stronger assumptions – on the existence of a least energy sign-changing solution of (16) was given by Castro-Cossio-Neuberger [**15**].

Proof of Theorem 18. Let $(u_n) \subset \mathcal{N}_{sc}$ be a sequence of functions such that $\Phi(u_n) \to c_{sc}$ as $n \to \infty$. Since $\Phi(u_n) = \Phi(u_n^+) + \Phi(u_n^-)$ and $\Phi(u_n^\pm) \geq c > 0$ for all n, we see that $\Phi(u_n^\pm)$ is bounded. Hence, by Proposition 14(a), $u_n^+ \rightharpoonup u_1 \neq 0$ and $u_n^- \rightharpoonup u_2 \neq 0$ and therefore $u_n \rightharpoonup \widetilde{u} := u_1 + u_2$ as $n \to \infty$. Since we may assume a.e. pointwise convergence along a subsequence, we find that $u_1 \cdot u_2 \equiv 0$. Moreover, for $u := s_{u_1}u_1 + s_{u_2}u_2 = u^+ + u^-$ we have, by Proposition 14(a),

$$2c \leq \Phi(u) = \Phi(s_{u_1}u_1) + \Phi(s_{u_2}u_2) \leq \liminf_{n\to\infty} [\Phi(u_n^+) + \Phi(u_n^-)] = c_{sc}.$$

Since $u \in \mathcal{N}_{sc}$, we conclude that $\Phi(u) = c_{sc}$.

Using the quantitative deformation lemma (see [**45**, Lemma 2.3]), we shall now prove that $\Phi'(u) = 0$. It follows from assumption (ii) of Theorem 16 that, for $s, t > 0$ and at least one of $s, t \neq 1$,

$$\Phi(su^+ + tu^-) = \Phi(su^+) + \Phi(tu^-) < \Phi(u^+) + \Phi(u^-) = c_{sc}. \tag{19}$$

If $\Phi'(u) \neq 0$, then there exist $\delta > 0$ and $\mu > 0$ such that

$$\|v - u\| \leq 3\delta \qquad \Rightarrow \qquad \|\Phi'(v)\| \geq \mu.$$

Let $D = [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$ and $g(s,t) = su^+ + tu^-$. It follows from (19) that $\Phi(g(s,t)) = c_{sc}$ if and only if $s = t = 1$ and $\Phi(g(s,t)) < c_{sc}$ otherwise. Hence

$$\beta := \max_{\partial D} \Phi \circ g < c_{sc}. \tag{20}$$

For $\varepsilon := \min\left\{\frac{c_{sc}-\beta}{4}, \frac{\mu\delta}{8}\right\}$, [**45**, Lemma 2.3] yields a deformation η such that
a) $\eta(1,v) = v$ if $v \notin \Phi^{-1}([c_{sc} - 2\varepsilon, c_{sc} + 2\varepsilon])$,
b) $\Phi(\eta(1,v)) \leq c_{sc} - \varepsilon$ for every $v \in E$ with $\|v - u\| \leq \delta$ and $\Phi(v) \leq c_{sc} + \varepsilon$.
c) $\Phi(\eta(1,v)) \leq \Phi(v)$ for all $u \in E$.
It is then clear that

$$\max_{(s,t)\in D} \Phi(\eta(1, g(s,t))) < c_{sc}.$$

We shall prove that $\eta(1, g(D)) \cap \mathcal{N}_{sc} \neq \emptyset$, contradicting the definition of c_{sc}. Let us define $h(s,t) := \eta(1, g(s,t))$ and

$$\Psi_0(s,t) := (\Phi'(su^+)u^+, \Phi'(tu^-)u^-),$$
$$\Psi_1(s,t) := \left(\frac{1}{s}\Phi'(h^+(s,t))h^+(s,t),\ \frac{1}{t}\Phi'(h^-(s,t))h^-(s,t)\right).$$

Since $\Phi'(su^\pm)u^\pm > 0$ if $0 < s < 1$ and < 0 if $s > 1$, the product formula for the degree now yields $\deg(\Psi_0, D, 0) = 1$. It follows from (20) and the property a) of η that $g = h$ on ∂D. Consequently, $\Psi_1 = \Psi_0$ on ∂D and $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Therefore $\Psi_1(s,t) = 0$ for some $(s,t) \in D$, hence $\eta(1, g(s,t)) = h(s,t) \in \mathcal{N}_{sc}$. We conclude that u is a critical point of Φ. $\square$

Next we consider a related boundary value problem for the p-Laplacian. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. It is well known (see e.g. [**22**] or Section 7.5A in [**23**]) that

$$\lambda_1 := \inf_{\substack{u \in W_0^{1,p}(\Omega)\\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx}$$

is attained and is the first Dirichlet eigenvalue of the p-Laplacian. Set

$$I_0(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda |u|^p)\,dx \equiv \frac{1}{p}\|u\|^p - I_1(u).$$

If $\lambda < \lambda_1$, then it is clear that I_0 is positively p-homogeneous and satisfies (12). However, $(pI_0)^{1/p}$ is not a norm and if $1 < p < 2$, then the unit sphere is not a $C^{1,1}$-submanifold of $E = W_0^{1,p}(\Omega)$. Our boundary value problem is now

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{21}$$

Recall from Section 2.2 that $p^* := Np/(N-p)$ if $N > p$ and $p^* := \infty$ otherwise. The following result seems to be new.

THEOREM 19. *Suppose that $\lambda < \lambda_1$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies (17) with 2^* replaced by p^* and 2 by p. Suppose further*
(i) $f(x,u) = o(|u|^{p-1})$ uniformly in x as $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|^{p-1}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,

(iii) $F(x,u)/|u|^p \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (21) has a ground state solution. Moreover, if f *is odd in* u*, then (21) has infinitely many pairs of solutions.*

Proof. We shall apply Theorem 13 to the functional

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{p}\int_\Omega \lambda|u|^p\,dx - \int_\Omega F(x,u)\,dx \;\equiv\; I_0(u) - I(u),$$

where

$$I_0(u) = \frac{1}{p}\|u\|^p - \frac{1}{p}\int_\Omega \lambda|u|^p\,dx \;\equiv\; \psi(u) - I_1(u)$$

and $I(u) = \int_\Omega F(x,u)\,dx$. It is clear that (A_1) holds with $\psi(u) = \frac{1}{p}\|u\|^p$ and $J = \psi'$ given by (8). Very similarly as in the proof of Theorem 16 we find that assumptions (i)-(iii) of Theorem 13 are satisfied. Moreover, both I' and I_1' are completely continuous by Theorem 5, so (iv) holds as well. In view of Remark 15 it therefore suffices to show that J satisfies

$$(J(v) - J(w))(v-w) \ge (\|v\|^{p-1} - \|w\|^{p-1})(\|v\| - \|w\|) \text{ for } v,w \in E,$$

but this property is well known, see e.g. [**22**] or (d) on p. 501 in [**23**]. □

The result is known if (ii) and (iii) are replaced by a condition of Ambrosetti-Rabinowitz type ($\mu > 2$ should be replaced by $\mu > p$ in (18)), see [**22**].

Also here one can show there exists a least energy sign-changing solution with $c_{sc} \ge 2c$ and each ground state solution satisfies $u \ge 0$ or $u \le 0$.

3.3. Elliptic problems in $\mathbb{R}^N$

Here we consider the problem

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0, & |x| \to \infty. \end{cases} \tag{22}$$

If V is bounded, f continuous and satisfying (17), but with 1 replaced by $|u|$, then

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx$$

is of class C^1 on $E \equiv H^1(\mathbb{R}^N)$ and critical points of Φ correspond to solutions of (22), see Theorem 6.

Theorem 20. *Suppose* $V \in C(\mathbb{R}^N, \mathbb{R})$*,* $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *satisfies (17) and*
(i) V*,* f *are 1-periodic in* $x_1, \dots, x_N$ *and* $V(x) > 0$ *for all* x*,*
(ii) $f(x,u) = o(u)$ *uniformly in* x *as* $u \to 0$*,*
(iii) $u \mapsto f(x,u)/|u|$ *is strictly increasing on* $(-\infty,0)$ *and* $(0,\infty)$*,*
(iv) $F(x,u)/u^2 \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (22) has a ground state solution.

This extends Theorem 2.1 in [**28**] where it was additionally assumed that f is of class C^1 with f_u satisfying a growth restriction. By (i), V is bounded and bounded away from 0, hence

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx \tag{23}$$

defines an equivalent norm and we have

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx \equiv \frac{1}{2}\|u\|^2 - I(u).$$

(23) remains valid, and so does Theorem 20, if $V > 0$ is replaced by the weaker condition that $\sigma(-\Delta + V) \subset (0,\infty)$, where σ is the spectrum in $L^2(\mathbb{R}^N)$. As has been noticed in Section 2.2, (17) and (ii) imply that f satisfies the proper growth restriction which is (7). Hence $\Phi \in C^1(E,\mathbb{R})$. By the periodicity of V and f, if u is a solution of (22), then so is $u(\cdot - y)$ for any $y \in \mathbb{Z}^N$. Two solutions which are not translates of each other by an element of $\mathbb{Z}^N$ will be called *geometrically distinct*. If f is odd in u, then (22) has in fact infinitely many geometrically distinct solutions under the assumptions of Theorem 20. We do not include the lengthy proof of this fact here and refer the reader to [**44**]. We remark that under stronger conditions there exist infinitely many geometrically distinct solutions also for non-odd f, see [**21**].

Let us note that if V is a positive constant and $f = f(u)$, then $\Phi(u) = \Phi(u(\cdot - y))$ for all $y \in \mathbb{R}^N$ and any translate $u(\cdot - y)$ of a solution $u \neq 0$ is again a solution. Hence the correct notion of geometrically distinct solutions here is the requirement that they are not translates of each other by any $y \in \mathbb{R}^N$. It follows that existence of a single nontrivial solution automatically leads to the existence of infinitely many geometrically distinct ones in the $\mathbb{Z}^N$-sense. However, as we shall see in Section 3.5, it is not necessarily true that the number of those which are $\mathbb{R}^N$-distinct is infinite.

We shall need the following result whose simple proof is left to the reader:

LEMMA 21. *If f satisfies (ii) and (iii) of Theorem 20, then $F(x,u) > 0$ and $\frac{1}{2}f(x,u)u > F(x,u)$ for all $u \neq 0$.*

PROOF OF THEOREM 20. We cannot apply Theorem 12 directly. It is easy to verify that (i)-(iii) of this theorem still hold, with the same proof as in Theorem 16 (note only that $u_n \rightharpoonup u$ in E implies $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$ and hence $u_n(x) \to u(x)$ a.e. after taking a subsequence). Therefore the results of Corollary 10 remain valid. However, I' is not completely continuous and Φ cannot satisfy the Palais-Smale condition on $\mathcal{N}$. Indeed, if $\Phi'(u) = 0$ and $u_n(x) := u(x - y_n)$, where $y_n \in \mathbb{Z}^N$, then (u_n) is a Palais-Smale sequence which converges weakly but not strongly to 0 if $|y_n| \to \infty$.

Let $(w_n) \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi'(w_n) \to 0$, hence also $\Phi'(u_n) \to 0$, where $u_n := m(w_n)$. We shall show that (u_n) is bounded. Suppose $\|u_n\| \to \infty$, then $v_n := u_n/\|u_n\| \rightharpoonup v$ in E and $v_n \to v$ a.e. after passing to a subsequence. By (ii) and (17), for each $\varepsilon > 0$ there is C_ε such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{q-1}. \tag{24}$$

If $v = 0$ and $v_n \to 0$ in $L^q(\mathbb{R}^N)$, then for each $s > 0$, $\int_{\mathbb{R}^N} F(x, sv_n)\,dx \to 0$ according to (24) and therefore

$$d \ge \Phi(u_n) \ge \Phi(sv_n) = \frac{s^2}{2} - \int_{\mathbb{R}^N} F(x, sv_n)\,dx \to \frac{s^2}{2}, \tag{25}$$

a contradiction for $s > \sqrt{2d}$. So $v_n \not\to 0$ in $L^q(\mathbb{R}^N)$ and it follows from P.L. Lions' lemma [**29**], [**45**, Lemma 1.21] that

$$\int_{B_1(y_n)} v_n^2\,dx \geq \delta \tag{26}$$

for some $\delta > 0$, $y_n \in \mathbb{R}^N$ and almost all n. Since Φ and $\mathcal{N}$ are invariant by translations of the form $v \mapsto v(\cdot - y)$, $y \in \mathbb{Z}^N$, we may assume translating v_n if necessary that the sequence (y_n) is bounded. Since $v_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$, (26) implies $v \neq 0$ and we get a contradiction since Fatou's lemma yields

$$0 \leq \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx \to -\infty \qquad \text{as } n \to \infty. \tag{27}$$

So (u_n) is bounded and we may assume $u_n \rightharpoonup u$ in E and $u_n \to u$ a.e. Hence u is a solution of (22), possibly the trivial one ($u = 0$). If $u_n \to 0$ in $L^q(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx = o(\|u_n\|)$ by (24) and the Hölder and Sobolev inequalities. So

$$o(\|u_n\|) = \Phi'(u_n)u_n = \|u_n\|^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx = \|u_n\|^2 + o(\|u_n\|) \tag{28}$$

and therefore $u_n \to 0$. However, this is a contradiction because $u_n \in \mathcal{N}$ and $\mathcal{N}$ is bounded away from 0. So $u_n \not\to 0$ in $L^q(\mathbb{R}^N)$ and applying P.L. Lions' lemma once more, we see that (26) holds with v_n replaced by u_n and as before we may assume translating u_n if necessary that $u_n \rightharpoonup u \neq 0$. Hence u is a nontrivial solution of (22) and in particular, $u \in \mathcal{N}$.

We still need to show that $\Phi(u) = c := \inf_{\mathcal{N}} \Phi$. Since we may assume passing to a subsequence that $u_n \to u$ a.e., Lemma 21 and Fatou's lemma imply

$$\begin{aligned} c + o(1) = \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n &= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\geq \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u)u - F(x,u)\right) dx + o(1) \\ &= \Phi(u) - \frac{1}{2}\Phi'(u)u + o(1) = \Phi(u) + o(1). \end{aligned}$$

Hence $\Phi(u) \leq c$. The reverse inequality is obvious. □

REMARK 22. Again, one sees by the argument of Remark 17 that $u > 0$ or $u < 0$ in $\mathbb{R}^N$ if u is a ground state solution.

REMARK 23. Also in the framework of Theorem 20 it is possible to replace $-\Delta$ by $-\Delta_p$ (and modify the other assumptions accordingly). A somewhat stronger result on the existence of a ground state for an equation involving the p-Laplacian has been obtained in [**30**] by means of different methods.

We add another result for a nonperiodic V corresponding to a potential well.

THEOREM 24. *Suppose* $V \in C(\mathbb{R}^N, \mathbb{R})$ *satisfies*
(i) $0 < \inf_{\mathbb{R}^N} V \leq \sup_{\mathbb{R}^N} V = V_\infty < \infty$ *with* $V_\infty := \lim_{|x|\to\infty} V(x)$.
Suppose furthermore that the nonlinearity $f = f(u)$ *in (22) does not depend on* x*, is continuous and satisfies*
(ii) $f(u) = o(u)$ *as* $u \to 0$*,*
(iii) $u \mapsto f(u)/|u|$ *is strictly increasing on* $(-\infty, 0)$ *and* $(0, \infty)$*,*

(iv) $F(u)/u^2 \to \infty$ *as* $|u| \to \infty$.
Then equation (22) has a ground state solution.

This extends Theorem 2.1 in [**28**] where it was additionally assumed that f is of class C^1 with f_u satisfying a growth restriction. The proof we will sketch here is also somewhat simpler than the one in [**28**] since we do not use a bounded domain approximation of problem (22).

Proof of Theorem 24. Since parts of the argument are similar to the proof of Theorem 20, we only give a brief sketch. We consider the associated limit energy functional

$$\Phi_\infty : E \to \mathbb{R}, \qquad \Phi_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx$$

and the corresponding minimax value

$$c_\infty := \inf_{w\in E\setminus\{0\}} \max_{s>0} \Phi_\infty(sw).$$

By assumption (i), we infer that $c \le c_\infty$. Moreover, by Theorem 20 and Remark 22, c_∞ is attained at a strictly positive or a strictly negative solution $u \in E$ of the equation $-\Delta u + V_\infty u = f(u)$. So if V is strictly smaller than V_∞ on a set of positive measure, we have $\Phi(su) < \Phi_\infty(su) \le c_\infty$ for all $s > 0$, and from this and assumption (iii) it easily follows that $c < c_\infty$. So we may assume without loss of generality that $c < c_\infty$ since otherwise we must have $V \equiv V_\infty$ and the assertion follows from Theorem 20. Now let (w_n) be a minimizing sequence for Ψ. Again we may assume $\Psi'(w_n) \to 0$, so that also $\Phi'(u_n) \to 0$ with $u_n := m(w_n)$. Arguing by contradiction, we suppose $\|u_n\| \to \infty$; then $v_n := u_n/\|u_n\| \rightharpoonup v$ in E and $v_n \to v$ a.e. after passing to a subsequence. If $v = 0$ and $v_n \to 0$ in $L^q(\mathbb{R}^N)$, then for each $s > 0$, $\int_{\mathbb{R}^N} F(sv_n)\,dx \to 0$ according to (24), and we obtain a contradiction as in (25). So $v_n \not\to 0$ in $L^q(\mathbb{R}^N)$ and therefore (26) holds again for some $\delta > 0$, $y_n \in \mathbb{R}^N$ and almost all n. If the sequence (y_n) is bounded, we obtain a contradiction as in the proof of Theorem 20. If, on the other hand, $|y_n| \to \infty$ for a subsequence, then $\widetilde{v}_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ for the functions $\widetilde{v}_n = v_n(\cdot - y_n)$, and as in (27) Fatou's lemma yields the contradiction

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(u_n)}{u_n^2} v_n^2\,dx = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(\widetilde{u}_n)}{\widetilde{u}_n^2}\widetilde{v}_n^2\,dx \to -\infty$$

as $n \to \infty$ with $\widetilde{u}_n := u_n(\cdot - y_n)$. So (u_n) is bounded. As in the proof of Theorem 20 we then find a new sequence of points $y_n \in \mathbb{R}^N$, $n \in \mathbb{N}$ such that $\widetilde{u}_n \rightharpoonup u \ne 0$ for the translated functions $\widetilde{u}_n := u_n(\cdot - y_n)$. We claim that the sequence (y_n) is bounded. Indeed, if $|y_n| \to \infty$ for a subsequence, the assumption on the asymptotic shape of V implies that u is a critical point of Φ_∞ and therefore

$$\begin{aligned} c + o(1) &= \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \int_{\mathbb{R}^N} \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2}f(\widetilde{u}_n)\widetilde{u}_n - F(\widetilde{u}_n)\right) dx \ge \int_{\mathbb{R}^N} \left(\frac{1}{2}f(u)u - F(u)\right) dx + o(1) \\ &= \Phi_\infty(u) - \frac{1}{2}\Phi'_\infty(u)u + o(1) = \Phi_\infty(u) + o(1) \ge c_\infty + o(1) \end{aligned}$$

which contradicts the inequality $c < c_\infty$. Hence the sequence (y_n) is bounded, so we may without loss of generality assume that $y_n = 0$ and therefore $\widetilde{u}_n = u_n$ for all n. Then we can conclude as in the proof of Theorem 20. $\square$

We remark that in a similar way one can treat elliptic systems like

$$\begin{cases} -\Delta_p u = F_u(x,u,v), & x \in \Omega \\ -\Delta_p v = F_v(x,u,v), & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded, F_u, F_v satisfy a suitable growth restriction and F, F_u, F_v satisfy suitably modified conditions (W_2)-(W_4) in Section 3.5. It is also possible to consider a system in $\mathbb{R}^N$, with $-\Delta_p$ replaced by $-\Delta u + u$ (and similarly for v) provided F is periodic in $x_1, \dots, x_N$. The details are left to the reader.

3.4. A multiplicity result for a singularly perturbed problem with domain topology

In this section we revisit the classical work of Benci and Cerami [**11**] on the multiplicity of positive solutions of the problem

$$\begin{cases} -\varepsilon^2 \Delta u + \mu u = f(u), & x \in \Omega, \\ \qquad u = 0 & x \in \partial\Omega. \end{cases} \tag{29}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\mu > 0$ is a fixed constant and the nonlinearity $f \in C(\mathbb{R},\mathbb{R})$ satisfies (17) and (ii)-(iv) of Theorem 24. The following multiplicity result in terms of the Lusternik-Schnirelman category has been obtained by Benci and Cerami under stronger assumptions. In particular, they require the nonlinearity to be of class $C^{1,1}$ and such that the Ambrosetti-Rabinowitz condition (18) holds. We recall that the category $\operatorname{cat}_X(A)$ of a subset A of a topological space X is defined as the minimal $k \in \mathbb{N}$ such that A is covered by k closed subsets of X which are contractible in X. As usual, if no such k exists, we put $\operatorname{cat}_X(A) = \infty$. We also write $\operatorname{cat}(X) := \operatorname{cat}_X(X)$.

THEOREM 25. *Suppose that f satisfies (17), (ii)-(iv) of Theorem 24 and that the underlying domain Ω is topologically nontrivial, i.e., it is not contractible. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem (29) admits at least* $\operatorname{cat}(\Omega) + 1$ *positive solutions.*

In the following we briefly sketch the proof of this result and refer to the papers indicated below for the details. Let $E := H_0^1(\Omega)$ and consider the functional

$$\Phi : E \to \mathbb{R}, \qquad \Phi(u) := \frac{1}{2}\|u\|^2 - I(u)$$

with

$$\|u\|^2 = \int_\Omega \left(\varepsilon^2 |\nabla u|^2 + \mu u^2\right) dx \quad \text{and} \quad I(u) = \int_\Omega F(u^+)\, dx.$$

Consider also the subset $S_+ := \{u \in E : \|u\| = 1, u^+ \neq 0\}$ of the unit sphere in E, where as before, u^+ denotes the positive part of u. For the sake of simplicity, we do not use ε-dependent notation for the functions and sets introduced in this section. It is easy to see that Φ is a C^1-functional, and from the maximum principle it follows that critical points correspond to positive solutions of (29). As a substitute for the conditions (A_2) and (A_3) introduced in Section 3.1, we note the following properties.

(I) For each $w \in E$ with $w^+ \neq 0$ there exists s_w such that if $\Phi'(sw)w > 0$ for $0 < s < s_w$ and $\Phi'(sw)w > 0$ for $s > s_w$.
(II) There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S_+$ and for each compact subset $\mathcal{W} \subset S_+$ there exists a constant $C_{\mathcal{W}}$ such that $s_w \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.
(III) The map $m : S_+ \to \mathcal{N}$, $m(u) = s_u u$ is a homeomorphism between S_+ and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = u/\|u\|$.

Here, as before, $\mathcal{N} := \{u \in E \setminus \{0\} : \Phi'(u)u = 0\}$ is the Nehari manifold. Similarly as in Section 3.1 we now consider the functional $\Psi : S_+ \to \mathbb{R}$ defined by $\Psi(w) := \Phi(\widehat{m}(w))$. By the same argument as in the proof of Proposition 9, we see that Ψ is a C^1-functional on the open subset S_+ of the smooth manifold S. Moreover, we have

$$\Psi'(w)z = \|m(w)\|\Phi'(m(w))z \quad \text{for all } w \in S_+ \text{ and } z \in T_w(S_+). \tag{30}$$

Hence nontrivial critical points of Ψ are in 1-1-correspondence with nontrivial critical points of Φ. In order to apply variational methods to the functional Ψ, we need the following crucial observation:

Lemma 26. *(i) Let $(u_n) \subset S_+$ be a sequence such that* $\operatorname{dist}(u_n, \partial S_+) \to 0$ *as $n \to \infty$ (where the distance is taken with respect to the norm $\|\cdot\|$). Then $\Psi(u_n) \to \infty$.*
(ii) Ψ satisfies the Palais-Smale condition in S_+, i.e. every sequence (u_n) in S_+ such that $\Psi(u_n)$ is bounded and $\Psi'(u_n) \to 0$ as $n \to \infty$ contains a subsequence which converges strongly to some $u \in S_+$.

Proof. (i) Let $s > 0$, and note that

$$\Phi(su_n) = \frac{s^2}{2} - I(su_n),$$

where as a consequence of (17) and similar arguments as in the proof of [**8**, Lemma 3.1] we have

$$\begin{aligned} |I(su_n)| &= \Big|\int_\Omega F(su_n^+)\,dx\Big| \leq C\Big(s^2 \int_\Omega |u_n^+|^2\,dx + s^q \int_\Omega |u_n^+|^q\,dx\Big) \\ &\leq \widetilde{C}\Big(s^2 \operatorname{dist}^2(u_n, \partial S_+) + s^q \operatorname{dist}^q(u_n, \partial S_+)\Big) \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

If follows that

$$\liminf_{n\to\infty} \Psi(u_n) \geq \liminf_{n\to\infty} \Phi(su_n) = \frac{s^2}{2} \qquad \text{for every } s > 0$$

and therefore $\Psi(u_n) \to \infty$, as claimed.

(ii) By the same argument as in the proof of Theorem 14, we see that Φ satisfies the Palais-Smale condition on $\mathcal{N}$. Combining this information with (i) and (30), we conclude that (ii) holds. □

Similarly as before, we now consider the least energy value

$$c = \inf_{u\in\mathcal{N}} \Phi(u) = \inf_{w\in E\setminus\{0\}} \max_{s>0} \Phi(sw) = \inf_{w\in S_+} \max_{s>0} \Phi(sw) = \inf_{w\in S_+} \Psi(w).$$

As a consequence of standard deformation arguments with respect to the flow of a pseudogradient vector field of Ψ on S_+ (see e.g. [**42**, Chapter 5]), we now derive an abstract multiplicity result for critical points of Ψ in terms of the Lusternik-Schnirelman category with respect to sublevel sets.

THEOREM 27. *If there exists $d \geq c$ and a compact set $K \subset S_+^d$ such that $\mathrm{cat}_{S_+^d}(K) \geq k$ for some $k \in \mathbb{N}$, where $S_+^d := \{u \in S_+ \; : \; \Psi(u) \leq d\}$, then S_+^d contains at least k critical points of Ψ. If furthermore $k \geq 2$ and there exists $e > d$ such that K is contractible in S_+^e, then there exists another critical point of Ψ in $S_+^e \setminus S_+^d$.*

The proof uses standard minimax arguments for the Lusternik-Schnirelman category like e.g. in [**3, 42, 45**]. We note here that by the definitions of Ψ and m and property (III) above, the following corollary is immediate.

COROLLARY 28. *If there exists $d \geq c$ and a compact set $K \subset S_+^d$ such that $\mathrm{cat}_{\mathcal{N}^d}(K) \geq k$ for some $k \in \mathbb{N}$, where $\mathcal{N}^d := \{u \in \mathcal{N} \; : \; \Phi(u) \leq d\}$, then $\mathcal{N}^d$ contains at least k critical points of Φ. If furthermore $k \geq 2$ and there exists $e > d$ such that K is contractible in $\mathcal{N}^e$, then there exists another critical point of Φ in $\mathcal{N}^e \setminus \mathcal{N}^d$.*

In view of Corollary 28, the proof of Theorem 25 is completed once we find, for each small enough $\varepsilon > 0$, values $d = d(\varepsilon) \geq c$, $e = e(\varepsilon) > d$ and a compact set $K \subset \mathcal{N}^d$ such that

$$\mathrm{cat}_{\mathcal{N}^d}(K) \geq \mathrm{cat}(\Omega), \tag{31}$$

$$K \text{ is contractible in } \mathcal{N}^e. \tag{32}$$

While it is easy to see that (32) holds for a given compact set $K \subset \mathcal{N}$ and large enough e (see [**11**, p. 43]), property (31) requires more work and the details are technical. The idea is to use the fact that for ε positive and close to zero and values d slightly larger than c (depending on ε) the functions in $\mathcal{N}^d$ are concentrated around points in Ω, and therefore the *barycenter map*

$$u \mapsto \frac{\int_\Omega x|\nabla u|^2 \, dx}{\int_\Omega |\nabla u|^2 \, dx}$$

can be used to exploit the effect of the domain topology. The construction of a set K satisfying (31) has been carried out in detail in [**11**], and up to one point the arguments only rely on properties of the nonlinearity f guaranteed by (17) and (ii)-(iv) of Theorem 24. The point where additional information is needed is the analysis of Φ-minimizing sequences on $\mathcal{N}$ in the case where $\varepsilon = 1$ and the underlying domain is the exterior of a ball in $\mathbb{R}^N$, see [**11**, p. 38]. Here it is crucial to note that these sequences are bounded, but this can be seen precisely as in the proof of Theorem 20. For the sake of brevity we do not give more details in this survey. As a final remark we only note that the arguments in [**11**] can be somewhat simplified by replacing the barycenter map by a generalized barycenter as defined in [**16**] or [**10**].

3.5. Newtonian systems

In this section we briefly discuss the Newtonian system of ordinary differential equations

$$-\ddot{q} + q = W_q(q,t), \quad q \in \mathbb{R}^N, \; t \in \mathbb{R} \tag{33}$$

under the following assumptions on W:

(W_1) $W \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $W_q \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^N)$ and W is 2π-periodic in t,
(W_2) $W_q(q,t) = o(|q|)$ uniformly in t as $q \to 0$,
(W_3) $s \mapsto s^{-1} W_q(sq,t) \cdot q$ is strictly increasing for all $q \neq 0$ and $s > 0$,
(W_4) $W(q,t)/|q|^2 \to \infty$ uniformly in t as $|q| \to \infty$.

Since $W(q,t)$ may be replaced by $W(q,t) - W(0,t)$, we may also assume without loss of generality that $W(0,t) = 0$ for all t.

THEOREM 29. *Suppose W satisfies* (W_1)-(W_4). *Then system (33) has a 2π-periodic ground state solution. If W is even in q, then (33) possesses infinitely many pairs of 2π-periodic solutions.*

Let $E := H^1(S^1, \mathbb{R}^N)$, where $H^1(S^1, \mathbb{R}^N)$ is the space of 2π-periodic $\mathbb{R}^N$-valued functions with the norm given by

$$\|q\|^2 = \int_0^{2\pi} (|\dot{q}|^2 + |q|^2)\, dt.$$

The functional corresponding to (33) is

$$\Phi(q) := \frac{1}{2}\|q\|^2 - \int_0^{2\pi} W(q,t)\, dt \equiv \frac{1}{2}\|q\|^2 - I(q).$$

Since there is an embedding $E \hookrightarrow C(S^1, \mathbb{R}^N)$, no growth restriction on W_q is necessary, and since this embedding is compact, I' is a completely continuous operator, see Remark 4. Now it is easy to see that the proof of Theorem 16 may be adapted to the present situation.

A (non-periodic) solution q of (33) will be called *homoclinic* (to 0) if $q \not\equiv 0$ and $q(t), \dot{q}(t) \to 0$ as $|t| \to \infty$.

THEOREM 30. *Suppose W satisfies* (W_1)-(W_4). *Then system (33) has a homoclinic solution which is a ground state.*

Here $E := H^1(\mathbb{R}, \mathbb{R}^N)$ with the usual norm given by

$$\|q\|^2 := \int_{\mathbb{R}} (|\dot{q}|^2 + |q|^2)\, dt$$

and

$$\Phi(q) := \|q\|^2 - \int_{\mathbb{R}} W(q,t)\, dt.$$

Then $\Phi \in C^1(E, \mathbb{R})$ and critical points $q \neq 0$ of Φ are homoclinic solutions of (33), see e.g. [**20**]. The proof of Theorem 30 parallels that of Theorem 20. We omit the details.

Also now it would be possible to apply the arguments of [**44**] in order to show the existence of infinitely many geometrically distinct pairs of homoclinics if W is even in q, and also here it is known that for not necessarily even W there exist infinitely many homoclinics under some stronger assumptions, see e.g. [**5, 20**].

Suppose the system (33) is autonomous, i.e., $W = W(q)$, and W satisfies (W_1)-(W_4). Then there exists a homoclinic q and also all its translates $q(\cdot - \theta)$, $\theta \in \mathbb{R}$, are homoclinics. Consider now the single equation

$$-\ddot{q} + q = q^3, \quad q, t \in \mathbb{R}.$$

The potential $V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$ is a W-shaped curve and it is easy to see by energy considerations that if q_0 is a homoclinic, then $A = \{\pm q_0(\cdot - \theta) : \theta \in \mathbb{R}\}$ is the set of all homoclinics. So there exist only two homoclinics which are geometrically distinct in the $\mathbb{R}$-sense (i.e., distinct in the real and not only formal sense) but infinitely many $\mathbb{Z}$-distinct ones, cf. the considerations preceding the proof of Theorem 20.

CHAPTER 4

Generalized Nehari manifold

4.1. Abstract setting

In this section we assume E is a Hilbert space and $\Phi \in C^1(E,\mathbb{R})$. Moreover, we are given an orthogonal decomposition

$$E = E^+ \oplus E^0 \oplus E^- \equiv E^+ \oplus F$$

such that $\dim E^0 < \infty$. We shall write

$$u = u^+ + u^0 + u^- = u^+ + v, \quad u^\pm \in E^\pm,\ u^0 \in E^0,\ v \in F.$$

So in contrast to our previous notation, from now on $u^\pm$ do *not* denote the positive and the negative part of a function u. Let

$$S^+ := S \cap E^+ = \{u \in E^+ : \|u\| = 1\},$$

$$E(u) := \mathbb{R}u \oplus F \equiv \mathbb{R}u^+ \oplus F \quad \text{and} \quad \widehat{E}(u) := \mathbb{R}^+ u \oplus F \equiv \mathbb{R}^+ u^+ \oplus F, \tag{34}$$

where as usual, $\mathbb{R}^+ = [0,\infty)$. We make the following assumptions on Φ:

- (B_1) $\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u)$, where $I(0) = 0$, $\frac{1}{2}I'(u)u > I(u) > 0$ for all $u \neq 0$ and I is weakly lower semicontinuous.
- (B_2) For each $w \in E \setminus F$ there exists a unique nontrivial (i.e., $\neq 0$) critical point $\widehat{m}(w)$ of $\Phi|_{\widehat{E}(w)}$. Moreover, $\widehat{m}(w)$ is the unique global maximum of $\Phi|_{\widehat{E}(w)}$.
- (B_3) There exists $\delta > 0$ such that $\|\widehat{m}(w)^+\| \geq \delta$ for all $w \in E \setminus F$, and for each compact subset $\mathcal{W} \subset E \setminus F$ there exists a constant $C_{\mathcal{W}}$ such that $\|\widehat{m}(w)\| \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

The inequality in (B_1) may seem artificial and restrictive but it will be satisfied in our applications, see Lemma 21. Let

$$\mathcal{M} := \{u \in E \setminus F : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in F\}.$$

If $u \neq 0$ and $\Phi'(u) = 0$, then $\Phi(u) = \Phi(u) - \frac{1}{2}\Phi'(u)u = \frac{1}{2}I'(u)u - I(u) > 0$ while $\Phi \leq 0$ on F. Hence $\mathcal{M}$ contains all nontrivial critical points of Φ and $\widehat{E}(w) \cap \mathcal{M} = \{\widehat{m}(w)\}$ whenever $w \in E \setminus F$. Since $\widehat{m}(w) = \widehat{m}(tw)$ for all $t > 0$, it is easy to see that if $F = \{0\}$, then (B_2), (B_3) are equivalent to (A_2), (A_3) and $\mathcal{M} = \mathcal{N}$. We shall call the set $\mathcal{M}$ the *generalized Nehari manifold*. It has been introduced by Pankov in [**35**]. We also note that if $\Phi \in C^2(E,\mathbb{R})$ and the restriction of $\Phi''(\widehat{m}(w))$ to $E(w)$ is negative definite for all $w \in E \setminus F$, then $\mathcal{M}$ is a manifold of class C^1 as follows from the implicit function theorem. According to our definitions, we have

$$\widehat{m} : E \setminus F \to \mathcal{M} \quad \text{and} \quad m := \widehat{m}|_{S^+} : S^+ \to \mathcal{M}. \tag{35}$$

It is easy to see that m is a bijection whose inverse m^{-1} is given by

$$m^{-1}(u) = \frac{u^+}{\|u^+\|}.$$

Since by (B_3), $\|m(w)^+\| \geq \delta$ for all $w \in E \setminus F$, $\mathcal{M}$ is bounded away from F and closed. Let

$$c := \inf_{u \in \mathcal{M}} \Phi(u).$$

It follows from (B_2) that $c > 0$ if it is attained. We shall show that if $u_0 \in \mathcal{M}$ and $\Phi(u_0) = c$, then u_0 is a critical point of Φ and hence it must be a ground state for the equation $\Phi'(u) = 0$.

Let us mention here that a reduction of an indefinite functional to a functional on E^+ is well known under stronger differentiability conditions, see [**14**] and e.g. [**1, 13, 38, 39**] (and [**2**] for a related finite-dimensional reduction). In [**38, 39**] a reduction in two steps has been performed: first to E^+ and then to a Nehari manifold on E^+.

Proposition 31. *Suppose Φ satisfies (B_1)-(B_3). Then:*
(a) The mapping $\widehat{m}$ is continuous.
(b) The mapping m is a homeomorphism between S^+ and $\mathcal{M}$.

Proof. (a) Suppose $(w_n) \subset E \setminus F$, $w_n \to w \notin F$. Since $\widehat{m}(w) = \widehat{m}(w^+/\|w^+\|)$, we may assume without loss of generality that $w_n \in S^+$. It suffices to show that $\widehat{m}(w_n) \to \widehat{m}(w)$ after passing to a subsequence. Write $\widehat{m}(w_n) = s_n w_n + v_n$ ($v_n = v_n^0 + v_n^- \in F$). By (B_3), $(\widehat{m}(w_n))$ is bounded. So taking a subsequence, $s_n \to s$ and $v_n \rightharpoonup v_* = v_*^0 + v_*^-$. Setting $\widehat{m}(w) = sw + v$, it follows from (B_2) that

$$\Phi(\widehat{m}(w_n)) \geq \Phi(s_n w_n + v) \to \Phi(sw + v) = \Phi(\widehat{m}(w))$$

and hence, using the weak lower semicontinuity of the norm and I,

$$\begin{aligned}\Phi(\widehat{m}(w)) &\leq \lim_{n\to\infty} \Phi(\widehat{m}(w_n)) = \lim_{n\to\infty}\left(\frac{1}{2}s_n^2 - \frac{1}{2}\|v_n^-\|^2 - I(\widehat{m}(w_n))\right)\\ &\leq \frac{1}{2}s^2 - \frac{1}{2}\|v_*^-\|^2 - I(sw + v_*) \leq \Phi(\widehat{m}(w)).\end{aligned}$$

Hence the inequalities above must be equalities. It follows that (v_n^-) is strongly convergent and so is (v_n^0) because $\dim E^0 < \infty$. Consequently, $v_n \to v_*$ and by the uniqueness property (B_2), $v_* = v$.

(b) This is immediate from (a) and the fact that m^{-1} is continuous. □

Let

$$\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R},\ \widehat{\Psi}(w) := \Phi(\widehat{m}(w)) \quad \text{and} \quad \Psi := \widehat{\Psi}|_{S^+}.$$

Proposition 32. *Suppose Φ satisfies (B_1)-(B_3). Then $\widehat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$ and*

$$\widehat{\Psi}'(w)z = \frac{\|\widehat{m}(w)^+\|}{\|w\|}\,\Phi'(\widehat{m}(w))z \quad \textit{for all } w, z \in E^+,\ w \neq 0.$$

Proof. Let $w \in E^+ \setminus \{0\}$, $z \in E^+$ and put $\widehat{m}(w) = s_w w + v_w$, $v_w \in F$. As in the proof of Proposition 9, we have

$$\begin{aligned}\widehat{\Psi}(w + tz) - \widehat{\Psi}(w) &= \Phi(s_{w+tz}(w + tz) + v_{w+tz}) - \Phi(s_w w + v_w)\\ &\leq \Phi(s_{w+tz}(w + tz) + v_{w+tz}) - \Phi(s_{w+tz} w + v_{w+tz})\\ &= \Phi'(s_{w+tz} w + v_{w+tz} + \tau_t s_{w+tz} tz) s_{w+tz} tz\end{aligned}$$

for all $|t|$ small enough and some $\tau_t \in (0,1)$, and also a similar reverse inequality holds. So continuing as in the above-mentioned proof, we obtain

$$\widehat{\Psi}'(w)z = s_w \Phi'(s_w w + v_w)z = \frac{\|\widehat{m}(w)^+\|}{\|w\|}\Phi'(\widehat{m}(w))z.$$

□

Corollary 33. *Suppose Φ satisfies (B_1)-(B_3). Then:*
(a) $\Psi \in C^1(S^+, \mathbb{R})$ and

$$\Psi'(w)z = \|m(w)^+\|\Phi'(m(w))z \quad \textit{for all } z \in T_w(S^+).$$

(b) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{M}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.
(c) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding values of Ψ and Φ coincide and $\inf_{S^+}\Psi = \inf_{\mathcal{M}}\Phi$.
(d) If Φ is even, then so is Ψ.

The argument is the same as in Corollary 10 except that the proof of (b) needs to be slightly modified. Now we have $E = T_w(S^+) \oplus E(w)$ for all $w \in S^+$ and setting $= m(w)$, (11) is replaced by

$$\|\Psi'(w)\| = \sup_{\substack{z\in T_w(S^+)\\ \|z\|=1}} \Psi'(w)z = \|u^+\| \sup_{\substack{z\in T_w(S^+)\\ \|z\|=1}} \Phi'(u)z = \|u^+\|\,\|\Phi'(u)\|,$$

where the last equality follows because $\Phi'(u)v = 0$ for all $v \in E(w)$ and $E(w)$ is orthogonal to $T_w(S^+)$. Since $\|u^+\| \ge \delta > 0$ for $u \in \mathcal{M}$, this gives the conclusion.

Remark 34. Similarly as in Remark 11, we now have the following variational characterization of the infimum of Φ over $\mathcal{M}$:

$$c = \inf_{u\in\mathcal{M}} \Phi(u) = \inf_{w\in E\setminus F}\ \max_{u\in\widehat{E}(w)} \Phi(u) = \inf_{w\in S^+}\ \max_{u\in\widehat{E}(w)} \Phi(u).$$

The following is an analogue of Theorem 12:

Theorem 35. *Suppose Φ satisfies (B_1), (B_2) and*
(i) $I'(u) = o(\|u\|)$ as $u \to 0$,
(ii) $I(su)/s^2 \to \infty$ uniformly for u on weakly compact subsets of $E \setminus \{0\}$ as $s \to \infty$.
(iii) I' is completely continuous.
Then equation $\Phi'(u) = 0$ has a ground state solution. Moreover, if I is even, then this equation has infinitely many pairs of solutions.

Proof. We show that (B_3) holds. By (B_1) and (i), we can find $\rho, \eta > 0$ such that $\Phi(w) \ge \eta$ whenever $w \in S_\rho(0) \cap E^+$. So for any $w \in E \setminus F$, $\Phi(\widehat{m}(w)) \ge \eta$. Since $\Phi(\widehat{m}(w)^+) \ge \Phi(\widehat{m}(w))$, $\|\widehat{m}(w)^+\| \ge \delta$ for some $\delta > 0$. That $\|\widehat{m}(w))\| \le C_{\mathcal{W}}$ for all w in a compact set $\mathcal{W} \subset S^+$ is an immediate consequence of (B_1), (ii) and the fact that $\Phi(\widehat{m}(w)) > 0$. Since $\widehat{m}(w) = \widehat{m}(w^+/\|w^+\|)$ for all $w \in E \setminus F$, this is also true if $\mathcal{W}$ is a compact subset of $E \setminus F$. We also note that $c = \inf_{\mathcal{M}} \Phi \ge \eta > 0$.

The rest of the argument is the same as in the proof of Theorem 12 taking Corollary 33 into account and assuming Proposition 36 below. □

We remark that it does not suffice to assume $s \mapsto I'(su)u/s$ is increasing for all $s > 0$ in order to ensure that (B_2) holds.

PROPOSITION 36. *Under the assumptions of Theorem 35, Φ satisfies the Palais-Smale condition on $\mathcal{M}$.*

PROOF. We modify some earlier arguments. Let $(u_n) \subset \mathcal{M}$ be a sequence such that $\Phi(u_n) \le d$ for some $d > 0$ and $\Phi'(u_n) \to 0$. If (u_n) is unbounded, we set $v_n := u_n/\|u_n\|$. Passing to a subsequence, we may assume $\|u_n\| \to \infty$ and $v_n \rightharpoonup v$. It follows from (ii) that if $v \ne 0$, then

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{I(\|u_n\|v_n)}{\|u_n\|^2} \tag{36}$$

and the right-hand side above tends to $-\infty$ (cf. (14)). Hence $v = 0$. By (36) and since $I \ge 0$, $\|v_n^+\| \ge \|v_n^-\|$. If $v_n^+ \to 0$, then also $v_n^- \to 0$ and therefore

$$\|v_n^0\|^2 = 1 - \|v_n^+\|^2 - \|v_n^-\|^2 \to 1.$$

Hence $v_n^0 \to v^0$ because E^0 is finite-dimensional. So $v \ne 0$, a contradiction. Therefore $v_n^+ \not\to 0$ and thus $\|v_n^+\| \ge \alpha$ for all n and some $\alpha > 0$, possibly after passing to a subsequence. We complete the proof of boundedness of (u_n) by noting that

$$d \ge \Phi(u_n) \ge \Phi(sv_n^+) \ge \frac{1}{2}\alpha^2 s^2 - I(sv_n^+) \to \frac{1}{2}\alpha^2 s^2 \tag{37}$$

for all $s > 0$, a contradiction again (cf. (15)). So (u_n) is bounded and

$$\Phi'(u_n) = u_n^+ - u_n^- - I'(u_n) \to 0.$$

Since I' is completely continuous and $\dim E^0 < \infty$, (u_n) has a convergent subsequence. $\square$

4.2. Elliptic equations – the indefinite case

Now we return to the problem (16) in a bounded domain, but this time for $\lambda \ge \lambda_1$. Let $E = H_0^1(\Omega)$ and let $E = E^+ \oplus E^0 \oplus E^-$ be the orthogonal decomposition corresponding to the spectrum of $-\Delta - \lambda$ in E. More precisely, denote the Dirichlet eigenvalues of $-\Delta$ by $\lambda_1, \lambda_2, \dots$ and a corresponding orthogonal (in E) set of eigenfunctions by $e_1, e_2, \dots$. Suppose $\lambda_k < \lambda = \lambda_{k+1} = \dots = \lambda_m < \lambda_{m+1}$, where $1 \le k < m$. Then

$$E^- = \operatorname{span}\{e_1, \dots, e_k\} \quad \text{and} \quad E^0 = \operatorname{span}\{e_{k+1}, \dots, e_m\}.$$

We also admit the cases $k = 0$ and $k = m \ge 1$ which respectively correspond to $E^- = \{0\}$ and $E^0 = \{0\}$. Let $u = u^+ + u^0 + u^- \in E^+ \oplus E^0 \oplus E^-$. In E we introduce an equivalent norm such that

$$\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx = \|u^+\|^2 - \|u^-\|^2.$$

Then the functional Φ corresponding to (16) is given by

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u), \tag{38}$$

where as before,

$$I(u) = \int_\Omega F(x,u)\,dx. \tag{39}$$

The following result is taken from [**44**] and is an extension of Theorem 16 to the case $\lambda \ge \lambda_1$:

THEOREM 37. *Suppose that* $\lambda \geq \lambda_1$, f *satisfies (17) and*
(i) $f(x,u) = o(u)$ *uniformly in* x *as* $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|$ *is strictly increasing on* $(-\infty, 0)$ *and* $(0,\infty)$,
(iii) $F(x,u)/u^2 \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (16) has a ground state solution. Moreover, if f *is odd in* u, *then (16) has infinitely many pairs of solutions.*

PROOF. Since I is weakly continuous by Theorem 3, it follows from (38) and Lemma 21 that (B_1) holds, and so do (i), (ii) of Theorem 35 as has been shown during the course of the proof of Theorem 16. Since also (iii) holds according to Theorem 3, the conclusion will follow from Theorem 35 if we can verify (B_2). This will be done in Proposition 39 below. $\square$

In Lemma 38 and Proposition 39 below we assume that the hypotheses of Theorem 37 are satisfied.

LEMMA 38. *Let* u, s, v *be real numbers such that* $s \geq -1$ *and let* $w := su+v \neq 0$. *Then*

$$f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,u+w) < 0$$

for all $x \in \Omega$.

PROOF. We fix $x \in \Omega$ and $u, v \in \mathbb{R}$. For $s \geq -1$ we put $z = z(s) := (1+s)u+v$ (so $z = u+w$) and

$$g(s) := f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,z).$$

We must show $g(s) < 0$ whenever $u \neq z$. Suppose $u = 0$. Then $z \neq 0$ and hence $g(s) = -F(x,z) < 0$ by Lemma 21. In what follows we assume $u \neq 0$. If $uz \leq 0$, we have, since $v = z - (1+s)u$,

$$\begin{aligned} g(s) &= f(x,u)\left[\left(\frac{s^2}{2}+s\right)u + (s+1)(z-(s+1)u)\right] + F(x,u) - F(x,z) \\ &< f(x,u)\left[\left(\frac{s^2}{2}+s\right)u + (s+1)(z-(s+1)u)\right] + \frac{1}{2}f(x,u)u - F(x,z) \qquad (40) \\ &= -\frac{1}{2}(s+1)^2 f(x,u)u + (s+1)f(x,u)z - F(x,z) \leq 0, \end{aligned}$$

where we have used Lemma 21 and the fact that $f(x,u)z \leq 0$ whenever $uz \leq 0$. Hence $g(s) < 0$ in this case. Suppose $uz > 0$. We note that

$$g(-1) = -\frac{1}{2}f(x,u)u + F(x,u) - F(x,v) < -F(x,v) \leq 0 \quad \text{and} \quad \lim_{s\to\infty} g(s) = -\infty$$

by Lemma 21. Moreover,

$$g'(s) = uz\left(\frac{f(x,u)}{u} - \frac{f(x,z)}{z}\right). \qquad (41)$$

Suppose that g attains its maximum on $[-1,\infty)$ at some point s with $g(s) \geq 0$. Then $g'(s) = 0$ and $u = z$ by (41) and (ii) of Theorem 37, so $g(s) = -\frac{1}{2}s^2 f(x,u)u \leq 0$. It follows that $g(s)$ may be 0 if $u = z$ (i.e., $w = 0$) but must be negative otherwise. $\square$

Recall the definition (34) of $\widehat{E}(u)$.

PROPOSITION 39. *(i)* $\widehat{E}(w) \cap \mathcal{M} \neq \emptyset$ *for any* $w \in E \setminus (E^0 \oplus E^-) \equiv E \setminus F$.
(ii) If $u \in \mathcal{M}$, *then*

$$\Phi(u+w) < \Phi(u) \quad \textit{whenever } u+w \in \widehat{E}(u),\ w \neq 0.$$

Hence u *is the unique global maximum of* $\Phi|_{\widehat{E}(u)}$.

PROOF. (i) Let $w \in E \setminus F$. Since $\widehat{E}(w) = \widehat{E}(w^+/\|w^+\|)$, we may assume $w \in S^+$. We claim that $\Phi \le 0$ on $\widehat{E}(w) \setminus B_R(0)$ provided R is large enough. Arguing by contradiction, we find a sequence (u_n) such that $\|u_n\| \to \infty$ and $\Phi(u_n) \ge 0$. Setting $v_n := u_n/\|u_n\|$, we can use (36) and the argument following it to conclude that $v_n \rightharpoonup 0$ and at the same time $\|v_n^+\| = \|s_n w\| = s_n$ is bounded and bounded away from 0. But then $v_n^+ \to sw$, $s > 0$, a contradiction.

By (i) of Theorem 37, $\Phi(sw) = \frac{1}{2}s^2 + o(s^2)$ as $s \to 0$. Hence $0 < \sup_{\widehat{E}(w)} \Phi < \infty$. Since Φ is weakly upper semicontinuous on $\widehat{E}(w)$ and $\Phi \le 0$ on $\widehat{E}(w) \cap F$, the supremum is attained at some point u_0 such that $u_0^+ \neq 0$. So u_0 is a critical point of $\Phi|_{\widehat{E}(w)}$ and hence $u_0 \in \mathcal{M}$.

Since $\dim F < \infty$, the argument above could be simplified. We have chosen not to do this because later on we shall encounter a situation where F is infinite-dimensional.

(ii) Let

$$B(v_1, v_2) := \int_\Omega (\nabla v_1 \cdot \nabla v_2 - \lambda v_1 v_2)\, dx, \quad v_1, v_2 \in E.$$

For $u \in \mathcal{M}$, let $u + w \in \widehat{E}(u)$. Then $u + w = (1+s)u + v$, where $s \ge -1$ and $v = v^0 + v^- \in F$. A calculation gives

$$\begin{aligned}
\Phi(u+w) - \Phi(u) &= \frac{1}{2}[B(u+w, u+w) - B(u,u)] + \int_\Omega (F(x,u) - F(x,u+w))\, dx \\
&= \frac{1}{2}[B((1+s)u+v, (1+s)u+v) - B(u,u)] + \int_\Omega (F(x,u) - F(x,u+w))\, dx \\
&= \frac{1}{2}\Big([(1+s)^2 - 1]B(u,u) + 2(1+s)B(u,v) + B(v,v)\Big) \\
&\qquad + \int_\Omega (F(x,u) - F(x,u+w))\, dx \\
&= -\frac{\|v^-\|^2}{2} + B(u, s(\frac{s}{2}+1)u + (1+s)v) + \int_\Omega (F(x,u) - F(x,u+w))\, dx \\
&= -\frac{\|v^-\|^2}{2} + \int_\Omega \Big(f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,u+w)\Big)\, dx.
\end{aligned}$$

In the last step we have used the fact that $z := s(\frac{s}{2}+1)u + (1+s)v \in E(u)$ and therefore

$$0 = \Phi'(u)z = B(u,z) - \int_\Omega f(x,u)z\, dx.$$

Since w is nonzero on a set of positive measure, the last integral above is negative according to Lemma 38 and hence $\Phi(u+w) < \Phi(u)$. □

We have completed the proof of Theorem 37. Next we turn to a result in [**44**] for the nonlinear Schrödinger equation (22) with V such that 0 is in a gap of the spectrum of $-\Delta + V$. In [**44**] it has been shown that (22) has a ground state solution and if, in addition, f is odd in u, then there are infinitely many geometrically distinct solutions. Here we only prove the existence of a ground state.

THEOREM 40. *Suppose $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies (17) and*
(i) V, f are 1-periodic in $x_1, \dots, x_N$, $0 \notin \sigma(-\Delta+V)$ and $\sigma(-\Delta+V) \cap (-\infty, 0) \neq \emptyset$,
(ii) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
(iii) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,
(iv) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.
Then equation (22) has a ground state solution.

PROOF. Let $E = H^1(\mathbb{R}^N)$. Since $0 \notin \sigma(-\Delta+V)$ and $\sigma(-\Delta+V) \cap (-\infty, 0) \neq \emptyset$, $E^0 = \{0\}$, $\dim E^\pm = \infty$ and there is an equivalent norm in E such that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx = \|u^+\|^2 - \|u^-\|^2.$$

Hence (38) holds except that in (39) Ω should be replaced by $\mathbb{R}^N$. Since $F \geq 0$, I is weakly lower semicontinuous as follows easily from Fatou's lemma and the fact that if $u_n \rightharpoonup u$ in E, then $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$ and hence a.e. after passing to a subsequence. This and Lemma 21 imply that (B_1) holds, and so does (B_2) by Proposition 39. (The only difference here is that integration is performed over $\mathbb{R}^N$ instead of Ω and $-\lambda v_1 v_2$ is replaced by $V(x)v_1v_2$ in the definition of the bilinear form B. Note also that in the setting of Theorem 37 we had $\dim E^- < \infty$ but as we have already mentioned in the proof of Proposition 39, this fact was not used.) That (B_3) is satisfied and $c = \inf_{\mathcal{M}} \Phi > 0$ follows as in the proof of Theorem 35.

Now it remains to combine the arguments of Theorem 20 and Proposition 36 as follows. We take a minimizing Palais-Smale sequence (w_n) for Ψ. Then (u_n), where $u_n := m(w_n)$, is a Palais-Smale sequence for Φ according to Corollary 33. Assuming $\|u_n\| \to \infty$ and setting $v_n := u_n/\|u_n\|$, we see as in (36) and the argument following it that $v_n \rightharpoonup 0$ after passing to a subsequence and $\|v_n^+\| \geq 1/\sqrt{2}$ (because $\|v_n^+\| \geq \|v_n^-\|$ and $\|v_n^+\|^2 + \|v_n^-\|^2 = 1$). If $v_n^+ \to 0$ in $L^q(\mathbb{R}^N)$, then using (24) we obtain a contradiction as in (37). Hence (26) holds with v_n replaced by v_n^+, and again we may assume, possibly after replacing v_n by $v_n(\cdot - y_n)$ for some $y_n \in \mathbb{Z}^N$, that (y_n) is a bounded sequence. But then $v^+ \neq 0$ and hence $v \neq 0$, a contradiction. We have shown that (u_n) is bounded. The rest of the argument is exactly the same as in Theorem 20 except that if $u_n \to 0$ in $L^q(\mathbb{R}^N)$, then (28) should read

$$o(\|u_n^+\|) = \Phi'(u_n)u_n^+ = \|u_n^+\|^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n^+\, dx = \|u_n^+\|^2 + o(\|u_n^+\|).$$

Hence $u_n^+ \to 0$ in E and

$$\liminf_{n\to\infty} \Phi(u_n) = \liminf_{n\to\infty} \left(\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - I(u_n) \right) \leq \frac{1}{2} \lim_{n\to\infty} \|u_n^+\|^2 = 0,$$

contradicting the fact that $\inf_{\mathcal{M}} \Phi > 0$. $\square$

Below we formulate two results for systems of equations. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the problem

$$\left\{ \begin{array}{ll} -\Delta u_1 = h(x,u_2), & x \in \Omega \\ -\Delta u_2 = g(x,u_1), & x \in \Omega \\ u_1 = u_2 = 0, & x \in \partial\Omega. \end{array} \right. \tag{42}$$

These systems, both in bounded and unbounded domains, have been considered in several recent papers, see e.g. [**38, 39**] and the references there. Our results seem to be new under the weak superlinearity conditions which we impose here.

Denote

$$G(x,u_1) := \int_0^{u_1} g(x,s)\,ds \quad \text{and} \quad H(x,u_2) := \int_0^{u_2} h(x,s)\,ds.$$

THEOREM 41. *Suppose g, h satisfy (17) and (i)-(iii) of Theorem 37. Then system (42) has a ground state solution. Moreover, if g is odd in u_1 and h odd in u_2, then (42) has infinitely many pairs of solutions.*

PROOF. Let $E := H_0^1(\Omega)\times H_0^1(\Omega)$ and

$$\Phi(u) := \int_\Omega \nabla u_1\cdot\nabla u_2\,dx - \int_\Omega (G(x,u_1)+H(x,u_2))\,dx \qquad \text{for } u=(u_1,u_2)\in E.$$

Then $\Phi\in C^1(E,\mathbb{R})$ and critical points of Φ are solutions of (42). The quadratic form

$$u \mapsto \int_\Omega \nabla u_1\cdot\nabla u_2\,dx$$

is indefinite and $E = E^+\oplus E^-$, where

$$E^\pm = \{u\in E : u_2 = \pm u_1\}. \tag{43}$$

So $\dim E^\pm = \infty$ and each $u\in E$ may be represented as

$$u = u^+ + u^- = \frac{1}{2}(u_1+u_2,\, u_1+u_2) + \frac{1}{2}(u_1-u_2,\, u_2-u_1), \quad \text{where } u^\pm\in E^\pm.$$

Hence we can write

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u)$$

with the norm $\|\cdot\|$ on E defined by $\|u\|^2 = \int_\Omega(|\nabla u_1|^2+|\nabla u_2|^2)$ for $u=(u_1,u_2)\in E$. Now it remains to repeat the argument of Theorem 37. The only point which requires explanation is that, since Lemma 38 holds for both g and h, we obtain as in Proposition 39(ii):

If $u\in\mathcal{M}$, then $\Phi(u+w)<\Phi(u)$ for any $w\neq 0$ such that $u+w\in\widehat{E}(u)$.

Indeed, we can write

$$u+w = (1+s)u+v \qquad \text{with } s\geq -1 \text{ and } v\in E^-.$$

Hence a computation similar to the one in the proof of Proposition 39(ii) gives

$$\begin{aligned}
\Phi(u+w)-\Phi(u) &= -\frac{\|v\|^2}{2} \\
&\quad + \int_\Omega \Big(\nabla u_1\cdot\nabla[s(\frac{s}{2}+1)u_2+(1+s)v_2]\,dx \\
&\quad + \int_\Omega \Big(\nabla u_2\cdot\nabla[s(\frac{s}{2}+1)u_1+(1+s)v_1]\,dx \\
&\quad + \int_\Omega \Big(G(x,u_1)-G(x,u_1+w_1)+H(x,u_2)-H(x,u_2+w_2)\Big)\,dx \\
&= -\frac{\|v\|^2}{2} + \int_\Omega \Big(g(x,u_1)[s(\frac{s}{2}+1)u_1+(1+s)v_1]+G(x,u_1)-G(x,u_1+w_1)\Big)\,dx \\
&\quad + \int_\Omega \Big(h(x,u_2)[s(\frac{s}{2}+1)u_2+(1+s)v_2]+H(x,u_2)-H(x,u_2+w_2)\Big)\,dx.
\end{aligned}$$

Since $w=(w_1,w_2)\neq 0$, at least one of the integrals above is negative and therefore $\Phi(u+w)<\Phi(u)$ as required. □

Finally we consider the problem

$$\begin{cases} -\Delta u_1+u_1=h(x,u_2), & x\in\mathbb{R}^N \\ -\Delta u_2+u_2=g(x,u_1), & x\in\mathbb{R}^N \\ u_1(x),u_2(x)\to 0, & |x|\to\infty. \end{cases} \tag{44}$$

THEOREM 42. *Suppose g,h satisfy (17) and (i)-(iv) of Theorem 40. Then system (44) has a ground state solution.*

Here $E=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$,

$$\Phi(u_1,u_2)=\int_{\mathbb{R}^N}(\nabla u_1\cdot\nabla u_2+u_1u_2)\,dx-\int_{\mathbb{R}^N}(G(x,u_1)+H(x,u_2))\,dx$$

and $E=E^+\oplus E^-$, with $E^\pm$ given by (43). The proof is by inspection of the arguments of Theorems 40 and 41. It can also be seen by inspection of the arguments in [**44**] that if g,h are odd in u_1 and u_2 respectively, then there exist infinitely many geometrically distinct solutions. However, if g and h are independent of x, see our earlier comments concerning $\mathbb{Z}^N$-distinct vs. $\mathbb{R}^N$-distinct.

REMARK 43. It is easy to see that one can admit more general linear parts in (42) and (44). We do not pursue the details.

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