

Homological algebra and algebraic topology

Problem set 9

due: Tuesday Nov 19 in class.

Problem 1 (3pt+1pt). Let p be a prime and let k be a positive integer. Suppose that X is a topological space with homology groups

$$H_0(X) \cong \mathbf{Z}, \quad H_k(X) \cong \mathbf{Z}/p\mathbf{Z}, \quad H_i(X) = 0, \quad i \neq k.$$

Calculate the cohomology groups $H^i(X; \mathbf{F}_p)$ and $H^i(X; \mathbf{Q})$ for all i .

(**Bonus + 1pt**: can you construct such a space?)

Problem 2 (2pt). Let M and N be abelian groups. Show that $\mathrm{Tor}_1^{\mathbf{Z}}(M, N)$ is always a torsion group. Moreover, show that $\mathrm{Tor}_1^{\mathbf{Z}}(M, N)$ contains an element of order n if and only if both M and N contain an element of order n .

Problem 3 (3pt). Let X be a topological space such that $\tilde{H}_k(X; \mathbf{F}_p) = 0$ for all primes p and all k . Show that the group $\tilde{H}_k(X)$ is uniquely divisible for every k . (An abelian group A is uniquely divisible if for every $a \in A$ and every non-zero integer n , there is a unique $b \in A$ such that $nb = a$.)

Problem 4 (2pt). Consider a chain complex of finite dimensional vector spaces over a field \mathbf{k} ,

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0.$$

Let $c_i = \dim_{\mathbf{k}} C_i$ and $h_i = \dim_{\mathbf{k}} H_i(C_*)$. Prove that

$$\sum_{i=0}^n (-1)^i c_i = \sum_{i=0}^n (-1)^i h_i.$$

(This number is the *Euler characteristic* of the chain complex.)