Homological algebra and algebraic topology
Problem set 8

due: Tuesday Nov 12 in class.

Problem 1 (2pt). Consider the subspaces

$$\Delta_t^{n+1} = \{(t_0, \ldots, t_{n+1}) \mid 0 \le t_i \le 1, \quad t_0 + \cdots + t_{n+1} = 1\} \subset \mathbf{R}^{n+2},$$
$$\Delta_x^{n+1} = \{(x_0, \ldots, x_n) \mid 0 \le x_0 \le \ldots \le x_n \le 1\} \subset \mathbf{R}^{n+1}.$$

Verify that the change of coordinates $x_i = t_0 + \cdots + t_i$, for $i = 0, 1, \ldots, n$, defines a homeomorphism $\Delta_t^{n+1} \to \Delta_x^{n+1}$.

In what follows we will use the x-coordinates for Δ^{n+1}.

Problem 2 (3pt). Consider the maps $\eta_0, \ldots, \eta_n \colon \Delta^{n+1} \to \Delta^n \times I$ defined by

$$\eta_i(x_0, \ldots, x_n) = \big((x_0, \ldots, \widehat{x}_i, \ldots, x_n), x_i\big).$$

Prove that the simplices $\Delta_i^{n+1} = \operatorname{im}(\eta_i)$ provide a triangulation of $\Delta^n \times I$, i.e.,

$$\Delta^n \times I = \bigcup_{i=0}^{n} \Delta_i^{n+1},$$

and the intersection of Δ_i^{n+1} and Δ_j^{n+1} is either empty or a common face of both. Draw a picture for $n = 2$.

Problem 3 (2pt). In this problem we will fill in the missing step in the proof of homotopy invariance for singular homology. Given a homotopy $h \colon X \times I \to Y$ between f and g, consider the maps $h_0, \ldots, h_n \colon S_n X \to S_{n+1} Y$ defined by

$$h_i(\sigma)(x_0, \ldots, x_n) = h\big(\sigma(x_0, \ldots, \widehat{x}_i, \ldots, x_n), x_i\big).$$

Verify the identities

$$\begin{aligned} d_0 h_0 &= f, \quad d_{n+1} h_n = g, \\ d_i h_j &= h_{j-1} d_i \quad (i < j), \\ d_j h_j &= d_j h_{j-1}, \\ d_i h_j &= h_j d_{i-1} \quad (i > j+1). \end{aligned}$$

(Hint: Begin by figuring out formulas for the face maps d_i in terms of the x-coordinates for Δ^{n+1}.)

Problem 4 (3pt). Let $n \neq 0$. Establish an exact sequence of chain complexes

$$0 \to C_*(X) \xrightarrow{n} C_*(X) \to C_*(X; \mathbf{Z}/n\mathbf{Z}) \to 0$$

and use this to derive a short exact sequence

$$0 \to H_i(X)/nH_i(X) \to H_i(X; \mathbf{Z}/n\mathbf{Z}) \to T_n(H_{i-1}(X)) \to 0$$

for every i. Here $T_n(A)$ denotes the n-torsion subgroup of A, consisting of all elements annihilated by n.