Homological algebra and algebraic topology
Problem set 11

due: Tuesday Dec 3 in class.

Problem 1 (3pt). For each permutation $\omega_0, \ldots, \omega_n$ of $0, 1, 2, \ldots, n$ consider the subspace
$$\Delta_\omega^n = \{(t_0, \ldots, t_n) \in \Delta^n \mid t_{\omega_0} \leq t_{\omega_1} \leq \cdots \leq t_{\omega_n}\} \subset \Delta^n.$$
Write down a linear homeomorphism $f_\omega \colon \Delta^n \to \Delta_\omega^n$ such that the formula
$$S(\sigma) = \sum_\omega \sigma \circ f_\omega$$
defines a chain map $S \colon C_*(X) \to C_*(X)$.

Problem 2 (2pt). Consider a pair of chain complexes of abelian groups $D_* \subseteq C_*$. Suppose that $S \colon C_* \to C_*$ is a chain map that satisfies the following conditions:
  (1) $S(D_*) \subseteq D_*$, and both chain maps $S \colon C_* \to C_*$ and $S_*|_{D_*} \colon D_* \to D_*$ induce isomorphisms on all homology groups.
  (2) For every $x \in C_n$ there is an m such that $S^m(x) \in D_n$.
Prove that the map $H_n(D_*) \to H_n(C_*)$ induced by the inclusion is an isomorphism for all n.

  Remark: For the inclusion $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$, where $\mathcal{U}$ is a family of subspaces of X, and the map S in Problem 1, one can show that the first condition is always satisfied, and that the second condition is satisfied if the interiors of the spaces in $\mathcal{U}$ cover X.

Problem 3 (2pt). Show that the inclusion of pairs
$$f \colon (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\}),$$
induces an isomorphism on all relative homology groups. Show that, despite this, f is not a homotopy equivalence of pairs.

Problem 4 (3pt+2pt). Let X be a topological space. Suppose there is an open cover,
$$X = U_0 \cup \cdots \cup U_n,$$
such that each intersection $U_{i_0} \cap \ldots \cap U_{i_k}$ is either empty or contractible.
  (1) Show that $\widetilde{H}_k(X) = 0$ for $k \geq n$.
  (2) Suppose there is an integer r such that $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$ for all $k \leq r$. Show that $\widetilde{H}_k(X) = 0$ for all $k < r$.
  (3) Give examples showing that the inequalities in (1) and (2) are sharp.
(**Bonus + 2pt:** Let u_k denote the number of subsets $\{i_0, \ldots, i_k\} \subseteq \{0, 1, 2, \ldots, n\}$ of size $(k+1)$ such that $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$. Prove that $\chi(X) = \sum_{k=0}^n (-1)^k u_k$.)